ETH Lecture 401-2673-00L Numerical Methods for CSE

Probevorlesung Maturandentage

Formerhaltende Interpolation

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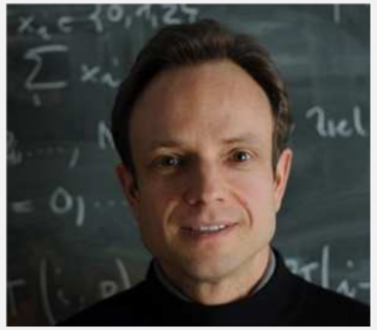


Aufgabe:

Modelliere die Beziehung physikalischer Grössen t, y durch eine (einfache) Funktion

$$f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$$

gestützt auf (genaue) Messwerte

$(t_i, y_i) \in \mathbb{R} \times \mathbb{R}$, $t_i \in D$, $t_i < t_{i+1}$, $i = 0, \dots, n$, $n \in \mathbb{N}$.

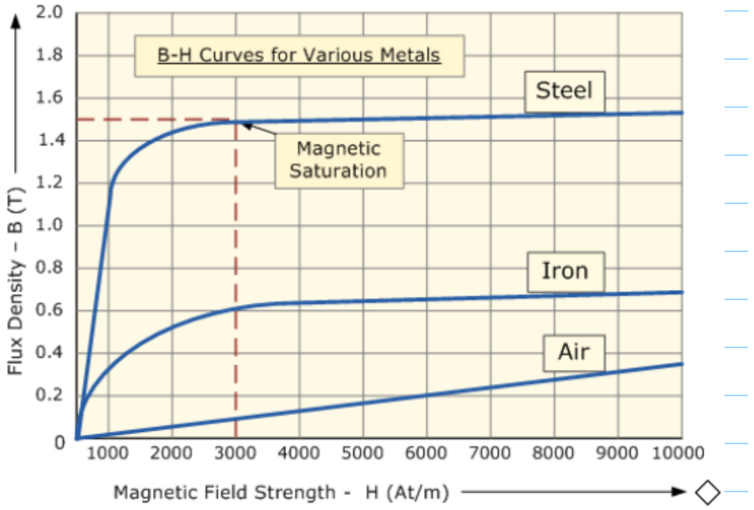
▷ Interpolationsbedingungen

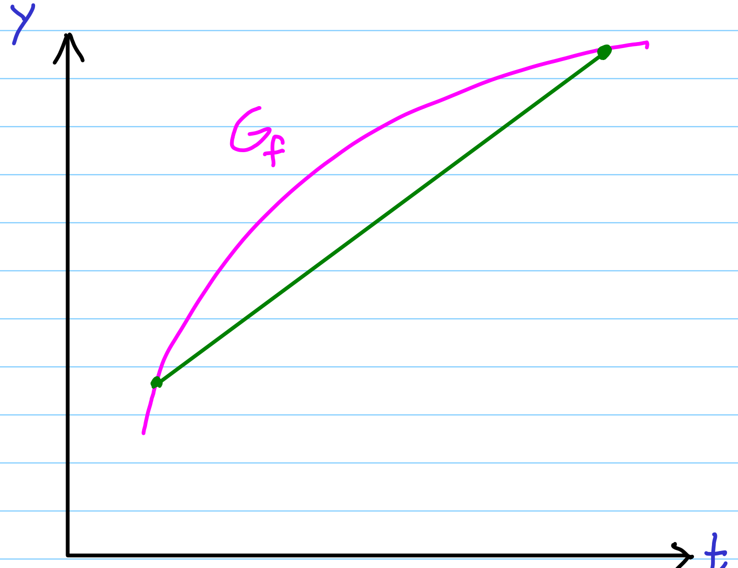
$$f(t_i) = y_i \,, \quad i = 0, \dots, n$$

Example 1.4.1 (Magnetization curves).

For many materials physics stipulates properties of the functional dependence of magnetic flux B from magnetic field strength H:

- $H \mapsto B(H)$ monotone (increasing),
- $H \mapsto B(H)$ concave
- $H \mapsto B(H)$ smooth



