Chapter 5

Probabilistic Design of Structures Submitted to Fatigue

5.1. Introduction

The fatigue behavior of materials submitted to cyclic loading is a random phenomenon at any scale of description. When considering materials without microscopic defects, the fatigue initiation sites usually correspond to the creation of slip bands within the superficial grains and are influenced by the size and the location of the grains as well as the roughness of the material surface. When considering materials with inclusions (e.g. carbides within nickel-based alloys) or defects (micro-cavities in cast steels), the latter may become preferential sites for crack initiation. The location and the size of these defects within an elementary material volume are naturally random ([BAT 10], Chapter 3).

Once the crack has been initiated at the microscopic scale, its trans-granular propagation in stage I is controlled by the crystalline orientation of the grain and its neighbors: the fine description of this phenomenon can be deterministically modeled at the grain scale. Nevertheless, when considering the crack propagation at the macroscopic scale of the structure, the phenomenon is still random.

At the specimen scale, the different mechanisms that are presented above, cannot be predicted: deterministically the lifetime of a specimen of a particular material (i.e. whose composition is perfectly controlled) subjected to an identical loading, varies from one specimen to another: this is the distribution observed by any

---

Chapter written by Bruno SUDRET.

experimenter, and which can be represented by a scatter plot within the $(\log N,\ S)$ plane, where S is the loading amplitude and N the number of cycles to failure measured according to the standards, e.g. [AFN 90].

If the fatigue phenomenon is considered through the propagation of a crack under a cyclic loading at the macroscopic scale, it is usually observed that the parameters controlling the propagation (e.g. Paris law's parameters [PAR 63]) are random, as shown, for instance in [VIR 78] for aluminum.

Thus the random aspect of the fatigue phenomenon seems to be occurring at all scales of description. However, if we pay a close attention to the regulations in force regarding the behavior of loaded structures under fatigue conditions, they are based on *deterministic rules* (RCC-M code for the nuclear industry [AFC 00], AC25.571-1 rules of the *Federal Aviation Administration* for the space industry, etc.). Relying on so-called conservative criteria (which means that the design performed according to these criteria shall ensure safety by construction), they are based on simplified engineering models. In addition, academic work on fatigue often tends to explain deterministically how the fatigue phenomena occur.

From these initial observations, it seems important to develop a coherent probabilistic fatigue phenomenon approach based on a fine description of the physics at the desired scale and on a rigorous treatment of uncertainty. For the last ten years, the uncertainty treatment in physical models has been studied in various probabilistic engineering mechanics (reliability of the structures, domains, stochastic finite elements). It is therefore suitable to apply the general uncertainty quantification methods to the fatigue problems of materials and structures: this is what this chapter focuses on.

A general framework dealing with the treatment of the uncertainties in mechanical models is proposed in section 5.2. Different statistical treatment methods of fatigue data are then described in section 5.3. A probabilistic design methodology is then proposed in section 5.4. The uncertainty issue regarding the propagation of existing cracks is addressed in section 5.5, by considering some non-destructive test data to update the predictions.

5.2. Treatment of hazard in mechanical models

5.2.1. *General scheme*

The treatment of uncertainties in mechanical models has been studied for the past 30 years and took different names such as *random mechanics* [KRE 83], *structural reliability* [DIT 96, LEM 05, MEL 99], *stochastic finite elements* [GHA 91] or *sensitivity analysis* [SAL 00, SAL 04] depending on the research area. Under these

different names, we can find a common methodology which allow us to come up with Figure 5.1 [SUD 07].

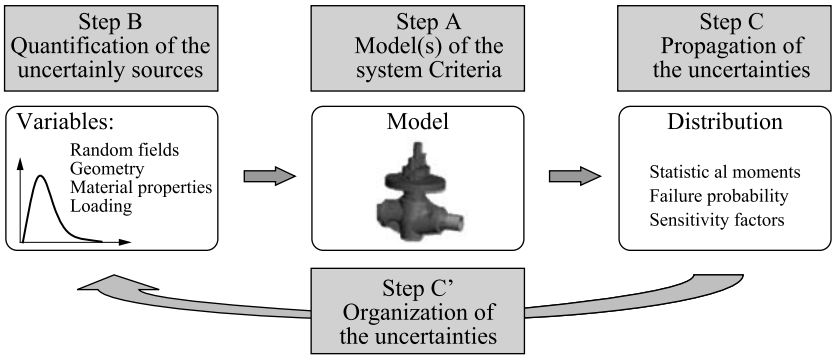

Figure 5.1. *Uncertainty treatment in mechanical models*

In the first step (called A), we usually define the *model* of the considered mechanical system, and especially its input parameters and its response (also named *quantity of interest*). We shall call $\boldsymbol{x}$ the vector of the input parameters, which may describe the geometry of the system (length of the elements, shape of the cross-sections, etc.), the behavior (elasticity moduli, parameters of the constitutive laws) and the applied loading. The response $\boldsymbol{y} = \mathcal{M}(\boldsymbol{x})$ is usually a vector gathering displacements, stress and strain components, internal variables (strain hardening, damage indicators), etc. It may also contain some post-processed quantities (e.g. amplitudes of extracted cycles by the Rainflow method, cumulative damage, etc.). Criteria related to these quantities of interest are also usually defined (e.g. acceptable threshold for a specific response in the context of reliability analysis).

5.2.2. *Probabilistic model of the input parameters*

As the model and its input parameters have been defined, we shall now focus on those parameters that are uncertain and define a *probabilistic modeling*. The sources of uncertainty, which can be multiple, are usually split into two categories:

– The *epistemic* uncertainties, which are due to a lack of knowledge (e.g. lack of precision of the measurements, lack of data which may lead to insufficient reliable statistical sampling, etc.). They can usually be *reduced* in a sense since an acquisition of additional data or more precise data allows us to decrease their number.

– The *random* uncertainties, which are intrinsic with respect to the observed parameter, and cannot be reduced. This is typically the case for the quantity “number

of cycles to failure of a material under cyclic loading with a given amplitude": the more specimens are tested in the same experimental conditions, the more likely it is to find some extreme values of this number of cycles (low or high). This random uncertainty can also show a spatial variability as is the case for the properties of geomaterials.

Building a probabilistic model of the parameters (step B) consists of defining the probability distribution function (PDF) of the random vector X of the input parameters. When we do not have enough data to model the variability of a parameter, we can rely on *expert judgment*: we assume a certain shape of the distribution of the considered parameter (e.g. Gaussian distribution, uniform, lognormal, Weibull, etc.) and then we set up the mean value and the standard deviation of the distribution. To do so, guidelines are available in [JOI 02].

When data is available, we may use common *statistical inference* techniques [SAP 06]. In general, we search for the best distribution among selected families (for instance, using the maximum likelihood method) and then carry out goodness-of-fit tests to validate or discard the different choices. For fatigue issues, this type of statistical analysis is performed to process test data in order to establish the probabilistic Wöhler curves, as we will see in section 5.3. When the available sample set has a small size, we may combine the data with some prior information on the distribution (that represents *expert judgment*) by using *Bayesian statistics* [ROB 92, DRO 04].

Sometimes, the variability of the parameters of interest cannot be directly measured (e.g. parameters of the model of crack initiation), but this variability can be apprehended through the measurements of quantities of interest which depend on it (e.g. number of cycles to failure). In these cases we have to rely on *probabilistic reverse methods*. Broadly speaking these inverse problems are unsolved yet. Some recent studies have recently been published in the field of fatigue though [PER 08, SCH 07]. Whatever the selected approach, step B eventually yields a probabilistic description of the input data as the probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$ of the random vector of the input parameters $\boldsymbol{X}$ (see Appendix A for the elementary notions of probability).

5.2.3. *Uncertainty propagation methods*

Once both the mechanical model and the probabilistic model of its input parameters have been established, we shall focus on the *propagation of uncertainty*

(step C in Figure 5.1). We are now ready to characterize the random response of the model, written as $\mathbf{Y} = \mathbf{M}(\mathbf{X})$[1].

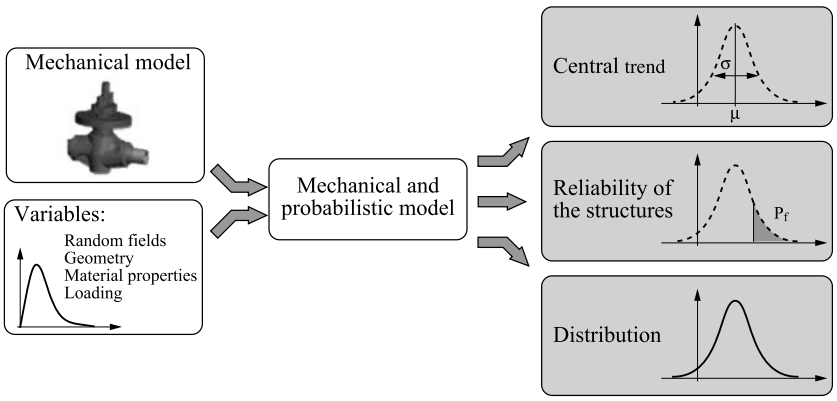

Figure 5.2. *Classification of the methods of uncertainty propagation*

Depending on what information we are looking for on Y (which is assumed to be a scalar quantity in this chapter for the sake of simplicity), we usually distinguish different types of analyses (Figure 5.2):

– *Analysis of the central tendency*, where we mainly focus on the mean value μ_Y and on the variance σ_Y^2 of the quantity of interest Y (higher order statistical moments may also be computed). If the distribution of the response Y was Gaussian, these two scalars would completely characterize it. However, this assumption is usually *wrong* in practice: thus the statistical moments shall be used for what they are, without deducing any information on the distribution of Y from their values.

– *Reliability analysis*, where we focus on the probability that the response is above a certain threshold $\bar{y}$. We then have to estimate the tail of the distribution of Y in this case. The associated probability (called failure probability) is usually low, with values ranging from 10^{-2} to 10^{-8}.

– *Distribution analysis*, where we are looking to characterize the probability density function of Y completely.

Specific methods have been proposed for every type of problem, which are given below. For more information, refer to [SUD 07]:

– The *Monte Carlo method* is the best known method of uncertainty propagation [RUB 81]. It allows us to resolve the problems presented above at least theoretically.

1. If the variability of some parameters of the model is equal to zero or is negligible, we can consider them as deterministic and gather them within a vector $\boldsymbol{d}$, and we shall write $Y = \mathcal{M}(\boldsymbol{X}, \boldsymbol{d})$.

It relies on the simulation of random numbers by specific algorithms which generate a sample set of input parameters conforming to the probabilistic model built in step B (i.e. according to the distribution $f_{\boldsymbol{X}}$). In general, this method is quite efficient for the calculation of the first statistical moments mean value and variance but gets computationally expensive when considering reliability or distribution analysis.

– The *perturbation method*, which was developed in mechanics in the 1980s, relies on a Taylor series expansion of model $\mathcal{M}$ around the nominal value of parameters $\boldsymbol{x}_0$ and allows the mean value and standard deviation of the quantities of interest to be efficiently estimated. The gradients of model $\mathcal{M}$ with respect to the input parameters need to be calculated. This method is nowadays used efficiently when these gradients are directly implemented into the finite element codes used in industrial application (e.g. Code_Aster [EDF 06]).

