Multi-level Monte Carlo Finite Volume methods for stochastic systems of hyperbolic conservation laws

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Abstract:

A large number of problems in physics and engineering such as global climate, propagation of tsunamis and avalanches, waves in the solar atmosphere, design of efficient aircraft, and the structural mechanics are modeled by systems of non-linear partial differential equations termed as *systems of balance laws*:

$$\begin{cases} \mathbf{U}_t(\mathbf{x},t) + \operatorname{div}(\mathbf{F}(\mathbf{x},\mathbf{U})) = \mathbf{S}(\mathbf{x},t,\mathbf{U}), \\ \mathbf{U}(\mathbf{x},0) = \mathbf{U}_0(\mathbf{x}), \end{cases} \quad \forall(\mathbf{x},t) \in \mathbf{D} \times \mathbb{R}_+. \tag{1}$$

Examples of conservation laws include the shallow water equations of oceanography, the Euler equations of aerodynamics, the Magnetohydrodynamics (MHD) equations of plasma physics and the equations of elasticity. In general, solutions of (1) develop *shock waves* in finite time even for smooth initial data. Hence, solutions are sought in the sense of distributions, additionally imposing *entropy conditions* to ensure uniqueness. For fluxes that are *non-linear* or with *varying coefficients*, analytical solution formulas are only available in very special cases. Apart from many other numerical methods, Finite Volume methods (FVM) emerged as the the most successful paradigm for practical computations in geophysics, aerodynamics and astrophysics. In FVM, the corresponding numerical fluxes are based on (approximate) solutions of Riemann problems at mesh cell interfaces. Higher order spatial accuracy is obtained by non-oscillatory reconstruction procedures such as TVD limiting, (W)ENO or by the Discontinuous Galerkin (DG) method; higher order temporal accuracy is obtained by SSP-RK method.

Uncertainty quantification. A Finite Volume scheme requires the initial data, fluxes, and source terms as inputs. These inputs are, in general, uncertain, i.e., initial condition $\mathbf{U}_0 = \mathbf{U}_0(x,\omega)$, source term $\mathbf{S} = \mathbf{S}(x,\omega,\mathbf{U})$ and fluxes $\mathbf{F} = \mathbf{F}(\mathbf{x},\omega,\mathbf{U})$ are random fields where $\omega \in \Omega$ and $(\Omega,\Sigma,\mathbb{P})$ denotes a complete probability space. Consequently, the solution $\mathbf{U}$ is sought as the *random entropy solution* of the *random balance law*:

$$\begin{cases} \mathbf{U}(x,t,\omega)_t + \operatorname{div}(\mathbf{F}(\mathbf{x},\omega,\mathbf{U})) = \mathbf{S}(\mathbf{x},\omega,\mathbf{U}), \\ \mathbf{U}(x,t,\omega) = \mathbf{U}_0(x,\omega), \end{cases} \quad (\mathbf{x},t) \in \mathbf{D} \times \mathbb{R}_+, \quad \omega \in \Omega. \tag{2}$$

Under certain assumptions on input data $\mathbf{U}_0$, $\mathbf{S}$, $\mathbf{F}$, the *existence* of the *k-th statistical moments* $\mathcal{M}^k(\mathbf{U})$ of the *random entropy solution* is established. The next step is the design of efficient numerical methods for the approximation of the *random balance law* (2). These methods include the stochastic Galerkin, stochastic collocation and stochastic Finite Volume. Currently these methods are not able to handle large number of uncertainty sources, are *intrusive* (existing deterministic solvers need to be reconfigured) and hard to parallelize. Hence, we focus on the sampling-type Monte Carlo methods.

Multi-Level Monte Carlo Finite Volume Method

Due to the slow convergence of the conventional Monte Carlo FVM sampling methods, we propose the Multi-Level Monte Carlo method (MLMC-FVM). MLMC was introduced by Giles for Itô SPDE. The key idea is to simultaneously draw MC samples on a hierarchy of nested grids:

0. **Nested meshes:** Consider *nested* meshes $\{\mathcal{T}_\ell\}_{\ell=0}^{\infty}$ of the domain $\mathbf{D}$ with corresponding mesh widths $\Delta x_\ell = 2^{-\ell}\Delta x_0$, where Δx_0 is the mesh width of the coarsest resolution.

1. **Sample:** For each level $\ell \in \mathbb{N}_0$, we draw M_ℓ independent identically distributed (i.i.d) samples $\mathbf{I}_\ell^i$ with $i = 1, \ldots, M_\ell$ from the random input data $\mathbf{I}(\omega)$ and approximate these by cell averages.

2. **Solve:** For each level ℓ and each realization $\mathbf{I}_\ell^i$, the balance law (1) is solved for $\mathbf{U}_{\Delta x_\ell}^{i,n}$ and $\mathbf{U}_{\Delta x_{\ell-1}}^{i,n}$ by the FVM method on meshes $\mathcal{T}_\ell$ and $\mathcal{T}_{\ell-1}$ with mesh widths Δx_ℓ and $\Delta x_{\ell-1}$, respectively.