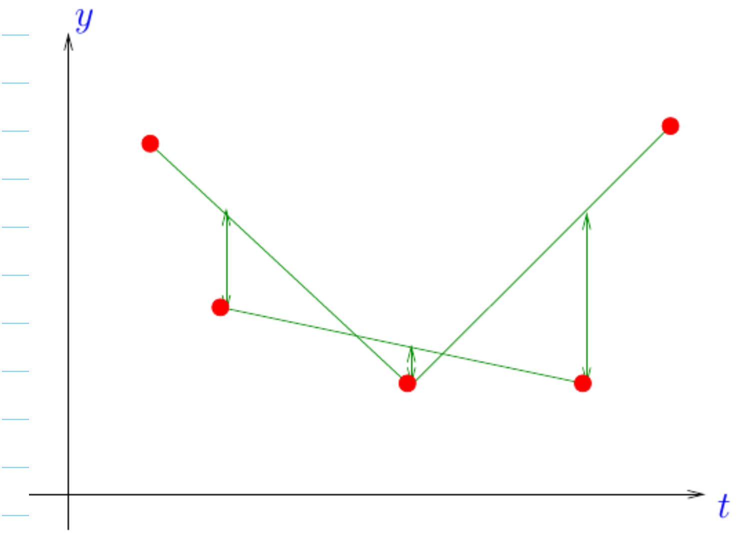


konkav

konvex

Formeigenschaften von Daten:

Definition 1.4.2 (monotonic data).
The data (t_i, y_i) are called monotonic*, when $y_i \geq y_{i-1}$ or $y_i \leq y_{i-1}$, $i = 1, \dots, n$.*



Definition 1.4.3 (Convex/concave data).
The data $\{(t_i, y_i)\}_{i=0}^n$ are called convex *(*concave*) if*

$$\Delta_j \overset{(\geq)}{\leq} \Delta_{j+1}\,, \quad j = 1, \dots, n-1 \quad ,$$
$$\Delta_j := \frac{y_j - y_{j-1}}{t_j - t_{j-1}}\,, \quad j = 1, \dots, n\,.$$

"connecting lines above data points"

Zurück zur Interpolation:

Daten $(t_i, y_i)_{i=0}^n$ ⟶ Funktion $f: D \subset \mathbb{R} \to \mathbb{R}$
$f(t_i) = y_i$

Goal: shape preserving interpolation:

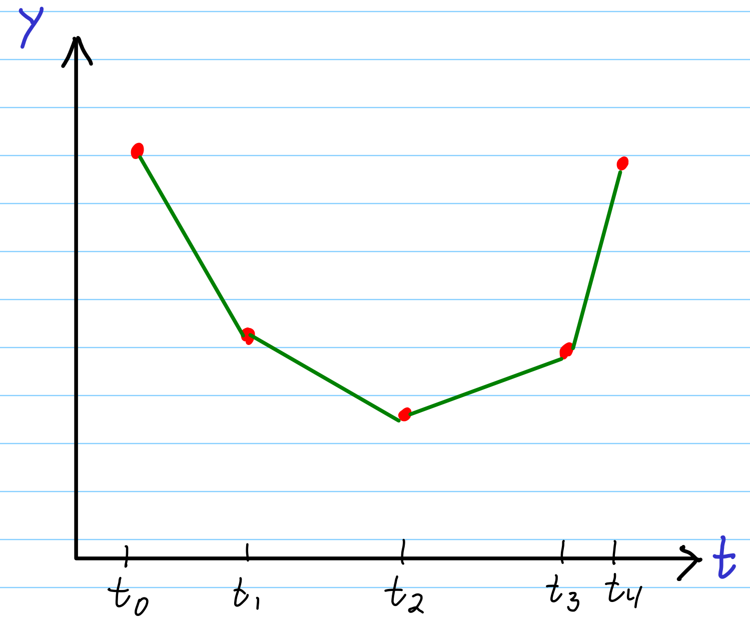
positive data	⟶	positive interpolant f,
monotonic data	⟶	monotonic interpolant f,
convex data	⟶	convex interpolant f.

Noch ambitionierter: Lokale Formerhaltung

$I \subset D$:

positive data in I	⟶	locally positive interpolant $f\vert_I$,
monotonic data in I	⟶	locally monotonic interpolant $f\vert_I$,
convex data in I	⟶	locally convex interpolant $f\vert_I$.

Option: Stückweise lineare Interpolation



• ⟷ (t_i, y_i)

— ≙ Streckenzug / Polygon

Theorem 1.4.4 (Local shape preservation by piecewise linear interpolation).
Let $s \in C([t_0, t_n])$ be the piecewise linear interpolant of $(t_i, y_i) \in \mathbb{R}^2$, $i = 0, \dots, n$, for every subinterval $I = [t_j, t_k] \subset [t_0, t_n]$:

if $(t_i, y_i)\vert_I$ are positive/negative	$\Rightarrow$	*$s\vert_I$ is positive/negative,*
if $(t_i, y_i)\vert_I$ are monotonic (increasing/decreasing)	$\Rightarrow$	*$s\vert_I$ is monotonic (increasing/decreasing),*
if $(t_i, y_i)\vert_I$ are convex/concave	$\Rightarrow$	*$s\vert_I$ is convex/concave.*