– The resolution of reliability problems, i.e. the estimation of the probability of failure historically led to some specific methods back in the 1970s. The FORM/SORM methods (respectively *first* and *second order reliability method*) were established in the 1980s and have been used by some industries (offshore, nuclear industry) for about 15 years. They are *approximation* methods of the distribution tail of the response and do not allow us to estimate the quality of the obtained result. They are usually coupled to some advanced simulation methods (directional simulation, importance sampling, subset simulation, etc.) [LEM 09].

– Initially introduced in the *spectral stochastic finite elements* method [GHA 91], representations of response Y by polynominal chaos are nowadays a promising approach for the treatment of uncertainties. The principle is to consider the random response Y within a suitable functional space of random variables, in which a foundation is built. Response Y is then completely represented through its "coordinates" in this basis, which are the coefficients of the polynominal chaos expansion. These coefficients may be calculated by some non-intrusive methods from a limited number of model evaluations, namely $\mathcal{Y} = \{y^{(i)} = \mathcal{M}(\boldsymbol{x}^{(i)}),\ i = 1, \ldots, N\}$. The analytical post-processing of the coefficients provides the distribution of Y at no additional calculation costs as well as its statistical moments, the probabilities of exceeding a threshold, etc. [SUD 07].

– Most of the methods of uncertainty propagation give, as by-products (with almost no additional calculation), some information on the relative significance of the input parameters of the model: we call these by-products *importance factor* or *sensitivity indices* depending on the methods. This hierarchization (or ranking) of the parameters according to their importance is called "step C" in Figure 5.1.

5.2.4. *Conclusion*

In this introductory part, we tried to focus on defining a general framework for dialing with uncertainties in mechanical models. We defined of the different necessary ingredients and then presented the most commonly used calculation

methods. The general scheme will now be applied to different problems related to the fatigue of materials and structures.

5.3. Plotting probabilistic *S–N* curves

5.3.1. *Introduction*

The experimental points obtained from fatigue testing are usually plotted in the "amplitude S/number of cycles to failure N" space in order to get the so-called Wöhler curves ([BAT 10], Chapter 2). In France, the statistical treatment obeys a standard [AFN 91] (mainly inspired by the work of Bastenaire [BAS 60]), which provides methods to obtain the endurance limit using the staircase method (Section 5 of the standard), the median Wöhler curve (Section 6), or some probabilistic $S - N$ curves (ESOPE method, Section 9).

The latter is summarized below in Section 5.3.2. A global approach for establishing probabilistic $S - N$ curves is proposed. This approach is based on the studies performed independently by Guédé [GUÉ 05], Perrin [PER 08], and Pascual and Meeker [PAS 97, PAS 99]. It has recently been used in industry by EDF and EADS [SCH 06, SCH 07].

Regardless of the approach, we consider a sample set of measurements:

$$\mathcal{E} = \{(S_i, N_i), i = 1, \ldots, Q\} \quad [5.1]$$

where N_i is the number of cycles to failure measured under alternate stress of amplitude S_i and Q is the size of the sample set. We consider that the sample set contains test results carried out in the same experimental conditions, allowing one to treat all the experimental points as a whole. We will also consider the data obtained in the case when the fatigue test has been stopped before failure has occurred. This so-called censored data is denoted by N_i^*.

5.3.2. *ESOPE method*

The ESOPE method, recommended by the AFNOR standard (A 03-405, Section 8) [AFN 91] is based on the following assumption: within a sample $\mathcal{E}$ as presented above, we *suppose* that the fraction $F(S, N)$ of the specimens, which failed *before* N cycles of amplitude S, has the following particular form:

$$F(S, N) = \Phi\left(\frac{S - \mu(N)}{\sigma}\right) \quad [5.2]$$

where σ is a distribution parameter, Φ is the standard normal (Gaussian) cumulative distribution function (see equation [5.31]) and $\mu(N)$ is a curve whose shape is

prescribed by the analyst and whose parameters are estimated from $\mathcal{E}$. We should note that the curve representing the median life time $N_{50\%}$ (corresponding to points such that $F(S,N) = 0.5$) corresponds to $S = \mu(N)$, according to equation [5.2], since $\Phi^{-1}(0.5) = 0$. The median curve gives, for each stress level S, the value $N_{50\%}$ such that there is as much chance for a specimen under loading S to fail before or after $N_{50\%}$ cycles.

On the other hand, the quantity $F(S,N)$ is an estimation of the cumulative distribution function $F_{N_S}(N;S)$ of the random variable $N_S(\omega)$ defined[2] as the number of cycles to failure of the considered material, under loading amplitude S. As a consequence, according to [5.2], the isoprobabilistic failure curves $N_p(S)$ defined as

$$\mathbb{P}\left(N_S(\omega) \leq N_p(S)\right) = p \qquad [5.3]$$

are also defined within the plane (N,S) by the following equation:

$$S = \mu(N) + \sigma\,\Phi^{-1}(p) \qquad [5.4]$$

It is clear they may be obtained by vertically switching the median curve by a quantity $\sigma\,\Phi^{-1}(p)$. In practice, we usually take a parametric form for this median lifetime, e.g. as proposed by the AFNORA 03-405 norm:

$$\mu(N) = a + b\,N^c \qquad [5.5]$$

We then estimate parameters a, b, c using the *maximum likelihood* method 5 (see section 5.7.3) by combining equations [5.2] and [5.5] to get:

$$F_{N_S}(N;S) = \Phi\left(\frac{S - [a + b\,N^c]}{\sigma}\right) \qquad [5.6]$$

From which the probability density function $f_{N_S}(N;S)$ can be derived. We can then write the likelihood of parameters a, b, c and estimate these parameters by maximizing the likelihood function.

NOTE: Equation [5.6] is often given in the following interpretation: "the variability of stress S, for a fixed life time N is normally distributed with a standard value σ." Actually, this assertion does not make any sense since, from an experimental point of view, the loading amplitude S is fixed (and not at all random!). Conversely, lifetime N under loading S is a random variable since it is related to the occurrence of cracking mechanisms at the microscopic scale that cannot be deterministically described at the macroscopic scale, as explained in the introduction.

2. Throughout the whole chapter, notation ω highlights the random property of the considered quantity. When there is no ambiguity, ω can be omitted for the sake of simplicity.

5.3.3. *Guédé-Perrin-Pascual-Meeker method (GPPM)*

The ESOPE method, which was described above, does not explicitly provide the probability density function of the random variables $N_S(\omega)$ modeling the lifetime of the specimens. Thus it is not suitable for a complete probabilistic assessment of a structure under fatigue loading, as we will see later on. The problem is better addressed by the direct modeling of the physically uncertain quantities, namely the number of cycles to failure of the specimens under loading amplitude S, i.e. by the inference of the random variables $N_S(\omega)$.

Several formulations were proposed in [SUD 03a, GUÉ 05] and were further elaborated by Perrin [PER 05]. The final formalism is actually rather close to the one proposed independently by Pascual and Meeker [PAS 97, PAS 99], which explains the GPPM acronym.

5.3.3.1. *Guédé's assumptions*

The approach by Guédé is split into different steps:

– The *choice of the distribution* for the variables $N_S(\omega)$: a lognormal distribution is usually selected (the logarithm of the number of cycles to failure is supposed to follow a Gaussian distribution). We should mention that other choices are possible (including the Weibull law). The hypothesis has to be validated *a posteriori* by performing some statistical tests.

– The *description of the parameters* of the distribution of $N_S(\omega)$ as a function of S. The initial work carried out by Sudret *et al.* [SUD 03a] assumes that the mean value $\lambda(S)$ of $\ln N_S$ can be written as follows (Stromeyer's formula):

$$\lambda(S) = A \ln(S - S_D) + B \qquad [5.7]$$

We may then consider either a constant standard deviation [LOR 05] (which is reasonable if the sample set contains mainly points in the low cycle domain), or a variable standard deviation which is a function of the amplitude of the loading S:

$$\sigma(S) = \delta \lambda(S) \qquad [5.8]$$

This last equation allows us to represent the large scattering of the data usually observed in the high cycle fatigue domain

– Regarding the type of dependency between the random variables $N_S(\omega)$ for different values of S, the hypothesis of *perfect dependency* has been adopted: it corresponds to the intuitive idea that if we could test the same specimen at different loading levels (which cannot actually be performed since fatigue tests are destructive), the fatigue strength would be uniformly good or bad, meaning that the number of cycles to failure would be, regardless of S, away from the median value with more or less the same proportions.

5.3.3.2. *Model identification*

The previous assumptions allow us to write the random variable $N_S(\omega)$ as follows:

$$\ln N_S(\omega) = \lambda(S) + \sigma(S)\,\xi(\omega) = (A\,\ln(S - S_D) + B)(1 + \delta\,\xi(\omega)) \qquad [5.9]$$

where $\xi(\omega)$ is a standard Gaussian variable (i.e. with a mean value equal to zero and a unit standard deviation). At this stage, the probabilistic model depends on the 4 parameters A, B (shape of the median curve), S_D (asymptote, which is considered as an infinite endurance limit) and δ (coefficient of varicetion of $\ln(N_s)$). Having a single variable $\xi(\omega)$ in [5.9] (and not one variable $\xi_S(\omega)$ for each amplitude) corresponds to the perfect dependency assumption described previously. From equations [5.7]–[5.7], we can get the probability density function of variable $N_S(\omega)$:

$$f_{N_S}(n, S; A, B, S_D, \delta) = \frac{1}{\delta\,[A\,\ln(S - S_D) + B]\,n}\varphi \times \left(\frac{\ln n - [A\,\ln(S - S_D) + B]}{\delta\,[A\,\ln(S - S_D) + B]}\right) \qquad [5.10]$$

where $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the standard normal probability density function.

For a given sample set $\mathcal{E}$ (see equation [5.1]), the *likelihood* of parameters A, B, S_D and δ reads:

$$\mathsf{L}(A, B, S_D, \delta; \mathcal{E}) = \prod_{i=1}^{Q} f_{N_S}(N_i, S_i; A, B, S_D, \delta) \qquad [5.11]$$

The maximum likelihood method consists of estimating the unknown parameters A, B, S_D, δ by *maximizing* the previous quantity (or by minimizing the log-likelihood $-2\ln\mathsf{L}$). The intuitive interpretation of the method is simple: it leads to the choice of the parameters which maximize the probability of having observed the available sample set $\mathcal{E}$. It is worth noting that the censored data N_i^* (no failure observed before N_i cycles with an amplitude S_i) can be used within equation [5.11] by replacing the probability density $f_{N_S}(Ni, S_i; A, B, S_D, \delta)$ by $1 - F_{N_S}(N_i^*, S_i; A, B, S_D, \delta)$, where the cumulative distribution function F_{N_S} is given by:

$$F_{N_S}(n, S; A, B, S_D, \delta) = \Phi\left(\frac{\ln n - [A\,\ln(S - S_D) + B]}{\delta\,[A\,\ln(S - S_D) + B]}\right) \qquad [5.12]$$

and $\Phi(x)$ is the standard normal cumulative distribution function.