3. **Estimate Statistics:** Fix $L < \infty$ corresponding to the highest level. Denoting MC estimator with $M = M_\ell$ by E_{M_ℓ}, the expectation of the random solution field $\mathbf{U}$ is estimated by

$$E^L[\mathbf{U}_{\Delta x_L}^n] := \sum_{\ell=0}^{L} E_{M_\ell}[\mathbf{U}_{\Delta x_\ell}^n - \mathbf{U}_{\Delta x_{\ell-1}}^n]. \tag{5}$$

To equilibrate the statistical and the spatio-temporal errors, we require $M_\ell = \mathcal{O}(2^{2(L-\ell)s})$ for $0 \leq \ell \leq L$. Notice that that the largest number of MC samples is required on the coarsest mesh level $\ell = 0$, whereas only a few MC samples are needed for $\ell = L$. Using such M_ℓ, we obtained the error vs. work estimate

$$\|\mathbb{E}[\mathbf{U}(t^n)] - E^L[\mathbf{U}_{\Delta x_L}^n](\omega)\|_{L^2(\Omega,\cdot)} \lesssim \begin{cases} (\text{Work})^{\min\{-s/(d+1),1/2\}} \cdot \log(\text{Work}) & \text{if } s \neq (d+1)/2, \\ (\text{Work})^{-1/2} \cdot \log(\text{Work})^{3/2} & \text{if } s = (d+1)/2. \end{cases} \tag{7}$$

The above estimate (7) shows that the MLMC-FVM is superior to the MC-FVM. For $s < (d+1)/2$, estimate (7) is exactly of the same order (modulo a log term) as the estimate for the *deterministic* FVM.

Parallel implementation. We have developed a massively parallel code ALSVID-UQ, which implements the MLMC-FVM algorithm to solve the systems of stochastic balance laws (2). We designed novel *static* and *adaptive* load balancing procedures and achieved linear (strong and weak) scaling up to 40 000 cores.

Numerical example. We consider three-dimensional Euler equations in domain $\mathbf{D} = [0,1]^3$ and the so-called *cloud-shock* initial data with 11 sources of uncertainty, i.e. with *random* initial shock at *random* location (near $x = 0.1$) heading towards high density cloud with uncertain shape of its boundary and uncertain inner density. The mean and variance for the density of the solution at time $t = 0.06$ are shown in Figure Figure 1. The results are from a MLMC-WENO run with 7 nested levels of resolution ($L = 6$) and the finest resolution is set to 1024^3 mesh. The flow in this case consists of the supersonic initial shock moving to the right, interacting with the high density bubble and leading to a complex flow pattern that consists of a leading bow shock, trailing tail shocks and a very complex center region possessing sharp gradients as well as turbulent like smooth features. Runtime: 5 hours on 21 844 cores.

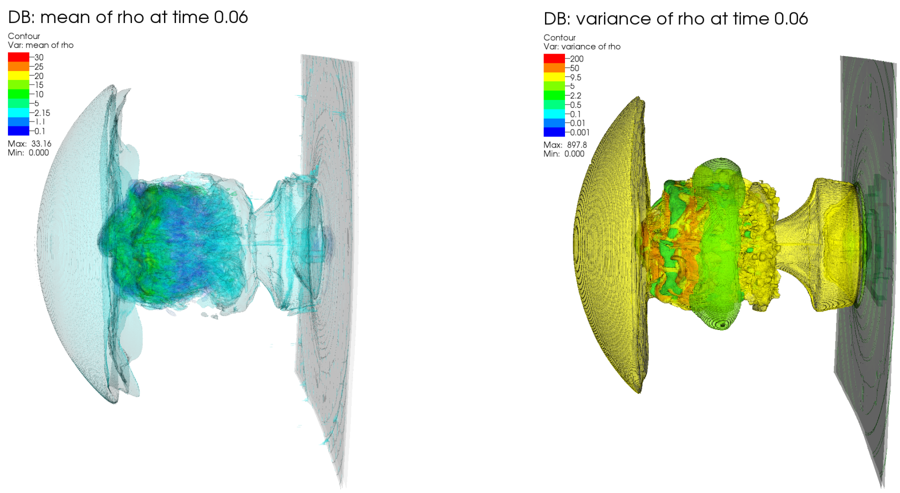

Figure 1: Mean and variance of density in the cloud-shock estimated with MLMC-FVM.

Short biography: I have obtained my BSc in Mathematics in Jacobs University Bremen, Germany, and my MSc in ETH Zürich. My scientific interests are: hyperbolic nonlinear stochastic partial differential equations, numerical analysis and simulations, massively parallel high performance computing, finite volume methods, multi-level Monte Carlo methods. More information under: `http://cv.sukys.lt`.

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