Einziger Makel : Fehlende Glattheit

Option : Stückweise kubische Interpolation

$s := f|_{[t_{i-1},t_i]}$ ist ein Polynom vom Grad 3 :

$$s(t) = f|_{[t_{i-1},t_i]}(t) = \underset{\downarrow}{a_i} t^3 + \underset{\downarrow}{b_i} t^2 + \underset{\downarrow}{c_i} t + \underset{\downarrow}{d_i}\,, \quad a_i, b_i, c_i, d_i \in \mathbb{R}$$

4 "Freiheitsgrade"

Given: data values $y_{i-1}, y_i \in \mathbb{R}$, slopes $c_{i-1}, c_i \in \mathbb{R}$

➣ There is a unique polynomial s of degree ≤ 3 with

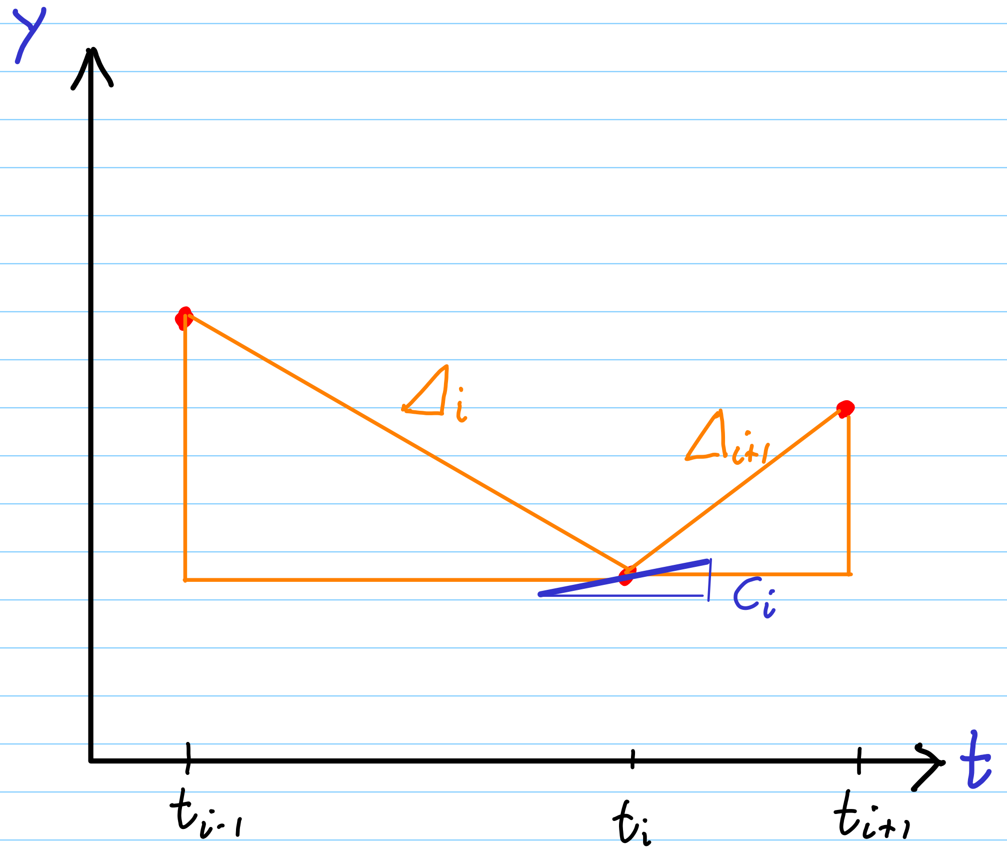
$$\underbrace{s(t_{i-1}) = y_{i-1} \quad , \quad s(t_i) = y_i}_{\text{Interpolationsbedingungen}} \quad , \quad \underbrace{s'(t_{i-1}) = c_{i-1} \quad , \quad s'(t_i) = c_i}_{\text{Fixierte Steigungen}} \; .$$

▷ "Automatisch" $f \in C^1$ (stetig differenzierbar)

Wie wählt man die Steigungen c_i ?

Naheliegend : gewichtetes Mittel

$$c_i = \begin{cases} \Delta_1 & \text{, for } i = 0\,, \\ \Delta_n & \text{, for } i = n\,, \\ \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}\Delta_i + \frac{t_i - t_{i-1}}{t_{i+1}-t_{i-1}}\Delta_{i+1} & \text{, if } 1 \leq i < n\,. \end{cases} \quad , \quad \Delta_j := \frac{y_j - y_{j-1}}{t_j - t_{j-1}}\,, \; j = 1, \ldots, n\,. \tag{1.4.6}$$



Example 1.4.7 (Piecewise cubic Hermite interpolation with averaged local slopes).