Once the parameters have been estimated from the sample set (they are denoted with a hat from now on), the $S-N$ curves can be naturally obtained from equation [5.9]. The equation of the iso-probability failure curve p is

$$N_p(S) = (\hat{A}\,\ln(S-\hat{S}_D)+\hat{B})(1+\hat{\delta}\,\xi_p) \qquad [5.13]$$

where $\xi_p = \Phi^{-1}(p)$ is the quantile of level p of the standard normal distribution. The median curve ($\xi_p = 0$) and, for instance, the quantiles at 5% and 95% ($\xi_p = \pm 1.645$) can then be easily plotted.

5.3.3.3. *GPPM method*

In the previous section parameter S_D was introduced as a fitting parameter in order to characterize an asymptotic behavior in the high cycle domain. However, this parameter could be considered from two different perspectives:

– It may be viewed as a *deterministic* fitting parameter, *i.e.* a stress amplitude such that a fatigue test carried out below this level would never lead to failure. This is what was assumed in the previous paragraph. Then the iso-probability curves have the same horizontal asymptote at S_D in this case.

– The fatigue limit may also be considered as a "true" material parameter, whose value differs from one specimen to the other: it would be the critical amplitude such as, *for the considered specimen*, failure never occurs for any cyclic loading with an amplitude lower than this value. In that case, S_D should be considered a random variable whose realization is different (and obviously unknown) for every tested specimen. In this context it is meaningful to try to infer the distribution of S_D from data together with the other parameters describing the probabilistic Wöhler curves: this is the goal of the GPPM model.

The same assumptions as in Section 5.3.3.1 are taken into account except for parameter S_D which is now modeled as a random variable, with a probability density function $f_{S_D}(s_D;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector of the distribution parameters (e.g. $\boldsymbol{\theta} = (\lambda_{S_D}, \zeta_{S_D})$ for a lognormal law). The probability density function given in equation [5.10] becomes *conditional* to these parameters, and is written $f_{N_S|S_D}(n, s_D, S; A, B, \delta)$. In order to calculate the likelihood (equation [5.11]), the unconditional distribution f_{N_S} has to be used; it can be obtained by performing the following integration:

$$f_{N_S}(n, S; A, B, \delta, \boldsymbol{\theta}) = \int f_{N_S|S_D}(n, s_D, S; A, B, \delta)\, f_{S_D}(s_D;\boldsymbol{\theta})\, ds_D \qquad [5.14]$$

In the end, the solution of the maximum likelihood problem provides both the estimators of the parameters controlling the failure iso-probability curves $\hat{A}, \hat{B}, \hat{\delta}$ and those of the distribution of the endurance limit S_D (see details in [PER 08]).

5.3.4. *Validation of the assumptions*

In both the ESOPE method and the GPPM approach, some hypotheses are made regarding the shape of the distributions. The theory of the goodness of fit tests [SAP 06, Chapter 14] allows these hypotheses to be validated or rejected. These hypotheses (usually called "null hypothesis H_0") are as follows:

ESOPE H_0: variable $S - [a + b\,N^c]$ follows a normal distribution with a standard value σ (see equations [5.2] and [5.5]). For every point of the sample $\{(N_i, S_i),\ i = 1,\ \ldots\ , Q\}$, the previous quantity is calculated and a sample is then obtained $\{\xi_i,\ i = 1,\ \ldots\ , Q\}$ where H_0 is tested.

Guédé H_0: variable $\xi(\omega) \equiv ((\ln N_S(\omega)/[A\ \ln(S - S_D) + B]) - 1)\ /\delta$ follows a standard normal distribution. For every point of the sample $\{(N_i, S_i),\ i = 1,\ \ldots\ , Q\}$, the previous quantity is calculated and a sample is then obtained $\{\xi_i,\ i = 1,\ \ldots\ , Q\}$ whose normality is tested.

GPPM H_0: for each amplitude S, the random variable $N_S(\omega)$ follows the distribution given by equation [5.14].

It is then possible to quantitatively compare the different methods of the various assumptions and to validate *a posteriori*.

5.3.5. *Application example*

To illustrate the different methods presented above, the results from Perrin [PER 08] are reported in this section. A sample set of 153 test results on austenite steel specimens at 20°C was presented. As presented in Figure 5.3, the different approaches give close results in terms of median curve.

The curves of the quantiles at 2.5% and 97.5% are given in Figure 5.4. A significant difference can be observed regarding the general shape of the associated confidence intervals. Indeed the ESOPE method seems to underestimate the variability in the low cycle domain.

For different amplitudes S, the probability density function of the lifetime $N_S(\omega)$ can be plotted using different methods [PER 08]. In particular the goodness-of-fit tests lead to the rejection of the ESOPE model, which is not the case with the two other models. The median and quantile curves obtained using both the Guédé and the GPPM methods are very similar, which is the same regarding the probability densities

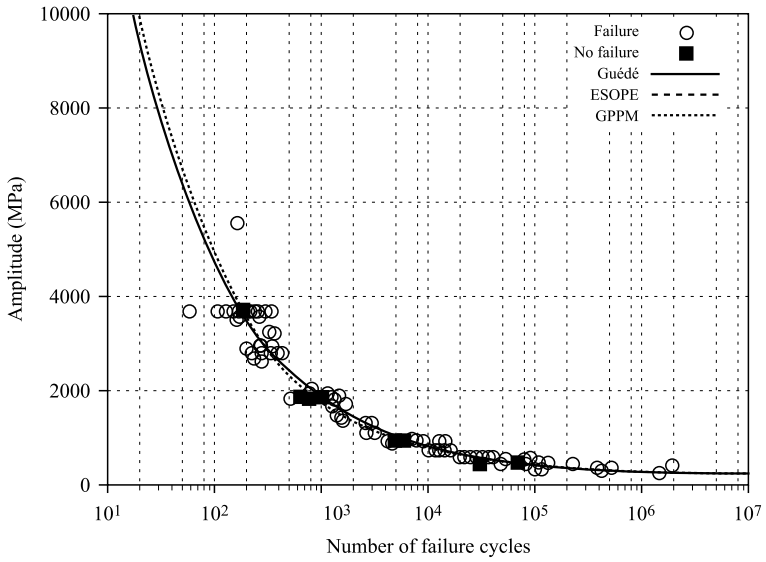

Figure 5.3. *Statistical treatment of fatigue tests on austenitic steel samples – median curves obtained with the ESOPE method and the Guédé and GPPM methods*

functions of the lifetime for some high amplitude levels. The difference between these approaches appears for the amplitudes close to the endurance limit. The latter one is estimated to be 236.9 MPa by the Guédé method and by a lognormal variable of mean value 257.8 MPa and standard value of 28.4 MPa by the GPPM method.

5.3.6. *Conclusion*

The statistical treatment of test data is a step that cannot be ignored in the design of sttuctures with respect to fatigue. In this section, the principles of the ESOPE method, which is the most commonly used in the industrial domain, were presented. An alternative formulation (named GPPM), based on the direct inference of the distributions of the lifetimes $N_S(\omega)$ was proposed and allows a random endurance limit to be considered, whose parameters are estimated jointly with the ones describing the median Wöhler curve. This approach leads to the treatment of an experimental scatter plot (including censored data) without having to rely on a specific amplitude levels and many points for each level. This is not the case for the ESOPE method which is based on the estimation of the fraction of broken specimens before N cycles, for each amplitude level S and thus requires a large amount of data at each level.

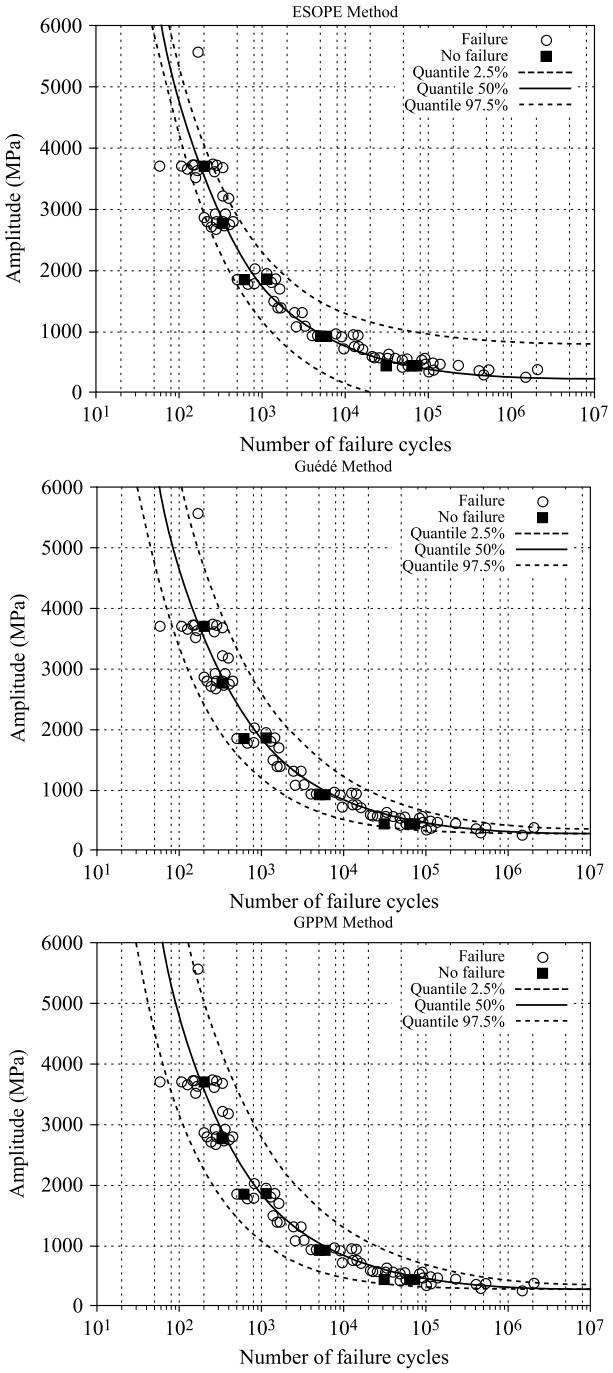

Figure 5.4. *Statistical treatment of fatigue tests on austenitic steel samples – quantile curves at 2.5% and 97.5% for the different methods*

5.4. Probabilistic design with respect to crack initation: case of a loaded pipe/under thermal fatigue conditions

5.4.1. *Introduction*

Defining Wöhler curves from the fatigue test data is usually the first step towards the design of structures subjected to fatigue. Broadly speaking, in order to design a structure, the applied loading (periodic or random loading) and its effects (stress analysis, extraction of the stress cycles) have to be characterized, and then the damage due to these stress cycles has to be calculated. In the case of a probabilistic analysis, the same procedure will be carried out including the sources of uncertainty at each step.

