Multi Level Monte Carlo methods with Control Variate for elliptic Stochastic Partial Differential Equations

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Abstract:

We consider the numerical approximation of a partial differential equation (PDE) with random coefficients. These type of problems can be found in many applications in which the lack of available measurements makes an accurate reconstruction of the coefficients appearing in the mathematical model unfeasible. In particular we focus on the model problem of an elliptic partial differential equation with random diffusion coefficient, modeled as a random field with limited spatial regularity. This approach is inspired by the groundwater flow problem which has a great importance in hydrology: in this context the diffusion coefficient describes the permeability of the subsoil and is often modeled as a lognormal random field. More precisely, we consider the problem defined on a physical domain $D \in \mathbb{R}^d$ and on the set of all possible events Ω:

$$\begin{cases} -\mathrm{div}(a(x,\omega)\nabla u(x,\omega)) = f(x), & x \in D, \omega \in \Omega, \\ u(x,\omega) = g(x) & x \in \Gamma_D \subset \partial D, \omega \in \Omega, \\ a(x,\omega)\nabla u(x,\omega)\cdot \mathbf{n} = 0 & x \in \Gamma_N \subset \partial D, \omega \in \Omega. \end{cases}$$

where $a(x,\omega) = e^{\gamma(x,\omega)}$ and $\gamma(x,\omega)$ is a Gaussian random field. Several models have been proposed in the literature for the covariance function of the log-permeability γ leading to realizations having varying spatial smoothness. In particular, a widely used covariance function is the exponential one, $\mathrm{cov}_\gamma(x_1,x_2) = \sigma^2 e^{-\frac{\|x_1-x_2\|}{l_c}}$, that has realizations with Hölder continuity $\mathcal{C}^{0,\alpha}$ with $\alpha < \frac{1}{2}$. In this work we focus on covariance functions belonging to the so called Matérn family. These covariance functions depend on a parameter ν that defines the spatial smoothness of the field, ranging from very low spatial regularity as in the exponential covariance case ($\nu = 0.5$) to very high spatial regularity as in the gaussian covariance case ($\nu \to \infty$).

Models with limited spatial smoothness pose great numerical challenges. The first step of their numerical approximation consists in building a series expansion of the input coefficient; here we use a Fourier expansion. Whenever the random field has low regularity, such expansions converge very slowly and this makes the use of deterministic methods such as Stochastic Collocation on sparse grids highly problematic since it is not possible to parametrize the problem with a relatively small number of random variables without a significant loss of accuracy. A natural choice is to try to solve such problems with a Monte Carlo (MC) type method. In formulas, let $Q(u)$ be the quantity of interest (QoI) related to the solution u of the elliptic stochastic PDE. The MC estimator of the QoI and its corresponding mean square error (MSE) are given by:

$$\hat{Q}^{MC}_{h,M} = \frac{1}{M}\sum_{i=1}^{M}\left(Q^i_h\right), \qquad e(\hat{Q}^{MC}_{h,M})^2 = \frac{\mathbb{V}\mathrm{ar}(Q_h)}{M} + \mathbb{E}[Q_h - Q]^2,$$

where Q_h denotes the approximate evaluation of Q computed from the finite element solution u_h of the PDE with mesh size h and Q^i_h are i.i.d. replica of Q_h corresponding to i.i.d. realizations of the

log-permeability field γ. On the other hand it is well known that the convergence rate of the standard Monte Carlo method is quite slow, making it impractical to obtain an accurate solution. Indeed, the computational cost of a Monte Carlo simulation is given by the number of samples of the random field multiplied by the cost needed to solve a single deterministic PDE, which requires a very fine mesh due to the roughness of the coefficient. Multilevel Monte Carlo methods (MLMC) have already been proposed in the literature in order to reduce the variance of the Monte Carlo estimator, and consequently reduce the number of solves on the fine grid. These methods introduce a sequence of increasingly fine grids $\mathcal{T}_{h_0}, ..., \mathcal{T}_{h_L}$ and, thanks to the linearity of the expectation operator, they split the total work on different levels in order to get a cheaper estimator $\hat{Q}^{MLMC}_{h,\{M_l\}}$ in terms of computational cost.
In this work we propose to use a MLMC approach combined with an additional control variate variance reduction technique on each level in order to solve the elliptic SPDE for different choices of the covariance function of the input field, within the Matérn family. The control variate is obtained as the solution of the PDE with a regularized version of the lognormal random field $\gamma^\epsilon(x,\omega) = \gamma(x,\omega) * \phi_\epsilon(x)$ with $\phi_\epsilon(x)$ a smooth convolution kernel. Since γ^ϵ is smooth, the mean of the correspondent QoI Q^ϵ_h can be successfully computed with a Stochastic Collocation method on each level. The solution of this regularized problem turns out to be highly positively correlated with the solution of the original problem on each level, which makes the control variate technique very effective.
Within this Monte Carlo framework the choice of a suitable regularized version of the input random field is the key element of the method; we propose to regularize the random field by convolving the log-permeability with a centered Gaussian kernel having ϵ^2 variance. We analyze the mean square error of the estimator and the overall complexity of the algorithm. The MLCV estimator of the QoI and its corresponding MSE bound are given by:

$$\hat{Q}^{MLCV}_{h,\{M_l\}} = \sum_{l=0}^{L} \frac{1}{M_l} \sum_{i=1}^{M_l} \left(Q^i_{h_l} - Q^i_{h_{l-1}} - (Q^{\epsilon,i}_{h_l} - Q^{\epsilon,i}_{h_{l-1}}) \right) + \mathbb{E}[Q^{\epsilon,SC}_{h_L}],$$

$$e(\hat{Q}^{MLCV}_{h_L,\{M_l\}})^2 \leqslant \sum_{l=0}^{L} \frac{\mathbb{V}\mathrm{ar}(Y^{CV}_{h_l})}{M_l} + 2\mathbb{E}[Q^\epsilon_{h_L} - Q^{\epsilon,SC}_{h_L}]^2 + 2\mathbb{E}[Q_{h_L} - Q]^2.$$

We also propose possible choices of the regularization parameter and of the number of samples per grid so as to equilibrate the space discretization error, the statistical error and the error in the computation of the expected value of the control variate by Stochastic Collocation. Numerical examples demonstrate the effectiveness of the method. A comparison with the standard Multi Level Monte Carlo method is also presented for different choices of the covariance function of the input field.

References

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Short biography – Currently I am a PhD student at EPFL in the Chair of scientific computing and uncertainty quantification. My research is devoted to the development of efficient numerical methods for flow and transport phenomena in heterogeneous random porous media. This project is funded by the Fonds National Suisse (FNS). I got my bachelor and master diplomas at Politecnico di Milano in Mathematical Engineering.