Data points:

- 11 equispaced nodes

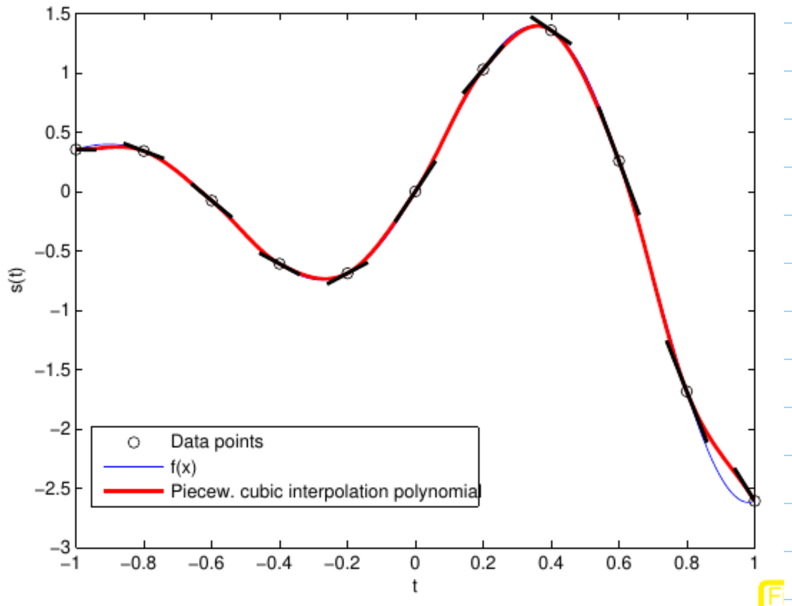
$$t_j = -1 + 0.2\,j, \quad j = 0, \dots, 10.$$

in the interval $I = [-1, 1]$,

- $y_i = f(t_i)$ with

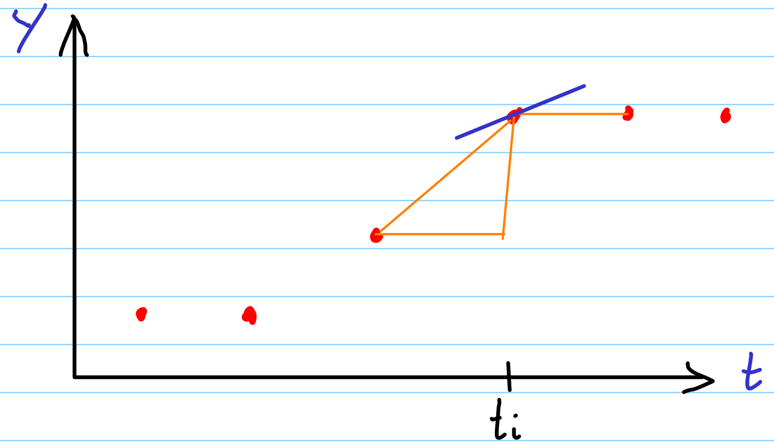
$$f(x) := \sin(5x)\,e^x \; .$$

Use of weighted averages of slopes as in (1.4.6).



Keine lokale Monotonieerhaltung

Problem "Überschiessen"



Mittelung :

$c_i > 0$!

Idee : Begrenzung der Steigung

$$c_i = \begin{cases} 0, & \text{wenn } \operatorname{sgn}(\Delta_i) \neq \operatorname{sgn}(\Delta_{i+1}), \\ \text{"geeignetes Mittel" von } \Delta_i, \Delta_{i+1} & \text{sonst}. \end{cases}$$

≈ 0, falls $\Delta_i \approx 0$ oder $\Delta_{i+1} \approx 0$

▷ Harmonisches Mittel

$$h(a,b) = \frac{2}{1/a + 1/b}$$

Special choice: *weighted harmonic mean* of local slopes

$$c_i = \frac{3(h_{i+1} + h_i)}{\frac{2h_{i+1}+h_i}{\Delta_i} + \frac{2h_i+h_{i+1}}{\Delta_{i+1}}}, \quad i = 1, \dots, n-1, \quad h_i := t_i - t_{i-1} \; . \qquad (1.4.9)$$

Theorem 1.4.10 (Monotonicity preservation of limited cubic Hermite interpolation)**.**
The cubic Hermite interpolation polynomial with slopes as in (1.4.9) *provides a* local monotonicity preserving C^1*-interpolant.*