To do so, the *deterministic* design of fatigue of a component of a nuclear plant (RCC-M code [AFC 00]) and its transposition to a probabilistic equivalent (Guédé's work [GUÉ 05, SUD 05]) is taken as an example. According to the general scheme for managing uncertainty presented in section 5.2.1, the deterministic model (step A) will be described first, then the different sources of uncertainties will be characterized (step B) and then the uncertainties will be propagated through the model to evaluate the reliability of the structure subjected to fatigue.

It is important to note that the whole methodology does not depend on the models used at each step of the deterministic calculation, which are much simplified here for the sake of clarity. The same general framework has been recently applied by Schwob [SCH 06, SCH 07] using some multiaxial fatigue criteria in collaboration with EADS and by Perrin [PER 06] in collaboration with Renault. It is also worth noting that the method called *stress – resistance* [THO 99], which was developed for the automotive industry by PSA Peugeot Citroën, applies the same concepts, just in a simplified manner.

5.4.2. *Deterministic model*

The main steps regarding the official standards for designing pipes subjected to fatigue are listed below [AFC 00]:

Description of the loading: The main issue here is the thermal fatigue induced by temperature fluctuations within the pipes. It is necessary to determine the temperature history at the inner walls of the pipes for each particular operating sequence. This temperature history can be obtained by computational fluid dynamics (CFD) or using some measurements on extrapolated scale models, etc.

Mechanical model: From the description of the structure's geometry (e.g. internal radius, thickness), of some material properties (Young's modulus, Poisson ratio,

parameters of the elasto-plastic constitutive laws), of the boundary conditions (fluid/structure heat transfer coefficient) and of the loading, the strain and stress fields are calculated as a function of time.

Extraction of fatigue cycles: From the stress tensor the equivalent stress history is obtained by means of the Tresca criterion. Then the cycles are extracted using the rainflow method [AMZ 94], yielding a sequence of amplitudes S_i, $i = 1, \ldots, N_c$. The values obtained are then corrected in order to consider the effect of the mean stress (Goodman line within Haigh diagram).

Choice of the design curve: In the RCC-M standard the design curve $N_d(S)$ can be obtained by modifying the median Wöhler curve $N_{bf}(S)$ obtained from some tests on specimens using *coefficients* so passage called *factors*, which empirically consider all the factors leading to a reduced lifetime for the structure compared to the specimens, and aiming to be conservative. The RCC-M code then defines the design curve with the following equation:

$$N_d(S) = \min(N_{bf}(S)/\gamma_N \ , \ N_{bf}(\gamma^S S)) \qquad [5.15]$$

The value of the reserve factors are respectively $\gamma^N = 20$ (reduction of the number of cycles in the low cycle domain) and $\gamma^S = 2$ (increase of the applied stresses in the high cycle domain).

Cumulative damage rule: Miner's linear cumulative damage rule is applied [MIN 45]: each cycle with an amplitude S_i is supposed to generate the elementary damage $d_i = 1/N_d(S_i)$, and this damage is supposed to accumulate at every cycle extracted using the rainflow method:

$$D = \sum_{i=1}^{N_c} d_i = \sum_{i=1}^{N_c} 1/N_d(S_i) \qquad [5.16]$$

In order to apply the design criterion, the cumulated damage has to remain lower than 1. When the loading applied to the structure is made of some sequences which are supposed to be identical, the cumulative damage D_{seq} can then be calculated for each single sequence. Then the lifetime of the structure can be cast as $T_d = 1/D_{seq}$, corresponding to a acceptable number of sequences.

The different steps of the calculation are presented in Figure 5.5.

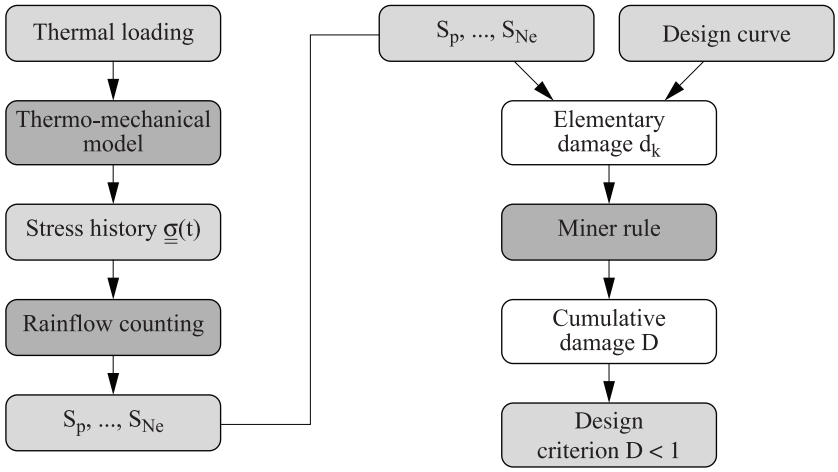

Figure 5.5. *Simplified deterministic scheme for the design of a pipe submitted to thermal fatigue*

5.4.3. *Probabilistic modeling of uncertainties*

The computational flowchart presented in the above section involves many parameters which are not well known in practice (e.g. the fluid/structure heat transfer coefficient) or are intrinsically random (e.g. the lifetime of the fatigue test specimens). It is then necessary to classify all the sources of uncertainty, and then to model them within a consistent probabilistic frame depending on the available data (Step B of the general scheme, Figure 5.1). By analyzing, the calculation process, step by step, the following parameters will be successively presented:

Thermal loading: To design some part of a circuit for a specific operating range, a thermal dimensioning signal has to be defined, either from some measurements on real structures or scale models, or from some operating scenarios or related thermohydraulic simulations. The uncertainty can be introduced by modeling the fluid temperature by a *random process*. To do so, the properties of the process can be identified from some long enough measurements (either within the time domain or within the frequency domain through the power *spectral density*). The problem can also be simplified if a sufficiently long deterministic temperature trajectory representing properly the thermal fluctuations can be determined.

Mechanical model: The parameters which are usually involved in the definition of the mechanical model (geometric dimensions, elastic or elasto-plastic properties of the materials) present a variability which can be modeled using random variables. The uncertainty in the geometry is related to the pipe manufacturing tolerances.

The uncertainties of the material properties can be characterized by some tests on the considered materials or, in the absence of available data found in the literature [JOI 02].

Design curve: The design curve was defined earlier from the specimen median Wöhler curve from the passage factors. In section 5.3, a meticulous statistical analysis methodology yielding the probabilistic Wöhler curves was presented. The passage factors shall now be studied.

Passage factors $\gamma^N = 20$ and $\gamma^S = 2$, defined above, have been introduced to the deterministic design in order to conservatively cover two effects of a very different nature: the natural variability of the lifetime under fatigue between laboratory specimens on the one hand and the structure within its real environment on the other hand. These factors can then be split as follows:

$$\begin{aligned} \gamma^N &= \gamma^N_{disp} \cdot \gamma^N_{passage} \\ \gamma^S &= \gamma^S_{disp} \cdot \gamma^S_{passage} \end{aligned} \qquad [5.17]$$

In these equations, γ_{disp} and $\gamma_{passage}$ correspond to the part connected to each effect. Some empirical decompositions can be found in the literature, for instance $\gamma^N_{disp} = 2$ when $\gamma^N = 20$ and $\gamma^S_{disp} = 1.19$ when $\gamma^S = 2$ [COL 98]. However, there is no real agreement regarding this topic as the data are mostly empirical and much connected to the type of material tested. The additional variables $\gamma^S_{passage}$ and $\gamma^N_{passage}$ are called specimen-to-structure passage factors and allow us to consider the effects of the size of the structure, its surface finish and the environmental conditions (especially regarding temperature and chemistry in nuclear engineering, etc.).

It is obvious that these passage factors that connect the crack initiation time of a specimen to that of a structure made of the *same* material (microstructure) cannot be physically measured. In order to get a rigorous probabilistic representation, it is then necessary to rely on some probabilistic inverse methods, which allow the factors and their distributions to be estimated from both data on specimens and structures (e.g. some pipe scale models that are typical of the considered real system, like the INTHERPOL tests described in [CUR 04, CUR 05])). Details about this identification technique are not reported in this chapter, see [PER 07a] for a detailed presentation.

5.4.4. *Random cumulative damage*

The cumulated damage D can be considered as the result of the computational chain, as shown in Figure 5.5. If we now consider the input parameters of each sub-model of this chain to be random (thermomechanical model, extraction of the cycles,

design curve, etc.), the cumulated damage becomes random. The natural definition of the random elementary damage related to a single cycle with a fixed amplitude S is:

$$d(S,\omega) \equiv 1/N^{struc}(S,\omega) = 1/\min\left[N_S(\omega)/\gamma^N_{passage}\ ,\ N_{\gamma^S_{passage}}(\omega)\right] \quad [5.18]$$

In this equation, $N^{struc}(S,\omega)$ is the lifetime of the structure under a loading with a constant amplitude S, which is connected to the probabilistic Wöhler curve $N_S(\omega)$ by the passage factors. If the thermomechanical calculation followed by the rainflow counting leads to a number of cycles N_c (possibly random), the cumulated random damage $D(\omega)$ is then written:

$$D(\omega) = \sum_{i=1}^{N_c} d(S_i(\omega),\omega) \quad [5.19]$$

In this equation, the random property of the damage comes from both:

– the randomness of the loading, which is propagated through the mechanical model for evaluating the amplitude of the cycles $\{S_k(\omega),\ k = 1,\ \ldots\ , N_c\}$;

– the randomness of the material fatigue strength, through the Wöhler curve.

In the case of a stationary random loading, for some loading sequences which are long enough, the number of extracted cycles N_c becomes high and can then be considered in a first approximation as being deterministic [TOV 01]. The amplitudes of the extracted cycles can then be continuously represented by their probability density function $f_S(s)$. This leads to the "continuous" definition of the random cumulated damage:

$$D(\omega)\ = \int_0^\infty \frac{N_c\, f_S(s)\, dS}{N^{struc}(S,\omega)} = N_c\, \mathrm{E}_S\left[\frac{1}{N^{struc}(S,\omega)}\right] \quad [5.20]$$

where $\mathrm{E}_S[.]$ stands for the mathematical expectation with respect to the probability density function of amplitudes $f_S(s)$. Assuming Miner's linear cumulative damage assumption and a significant number of independent extracted cycles, the cumulative damage defined by equation [5.19] has been proven to converge towards the one defined by equation [5.20] [SUD 03b].

The previous continuous formulation can be successfully applied to a fatigue assessment in the frequency domain. From the power spectral density (PSD) of the loading, and if the mechanical model is linear, the PSD of the resulting equivalent stress can be obtained. Some empirical formulae, such as those proposed by Dirlik [BEN 06, DIR 85], allow the probabilistic density of the extracted cycles using the rainflow method to be constructed, which can then be substitued into equation [5.20]. Readers are referred to the work by Guédé [GUÉ05, chapter 6, GUÉ07].

5.4.5. *Application: probabilistic design of a pipe under thermal fatigue conditions*

5.4.5.1. *Problem statement and deterministic model (step A)*

To illustrate the different concepts presented in this section, a piece of pipe, that is of a circuit of a pressurized water reactor, will be considered. The results obtained by [GUÉ 05], Chapter 7, are reported. The reader can check this reference for a comprehensive parametric analysis of the problem under different loadings, as well as for a comparison of the probabilistic approaches within both time and frequency domains.

Let us consider a section of a pipe with a radius R_{int} and a thickness t, subjected to a fluid temperature at the inner wall modeled by a random Gaussian process $\theta(t, \omega)$, whose average temperature is equal to 130°C and standard deviation is equal to 20°C. This process is a pseudo-white noise whose power spectral density is constant within the [0 ; 5 Hz] interval. From these data, a realization $\theta(t)$ is simulated on a long enough time interval ([0 ; 360 s] here) and this signal is then considered as being periodically reproduced. The first 10 seconds of the signal are represented in Figure 5.6. The component service duration is considered to be equal to $N_{seq} = 10,000$ sequences of 360 seconds.

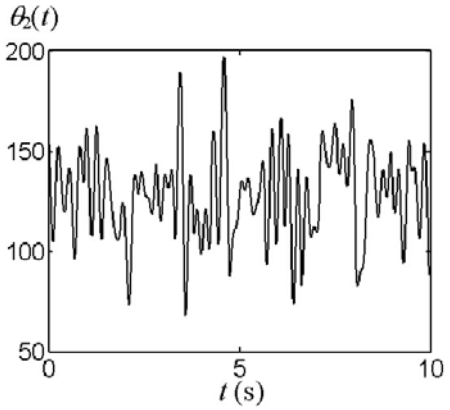

Figure 5.6. *Probabilistic design of a pipe submitted to thermal fatigue temperature history*

The stress state within the pipe can be calculated using a 1D axi-symmetrical model with generalized plane strains (the strain component ε_{zz} is assumed to be constant within the thickness). According to the RCC-M code, an elastic calculation is carried out. The fluid temperature is transferred to the inner wall of the pipe through a fluid/structure heat transfer coefficient. The external wall of the pipe is insulated

(preventing the loss of heat). Due to the simplified form of the model, the obtained stress tensor is diagonal and its components are varying synchronously in time. The orthoradial stress time history $\sigma_{\theta\theta}(t)$ may be considered to perform the rainflow counting. The amplitudes of the extracted cycles are corrected by the Goodman line within the Haigh diagram in order to consider the average stress.

5.4.5.2. *Probabilistic model (step B)*

The different random variables modeling the uncertainties on the parameters of the model are presented in Table 5.1. The choice of the probability distribution functions and their parameters (step B) was performed as follows[3]:

– The properties of the materials are modeled by some lognormal laws which are usually well adapted to this type of parameter (being positive). The coefficients of variation have to be fixed by experts. The geometry parameters are also modeled by lognormal distributions, with a variation coefficient made high on purpose.

– The variables physically bounded (Poisson ratio, passage factors) are modeled by Bêta distributions.

– The probabilized Wöhler curve (equation [5.9]) was established by the Guédé method (section 5.3.3.1) and is written as follows:

$$N(S,\omega) = \exp\left[(-2.28\log(S-185.9)+24.06\,)\,(1+0.09\,\xi\,(\omega))\right] \qquad [5.21]$$

where $N(S,\omega)$ is the crack initiation time of the specimens.

5.4.5.3. *Reliability and sensitivity analysis (step C&C')*

The probability of crack initiation is studied for a prescribed number of operating sequences N_{seq}. To do so, the following limit state function g is defined (the negative values of g correspond to the values of the parameters leading to failure):

$$g\,(N_{seq}, \boldsymbol{X}) = 1 - N_{seq}\; d_{seq}(\boldsymbol{X}) \qquad [5.22]$$

where $d_{seq}(\boldsymbol{X})$ stands for the random cumulative damage related to a sequence of 360 seconds of operation, and $\boldsymbol{X}$ stands for the vector of the random variables listed in Table 5.1.

By using the FORM method (Appendix A, section 5.7), a probability of crack initiation is obtained: $P_f = 8.59.10^{-2}$ for a service life of 10,000 sequences. This result is confirmed by some importance sampling, which finally give $P_f = 7.65.10^{-2}$.

The form method used for the calculation of the initiation probability also provides the *importance factors* of the different random variables, which allows the input

3. The data used for the calculation (especially the distribution parameters) does not represent any real structure.

Parameter	Distribution	Mean value	V.C †
Internal radius R_{int}	lognormal	127.28 mm	5%
Thickness t	lognormal	9.27 mm	5%
Heat capacity ρC_p	lognormal	4,024,000 J/kg	10%
Thermal conductivity λ	lognormal	16.345 W.m^{-1}.K^{-1}	10%
Heat transfer coefficient H	lognormal	20,000 W.K^{-1}.m^{-2}	30%
Young module E	lognormal	189,080 MPa	10%
Poisson rotio ν	Bêta [0.2; 0.4]	0.3	10%
Thermal dilatation coefficient α	lognormal	16.95 10^{-6}	10%
Yield stress S_y	lognormal	190 MPa	10%
Ultimate strength S_u	lognormal	496 Mpa	10%
Passage factor $\gamma^N_{passage}$	Bêta [7; 11]	9.39	10%
Passage factor $\gamma^S_{passage}$	Bêta [1; 2]	1.68	10%
Scattering of the fatigue data ξ (equation [5.9])	Gaussian	0	Standard value: 1

† coefficient of variation, equal to the ratio of the standard deviation and the mean value.

Table 5.1. *Probabilistic design of a pipe submitted to thermal fatigue: probabilistic model of the parameters*

Parameter	Importance factor (%)
Scattering of fatigue data ξ	40.3
Heat transfer coefficient H	20.8
Passage factor $\gamma^S_{passage}$	13.7
Young's module E	8.9
Thermal dilatation coefficient α	8.9
Other variables	7.4

Table 5.2. *Probabilistic design of a pipe submitted to thermal fatigue: importance factors*

parameters of the model to be ranked according to their respective contribution to the fatigue. These normalized factors (given in percentages) are listed by decreasing order in Table 5.2.

Beyond the strict numerical values of these importance factors (which depend on the probability density functions chosen for modeling the input parameters), the orders of magnitude as well as the obtained classification lead to some observations. It clearly appears that the main parameters that mainly explain the variability of the structure crack initiation time is the scattering of the endurance of the specimens (modeled in the probabilistic Wöhler curves). Then there are the fluid/structure transfer coefficient, the passage factor $\gamma^S_{passage}$ increasing the stress amplitudes in the high cycle domain,

and finally the Young's modulus and the thermal dilatation coefficient. The other parameters have a negligible importance, which means that their variability does not contribute to the variability of the cumulative damage (and thus to the initiation probability). As a consequence, they can be considered to be deterministic in this type of analysis.

5.4.6. *Conclusion*

In this section, we propose a general frame to address the issue of uncertainties in the design of loaded the structure submitted to fatigue, stressing on every uncertainty source observed in the computational chain. As an illustrative example, the probability of crack initiation within a pipe submitted to thermal fatigue was estimated. A quantification of the influence of every uncertain parameter on this probability was obtained. Even if caution is required for the values, some usefully qualitative conclusions can still be drawn for the comprehension of the problem.

5.5. Probabilistic propagation models

5.5.1. *Introduction*

As observed for crack initiation, the propagation of pre-existing cracks under cyclic loading shows some randomness. An extensive experimental study performed by Virkler *et al.* [VIR 78] on some alumina 2024-T3 specimens clearly shows the scattering of crack propagation among identical specimens (Figure 5.7): 68 pre-cracked (initial size $a_0 = 9$ mm) rectangular sheets with a length of $L = 558.8$ mm, a width of w =152.4 mm and a thickness of t =2.54 mm) are loaded under tension-compression ($\Delta\sigma = 48.28$ MPa, $R = 0.2$). The tests are stopped when the length of the crack reaches 49.8 mm. It can be observed that this final length is reached between 223,000 and 321,000 cycles depending on the specimens with a almost continuous distribution in between.

By observing Figure 5.7, two types of randomness can be noticed regarding crack propagation:

– a global scattering of the curves, whose shapes are similar although the number of cycles varies;

– for each curve, a local irregularity, which shows that the successive increments of the crack size along a given trajectory are also random.

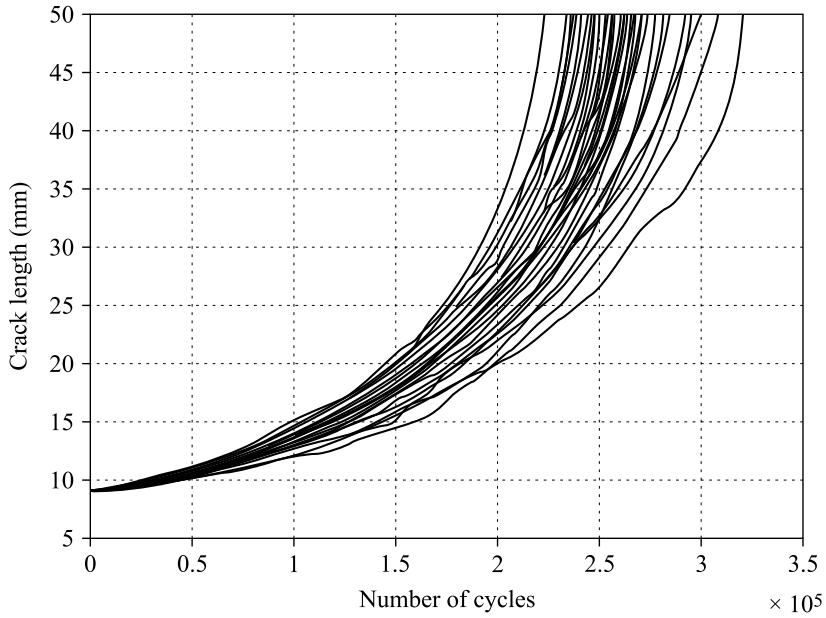

Figure 5.7. *Crack propagation – experimental propagation curves according to Virkler et al. [VIR 78]*

5.5.2. *Deterministic model*

Crack propagation under cyclic loading is usually modeled using the Paris-Erdogan law [PAR 63] ([BAT 10], Chapter 6):

$$\frac{da}{dN} = C\,(\Delta K)^m \qquad [5.23]$$

In this equation, a is the length of the crack, ΔK is the amplitude of the stress intensity factor for a cycle with an amplitude equal to $\Delta\sigma$ and (C, m) are the typical parameters of the studied material. In the case of a sheet with a width of w bearing a crack in its core, the amplitude of the stress intensity factor is given by:

$$\Delta K = \Delta\sigma\ F\left(\frac{a}{w}\right)\sqrt{\pi a} \qquad F\left(\frac{a}{w}\right) = \frac{1}{\sqrt{\cos\left(\pi\frac{a}{w}\right)}} \quad \text{for} \quad \frac{a}{w} < 0.7 \quad [5.24]$$

where $F\left(\frac{a}{w}\right)$ is the Feddersen correction factor.

To reproduce the global scattering of the experimental curves using simulation, the parameters of Paris' law [5.23] can be made random: this is the approach used

from now on in this chapter. For each sample of parameters $(C,\ m)$, a propagation curve is obtained and this curve can reproduce the general shape of the experimental curves. However, the modeling of the irregularities of the curves needs the Paris-Erdogan law to be modified by introducing a random process for modulating the size increments during the propagation itself. This type of approach has been studied by Ditlevsen and Olesen [DIT 86], Yang and Manning [YAN 96]. A detailed review can be obtained from [ZHE 98].

5.5.3. *Probabilistic model of the data*

For each crack propagation curve, the best-fit couple of parameters (C, m) may be estimated using an optimization procedure, which leads to a sample set. A statistical treatment of this sample set can then be carried out to infer the best probability density functions. Kotulski [KOT 98] shows that parameters m and $\log C$ may be reasonably represented by truncated normal distributions (Table 5.3).

Parameter	Distribution	Boundaries	Average	Variation coefficient
m	normal truncated	$[-\infty;\ 3.2]$	2.874	5.7%
$\log C$	normal truncated	$[-28;\ +\infty]$	-26.155	3.7%
Correlation coefficient: $\rho = -0.997$				

Table 5.3. *Crack propagation – probability distribution functions of* $(\log C\ ,\ m)$ *of Paris law for the Virkler tests [KOT 98]* *(*da/dN *given in mm/cycle)*

A strong correlation between both m and $\log C$ parameters can be observed (Figure 5.8), which leads us to think that a single and unique underlying parameter might exist whose variability from one specimen to another explains the dispersion of the propagation curves.

5.5.4. *Propagation prediction*

From the Paris model and the probabilistic description of the propagation parameters given in Table 5.3, a cluster of propagation curves can be obtained using the Monte Carlo simulation. This leads to a confidence interval (e.g. at a confidence level of 95%) regarding the crack length as a function of the number of applied loading cycles. Figure 5.9 shows the obtained median propagation curve as well as the 2.5% and 97.5% quantiles.

It appears that the scattering observed from the simulated curves has the same order of magnitude as that observed on the experimental curves, which validates the

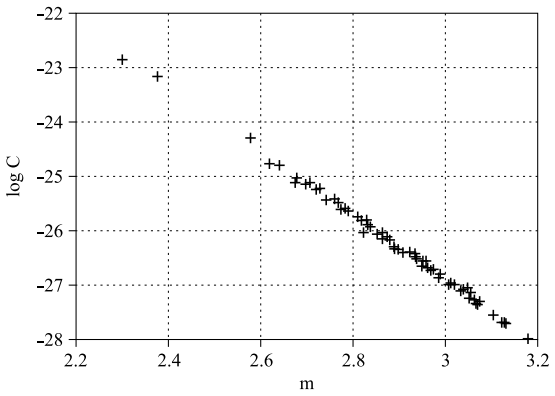

Figure 5.8. *Crack propagation – sample of the parameters of the Paris-Erdogan law for the Virkler tests (da/dN given in mm/cycle)*

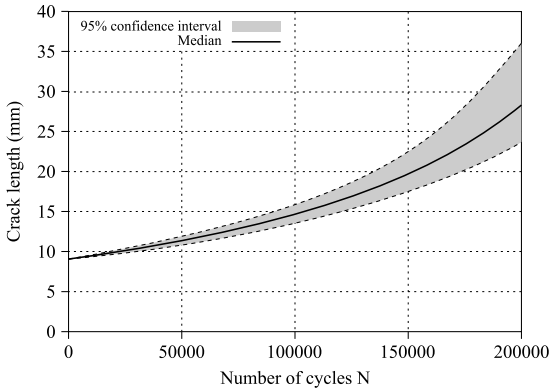

Figure 5.9. *Crack propagation – prediction of the median propagation curve and of the 95% confidence interval*

statistical treatment of the $(\log C\,,\, m)$ data previously carried out (see [BOU 08] for a detailed investigation of the influence of the correlation between $\log C$ and m on the predictions).

Nevertheless, the results presented in Figure 5.9 do not give much information in an industrial context where the best prediction of the length of the crack, as a function of the number of applied cycles, is required. The graph only leads to the conclusion that a crack with an initial size of 9 mm will reach a size ranging from 23.5 and 35.8 mm after 200,000 cycles with a probability of 95%. It is then obvious that a better estimation is required to assess, for instance, an inspection planning. The following section will focus on how to combine the previous results with the measurements

performed on a structure of interest during the initial phase of propagation, in order to reduce the confidence interval of the prediction.

5.5.5. *Bayesian updating crack propagation models*

5.5.5.1. *Introduction*

Bayesian statistics allows us to combine some *prior* information on the parameters of a probabilistic model with some measurement data. The reader who is unfamiliar with these concepts can get more information from Appendix A in section 5.7. In the context of probabilistic mechanics, the Bayesian framework can be applied to consistently combine, on the one hand, the predictions of a model whose uncertain input parameters are modeled by random variables, and on the other hand, the measurements of the response of the real mechanical system which was modeled. For the precise example of crack propagation, some measurements of the crack length obtained for different numbers of cycles *at the early stage of propagation* may be introduced in order to update the prior prediction which revealed inaccurate (as seen from the previous section).

5.5.5.2. *Ingredients for a Bayesian approach to crack propagation*

The probabilistic propagation model developed in section 5.5.4 is made of the crack propagation model (Paris-Erdogan law) and the probabilistic model of the parameters. The truncated normal distributions (Table 5.3) are considered as *prior* information (regarding the Bayesian vocabulary) on the propagation parameters in the case of 2024-T3 aluminum.

A particular specimen is now studied and the test trajectory corresponding to the slowest propagation is for the sake of illustration chosen. Figure 5.10 clearly shows that this test is singular since the propagation curve is largely outside the 95% confidence interval on the prior prediction.

To compare the observations and the predictions, a *measurement/model* error is usually defined, which considers the measurement of a physical quantity never to be perfectly accurate and the entire mathematical model of the real world to always be more or less imperfect. Thus the following link between observations and prediction is used:

$$y_{obs} = \mathcal{M}(\tilde{\boldsymbol{x}}) + e \qquad [5.25]$$

where y_{obs} is the measured value and $\mathcal{M}(\tilde{\boldsymbol{x}})$ is the prediction of the model for the "true value" $\tilde{\boldsymbol{x}}$ of the vector of the input parameters. This true value is usually unknown, but it is assumed that it corresponds to a specific realization of vector $\boldsymbol{X}$. The measurement/model error e is supposed to be a realization of a random variable of prescribed distribution (usually Gaussian, with mean value equal to zero and with

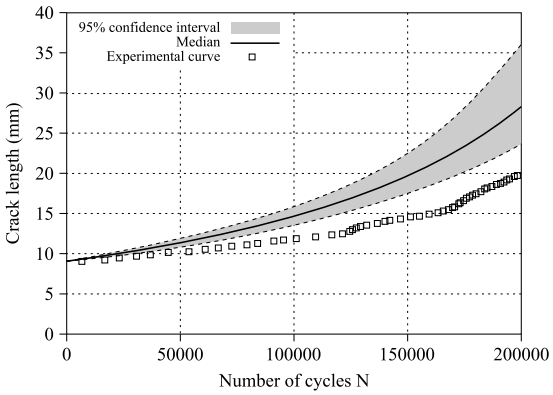

Figure 5.10. *Crack propagation – a priori prediction of the propagation (median curve and 95% confidence interval) and slowest experimental propagation curve*

standard deviation σ_e). These assumptions lead us to think that y_{obs} is a realization of a random variable Y_{obs} whose conditional distribution with repect to $\boldsymbol{X} = \tilde{\boldsymbol{x}}$ reads:

$$Y_{obs}|\boldsymbol{X} = \tilde{\boldsymbol{x}} \ \sim \ \mathcal{N}\left(\mathcal{M}(\tilde{\boldsymbol{x}}); \sigma_e^2\right) \qquad [5.26]$$

The previous equation then allows the formulation of a likelihood function for the observations and calculation of an *a posteriori* for vector $\boldsymbol{X}$ within the Bayesian framework.

5.5.5.3. *Bayesian updating methods*

The theoretical aspects of the Bayesian updating of mechanical models from observations go beyond this chapter's main topic. The interested reader can get more information from [PER 08]. To summarize, two main types of resolution methods can be distinguished regarding this issue:

– The methods which will update the *prior* probabilistic model of the input parameters of the model (here, the prior probability density functions of $\log C$ and m) from the measurements of the system response, and which then allow an *a posteriori* distribution to be estimated [PER 07b]. The propagation of this new *a posteriori* probabilistic model will allow an updated confidence interval to be calculated on the propagation curve. From an algorithmic point of view, the Markov chain Monte Carlo methods (MCMC) [ROB 96] are well adapted to the simulation of the *a posteriori* distributions.

– The methods which directly deal with the updating of the model response by defining a confidence interval, conditionally to the observations. These methods rely on the FORM approximation method that is used in structural reliability analysis [PER 07c, SUD 06].

5.5.5.4. *Application*

The evolution of the crack length during the first part of the propagation is supposed to be measured for a few values of the number of cycles (lower than 100,000). The Bayesian approach will allow the prediction to be updated, which means that a 95% confidence interval on the propagation curve will be re-estimated considering the measurement data. In this case, five measured values reported in Table 5.4 are considered.

Crack length (mm)	Number of cycles
9.4	16,345
10.0	36,673
10.4	53,883
11.0	72,556
12.0	101,080

Table 5.4. *Crack propagation measurements of the crack length on a particular sample at an early stage of propagation*

The updated median curve and the 95% confidence interval are plotted in Figure 5.11. It clearly appears that the updated prediction agrees with the observations, and that the confidence interval is significantly reduced compared to the *prior* prediction. The length of the crack at 200,000 cycles, predicted once updating has been performed, ranges from 17.4 to 19.8 mm, with a 95% probability, the measured value corresponding to the highest value.

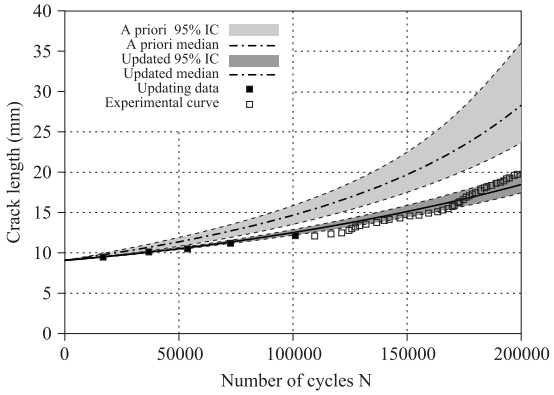

Figure 5.11. *Crack propagation – a priori and posteriori prediction of the crack propagation curve (median curve and 95% confidence interval). The squares correspond to the measurements used for the updating; the black and white squares correspond to the rest of the considered experimental curve*

It is important to note that the probabilistic formalism previously presented allows us to represent the model error: indeed, the experimental measurements of the crack length are very accurate and the measurement error can thus be considered as equal to zero. In contrast, the simplified Paris law model does not clearly allow the irregularities from the propagation curve to be reproduced. However, the model error introduced within the method (standard value $\sigma_e = 0.2$ mm for the numerical application) allows a satisfying confidence interval to be obtained. Such a result would not be obtained using the least squares fitting method, of a Paris curve on the five measurements gathered from Table 5.4.

5.5.6. *Conclusion*

In this section, the general scheme of the uncertainty modeling was applied to crack propagation using the classical Paris-Erdogan model. The observed scattering on the crack propagation rate for identical specimens was well reproduced by propagating the uncertainties identified on parameters $(\log C,\, m)$ through the Paris law. The 95% confidence interval tends to become very wide when the number of cycles increases. Nevertheless, considering the auscultation data (in this case, crack lengths measured at an early stage of propagation) within a Bayesian framework allows the predictions to be significantly improved as it gives an updated median curve in accordance with the observations and strongly reduces the 95% confidence interval.

The probabilistic approaches present the advantage to be used for any type of underlying physical model. Thus, the use of the extended finite element method (X-FEM) applied to crack propagation can be coupled with the methods of uncertainty modeling as presented in section 5.2, see [NES 06, NES 07].

5.6. Conclusion

The random nature of the fatigue phenomenon within materials and structures has been well-known for many years. Nevertheless, the consistent and rigorous integration of all the uncertainty sources regarding the design of realistic structures is an interesting topic. In this chapter, a general methodology of the uncertainty treatment was described. This methodology can be applied to mechanics and also to any domain where the numerical simulation of physical phenomena is necessary (computational fluid, dynamics, thermal problems, neutronics, electromagnetism, chemical engineering, etc.). This general scheme can be applied to the design of the fatigue of mechanical parts; either at the crack initiation phase ($S - N$) approach) or at the propagation step of existing cracks (Paris-Erdogan approach).

From a general point of view, a certain French cultural reluctance to use the probabilistic methods in industry can be observed, especially regarding the behavior of

structures. Most of the design codes (nuclear, aerospace, civil engineering industries) are mainly deterministic (based on *conservative* design ensured by safety factors), even if some parts of the codified rules (especially the choice of characteristic values for the calculation parameters) do have some probabilistic interpretation. As fatigue analysis is a field where randomness is observed at various levels (material strength, loading, etc.) and cannot be reduced, it seems important that the uncertainty quantification methods are used as a routine in this field in the future.

5.7. Appendix A: probability theory reminder

The aim of this appendix is to recall the basics of probability theory required to understand this chapter. It is not a mathematical course on this topic. For further information on the statistical methods used in this chapter, readers should refer to the books by Saporta [SAP 06] and O'Hagan and Forster [OHA 04].

5.7.1. *Random variables*

The classical axiomatic approach of the probability theory consists of building an abstract probability space triplet $(\Omega,\, \mathcal{F},\, \mathbb{P})$, where Ω stands for random experiment, $\mathcal{F}$ stands for which means set of sub-ensembles of steady Ω by switching to the complementary and the finite union, and $\mathbb{P}\,()$ stands for probability measurement. This last allows each event $A \in \mathcal{F}$ to be connected to its probability $\mathbb{P}\,(A)$ which is a real number ranging from 0 to 1.

In probabilistic engineering mechanics *random variables* (and therfore the random vectors) are used to model the uncertainty of the parameters of the mathematical model which describes the mechanical system. A random (real) variable $X(\omega)$ is defined as an application $X \,:\, \Omega \longmapsto \mathcal{D}_X \subset \mathbb{R}$. A *realization* of a random variable $x_0 \equiv X(\omega_0) \in \mathcal{D}_X$ is one of the possible values that the parameter, modeled by X, can take. The term of discrete or continuous variable is used depending on the support $\mathcal{D}_X$ (meaning the set of all the possible realizations of the variable) which can be discrete or be continuous.

A random variable $X(\omega)$ is entirely defined by its cumulative distribution function, written as $F_X(x) : \mathcal{D}_X \longmapsto [0\,,\, 1]$:

$$F_X(x) = \mathbb{P}\,(X(\omega) \leq x) \qquad [5.27]$$

The cumulative distribution function evaluated at point x is then the probability random variable[4] X takes values lower than or equal to x. For a continuous random

4. From now on, the dependency on ω is omited for the sake of simplicity. The random variables are denoted by capital letters while lowercase letters are used for realizations.

variable, the *probability density function* is defined by:

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad [5.28]$$

Therefore, quantity $f_X(x)\,dx$ stands for the probability that X takes a value ranging from x to $x + dx$. By definition, the integral of f_X on its definition domain is equal to 1. This is also the limit of $F_X(x)$ when x tends towards the upper boundary of $\mathcal{D}_X$. The classical axiomatic approach of the probability theory consists of building an abstract probability space defined by the triplet $(\mathbf{\Omega}, \mathbf{F}, \mathbf{P})$, where $\mathbf{\Omega}$ stands for the outcome space of the random experiment, $\mathbf{F}$ stands for the σ-algebra of events (which means the set of all subsets of $\mathbf{\Omega}$ that is stable by union and complementary operations) and $\mathbf{P}()$ stands for the probability measure, which allows us to assign a probability $\mathbf{P}(\mathbf{A})$ to each event $\mathbf{A} \in \mathbf{F}$. For instance, the uniform distribution of an interval $[a\,,\,b]$ has a probability density function $f_U(x) = 1/(b-a)$ if $x \in [a\,,\,b]$, and 0 otherwise. A Gaussian distribution (also called normal distribution) $\mathcal{N}(\mu\ \sigma)$ is defined by the following probability density for any $x \in \mathbb{R}$:

$$f_{\mathcal{N}}(x) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right) \qquad [5.29]$$

where $\varphi(x)$ is the standard normal probability density function which can be written as:

$$\varphi(x) = \frac{1}{\sqrt{2\,\pi}} e^{-x^2/2} \qquad [5.30]$$

The normal cumulative distribution function, usually written as Φ, is defined by:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\,\pi}} e^{-t^2/2}\,dt \qquad [5.31]$$

It has no analytical expression, however it is tabulated in such as Excel, Matlab, Scilab, etc.

In this chapter, the lognormal distributions are used. By definition, a random variable follows a lognormal distribution if its logarithm follows a Gaussian distribution. The following equation is then used:

$$X \sim \mathcal{LN}(\lambda, \zeta) \quad : \quad X = e^{\lambda + \zeta\,\xi} \qquad \text{with } \xi \sim \mathcal{N}(0,1) \qquad [5.32]$$

The probability density of a variable $\mathcal{LN}(\lambda, \zeta)$ is written as:

$$f_{\mathcal{LN}}(x) = \frac{1}{\zeta x}\varphi\left(\frac{\ln x - \lambda}{\zeta}\right) \qquad [5.33]$$

The most likely value of a random variable corresponds to the realization x_0 which maximizes the probability density $f_X(x)$: this is what is called the *mode*. The usual distributions present a single maximum (they are called uni-modal) but some multi-modal distribution can also be built.

5.7.2. *Expected value, moments, and quantiles*

The *expected value* of a random variable is defined as:

$$\mathrm{E}[X] \equiv \int_{\mathcal{D}_X} x\, f_X(x)\, dx \qquad [5.34]$$

This is what is usually called the *average* of X, which is also written as μ_X. As long as the integral is defined, the expected value of a function $g(X)$ can usually be defined by:

$$\mathrm{E}[g(X)] \equiv \int_{\mathcal{D}_X} g(x)\, f_X(x)\, dx \qquad [5.35]$$

The statistical *moments* m_k (respectively centered moments μ_k) correspond to the particular case where $g(X) = X^k$ (respectively $g(X) = (X - \mu_X)^k$, $k \in \mathbb{N}$):

$$m_k = \int_{\mathcal{D}_X} x^k\, f_X(x)\, dx \qquad [5.36]$$

$$\mu_k = \int_{\mathcal{D}_X} (x - \mu_X)^k\, f_X(x)\, dx \qquad [5.37]$$

Variance $\sigma_X^2 \equiv \mu 2 = \mathrm{E}_{(X-\mu X)^2}$ is the centered moment of the second order. The standard value of X, written as σ_X, is the square root of the variance, and the coefficient of variation (given as a percentage) is the ratio $CV_X = \sigma_{X/\mu_X}$.

The *quantiles* of a random variable are defined from the inverse cumulative distribution function. The quantile x_q with an order of q is defined by:

$$x_q \,:\, \mathbb{P}(X \leq x_q) = q \qquad [5.38]$$

which can also be written as:

$$x_q = F_X^{-1}(q) \qquad [5.39]$$

if the cumulative distribution function F_X is stricly increasing.

5.7.3. *Maximum likelihood*

Step B of the general scheme of the treatment of the uncertainties consists of proposing a probabilistic model of the input parameters of the mechanical model; that is to say, to prescribe a probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$ for the vector $\boldsymbol{X}$ of these parameters.

In this chapter, only the case where the input variables can be considered as independent is studied : $f_{\boldsymbol{X}}(\boldsymbol{x})$ is then the product of the marginal probability density functions f_{X_i}. If we can rely on experimental data $\mathcal{E} = \{x^{(1)}, \ldots, x^{(Q)}\}$ for the parameters, the statistical inference techniques can be used in order to determine a probability distribution that is consistent with the available data. The parametric inference consists of assuming a specific form for the probability density f_X (e.g. Gaussian, lognormal, etc.) and then in estimating the parameters of this density so that it better reproduces the sample. Let $f(X;\boldsymbol{\theta})$ be this density where $f()$ is a known function (e.g. equation [5.29]) and $\boldsymbol{\theta}$ is the vector of parameters that shall be estimated (e.g. $\boldsymbol{\theta} = (\mu\,,\,\sigma)$). The likelihood function of the sample set is defined as:

$$\mathsf{L}(\boldsymbol{\theta}\,;\,\mathcal{E}) = \prod_{i=1}^{Q} f(x^{(i)};\boldsymbol{\theta}) \qquad [5.40]$$

Once the sample is known, this likelihood function only depends on $\boldsymbol{\theta}$. The maximum likelihood principle indicates that the best choice of the parameters (called $\hat{\boldsymbol{\theta}}$) is then the one which maximizes the previous function, or equivalently minimizes the log-likelihood:

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \left[-\log \mathsf{L}(\boldsymbol{\theta}\,;\,\mathcal{E})\right] = \arg\min_{\boldsymbol{\theta}} \left[-\sum_{i=1}^{Q} \log f(x^{(i)};\boldsymbol{\theta})\right] \qquad [5.41]$$

Usually, from the observation of sample set $\mathcal{E}$ using the tools of descriptive statistics (e.g. histograms [SAP 06, Chapter 5]), different choices are possible for $f(X;\boldsymbol{\theta})$. The best parameters are then estimated for each choice and the hypotheses are validated *a posteriori* using goodness-of-fit tests (e.g. the Kolmogorov-Smirnov test, the Anderson-Darling test, χ^2, etc. [SAP 06, Chapter 14]). The most relevant distribution can eventually be selected using likelihood criteria such as AIC (*Akaike Information Criterion*) or BIC (*Bayesian Information Criterion*) [SAP 06, Chapter 19]).

5.7.4. *Bayesian inference*

Bayesian statistics is a branch of statistics coming from Bayes' theorem which is an elementary result of the probability theory, which states that for two random events A and B:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\ \mathbb{P}(A)}{\mathbb{P}(B)} \qquad [5.42]$$

where $\mathbb{P}(A|B)$ is the *conditional probability* of A *knowing* B and is equal by definition to $\mathbb{P}(A \cap B)/\mathbb{P}(B)$. The previous result can easily be applied to the probability density functions (respectively conditional probability density functions) of random variables.

The Bayesian approach consists in integrating some *prior* information on the parameters $\boldsymbol{\theta}$ to be estimated. Not only a family of distribution $f(X;\boldsymbol{\theta})$ is chosen but also a probability density function $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ (called *prior distribution*) is defined for the so-called *hyper-parameters* $\boldsymbol{\theta}$. For instance the Gaussian distribution $f(X;\boldsymbol{\theta})$ with a mean value μ and a standard value σ is selected and in addition, the mean value μ is assumed to *a priori* vary between a lower and an upper bound, which comes to modeling the hyper-parameter μ by a uniform variable between these two bounds. The *a posterior distribution* $f_{\boldsymbol{\Theta}}$ can then be deduced by combining the prior distribution and the likelihood function:

$$f''_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = c\,\mathsf{L}(\boldsymbol{\theta}\,;\,\mathcal{E})\,p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \qquad [5.43]$$

where L is the likelihood function defined in equation [5.40] and c is a normalization factor which ensures that $f''_{\boldsymbol{\Theta}}$ is a probability distribution function. This equation is similar in principle to equation [5.42] and can be read as follows: the distribution of $\boldsymbol{\Theta}$ conditionally to the observations $\mathcal{E}$ is equal to the prior distribution multiplied by the likelihood function.

Focusing on the initial problem, which is to propose the best possible probabilistic model for parameter X, the mean value or the *a posteriori* mode of $f''_{\boldsymbol{\Theta}}$ can be chosen as the best parameter.

In the case of updating the predictions of a model $Y = \mathcal{M}(\boldsymbol{X})$ by some observations $\mathcal{Y}_{obs} = \{y^{(1)}, \ldots, y^{(Q)}\}$ (see section 5.5.5), the *prior* probabilistic model is defined for X (which acts as hyper-parameters $\boldsymbol{\Theta}$ in the above description) and the *a posteriori* distribution $f''_{\boldsymbol{X}}$ is eventually propagated through the physical model $\mathcal{M}$.

5.7.5. *Reliability analysis and FORM method*

A structural reliability problem is set up from the following ingredients: a mechanical model $\mathcal{M}$, a probabilistic model for its input parameters $\boldsymbol{X}$ (i.e. a probability density function $f_{\boldsymbol{X}}(x)$, and a failure criterion[5] (see [LEM 08] for a pedagogical introduction to structural reliability). It can be mathematically formulated using a *limit state function* (also called *performance function*) $g(\boldsymbol{X})$ which shall take negative (respectively positive) values for the realizations of $\boldsymbol{X}$ such that the mechanical system fails (respectively does not fail). The set of points x such that $g(x)= 0$ defines the *limit state surface*. A common situation corresponds to the case when a quantity of interest Y (obtained as the response of a mechanical model) shall not be greater than a prescribed threshold $\bar{y}$. The associated limit state function then reads:

$$g(\boldsymbol{X}) = \bar{y} - \mathcal{M}(\boldsymbol{X}) \qquad [5.44]$$

The failure probability is then given by:

$$P_f \equiv \mathbb{P}\left(g(\boldsymbol{X}) \leq 0\right) = \mathbb{P}\left(\mathcal{M}(\boldsymbol{X}) \geq \bar{y}\right) = \int_{\mathcal{D}_f} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \qquad [5.45]$$

where $\mathcal{D}_f = \{\boldsymbol{x} : g(\boldsymbol{x}) \leq 0\}$ stands for the failure domain. The Monte Carlo simulation method allows us to evaluate the above failure probability quite easily: the input random vector is *sampled*, which means that an artificial sample set of input vectors is created according to $f_{\boldsymbol{X}}$, say $\mathcal{X} = \{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N)}\}$. For each random vector $\boldsymbol{x}^{(i)}$ the response $\mathcal{M}(\boldsymbol{x}^{(i)})$ and then the criterion $g(\boldsymbol{x}^{(i)})$ are evaluated. The failure probability is estimated by the number of times N_f the calculation leads to a negative value of $g(\boldsymbol{x}^{(i)})$, divided by the total number of samples N:

$$\hat{P}_f = \frac{N_f}{N} \qquad [5.46]$$

This method is not applicable in practice for problems where the failure probability to be evaluated is low (say from 10^{-2} to 10^{-6}): indeed, about $N = 400 \times 10^k$ realizations of X are needed to get a 5% accurate estimation of a failure probability with an order of magnitude 10^{-k}. The Monte Carlo simulation is therefore not possible, due to its cost, when each evaluation of M corresponds to a calculation relying on finite elements that may take several minutes or hours.

The FORM method is an approximation method which provides an estimation of P_f at low computational cost. The vector of parameters $\boldsymbol{X}$ is first transformed into a

5. Failure corresponds here to the "non-fulfilling of a performance assigned to the system" and not necessarily to the collapse of the considered mechanical system.

vector of standard normal random variables $\boldsymbol{X} = \mathcal{T}(\boldsymbol{\xi})$, which allows equation [5.45] to be recast as:

$$P_f \equiv \mathbb{P}\left(g(\mathcal{T}(\boldsymbol{\xi})) \leq 0\right) = \int_{\mathcal{D}_f = \{\boldsymbol{\xi} : g(\mathcal{T}(\boldsymbol{\xi})) \leq 0\}} (2\pi)^{-n/2}\, e^{-\|\boldsymbol{\xi}\|^2/2}\, d\boldsymbol{\xi} \qquad [5.47]$$

where n is the dimension of $\boldsymbol{\xi}$. In this equation, the integrands exponentially decrease with $|\boldsymbol{\xi}|^2$. Therefore, the points of the integration domain (failure domain) that contribute the most to the integral are those points that are close to the origin of this standard normal space. Thus the *design point* $\boldsymbol{\xi}^*$ is first computed, which is the point of the failure domain that is the closest to the origin of the space. The limit state surface is then linearized around $\boldsymbol{\xi}^*$. The failure probability is then proven to be equal to:

$$P_{f,\mathsf{FORM}} = \Phi(-\beta) \qquad [5.48]$$

where $\beta = |\boldsymbol{\xi}^*|$ is the Hasofer-Lind reliability index.

For a reliability problem in which the dimension of the input vector $\boldsymbol{X}$ is lower than 10, the FORM method usually yields good results for a number of evaluations of g (which are necessary to find the designing point $\boldsymbol{\xi}^*$) lower than 100. This number is *independent* of the order of magnitude of P_f in contrast to Monte Carlo simulation. However FORM only provides an *approximation* of P_f and it is not possible to prove that this FORM result is close to the true value of P_f or not, and/or that the result is conservative or not. Some additional techniques like *importance sampling* shall be used to validate the FORM results [LEM 05].

5.8. Bibliography

[AFC 00] AFCEN, *Règles de Conception et de Construction des Matériels des Ilots Nucléaires*, (RCCM), June 2000, Paris.

[AFN 90] AFNOR, *Produits Métalliques – Pratique des Essais de Fatigue Oligocyclique, Norme A03-403*, AFNOR, 1990.

[AFN 91] AFNOR, *Produits Métalliques – Essais de Fatigue: Traitement Statistique des Données, Norme A03-405*, AFNOR, 1991.

[ALE 04] F. ALEXANDRE, Aspects probabilistes et microstructuraux de l'amorçage des fissures de fatigue dans l'alliage Inco 718, PhD Thesis, Ecole Nationale Supérieure des Mines de Paris, 2004.

[AMZ 94] C. AMZALLAG, J.-P. GEREY, J.-L. ROBERT, J. BAHUAUD, "Standardization of the Rainflow counting method for fatigue analysis", *Int. J. Fatigue*, vol. 16, pp. 287-293, 1994.

[BAS 60] F. BASTENAIRE, Etude statistique et physique de la dispersion des résistances et des endurances à la fatigue, PhD Thesis, Faculté des Sciences de l'Université de Paris, 1960.

[BAT 10] C. BATHIAS, A. PINEAU, *Fatigue of Materials: Fundamentals*, ISTE, London and John Wiley & Sons, New York, 2010.

[BEN 06] D. BENASCIUTTI, R. TOVO, "Comparison of spectral methods for fatigue analysis of broad-band Gaussian random processes", *Prob. Eng. Mech.*, vol. 21, pp. 287-299, 2006.

[BOU 08] J.-M. BOURINET, M. LEMAIRE, "FORM sensitivities to correlation: application to fatigue crack propagation based on Virkler data", *Proc. 4th Int. ASRANet Colloquium*, Athens, 2008.

[COL 98] COLLECTIF, *Re-evaluation of Fatigue Analysis Criteria, EE/S 98.317 Report*, Framatome, 1998, (Final report to CEC-DG XI contract B4-3070/95/000876/MAR/C2).

[CUR 04] F. CURTIT, "INTHERPOL thermal fatigue tests", *Proc. 3rd Int. Conf. Fatigue of Reactor Components*, NEA/CSNI/R, Seville, 2004.

[CUR 05] F. CURTIT, J.-M. STÉPHAN, "Mechanical aspect concerning thermal fatigue initiation in the mixing zones piping", *Trans. 18th Int. Conf. on Struct. Mech. in Reactor Tech. (SMiRT 18)*, Beijing, China, 2005.

[DIR 85] T. DIRLIK, Application of computers in fatigue analysis, PhD thesis, University of Warwick, United Kingdom, 1985.

[DIT 86] O. DITLEVSEN, R. OLESEN, "Statistical analysis of the Virkler data on fatigue crack growth", *Eng. Fract. Mech.*, vol. 25, pp. 177-195, 1986.

[DIT 96] O. DITLEVSEN, H. MADSEN, *Structural Reliability Methods*, J. Wiley and Sons, Chichester, 1996.

[DRO 04] J. DROESBEKE, J. FINE, G. SAPORTA, *Méthodes Bayésiennes en Statistique*, Editions Technip, 2004.

[EDF 06] EDF, R&D DIVISION, Code_Aster: *Analyse des Structures et Thermo-mécanique pour des Études et des Recherches, V.7*, 2006, http://www.code-aster.org.

[GHA 91] R. GHANEM, P. SPANOS, *Stochastic Finite Elements – A Spectral Approach*, Springer Verlag, 1991.

[GUÉ 05] Z. GUÉDÉ, Approche probabiliste de la durée de vie des structures soumises à la fatigue thermique, doctorate thesis, Université Blaise Pascal, Clermont-Ferrand, 2005.

[GUÉ 07] Z. GUÉDÉ, B. SUDRET, M. LEMAIRE, "Life-time reliability based assessment of structures submitted to thermal fatigue", *Int. J. Fatigue*, vol. 29, no. 7, pp. 1359-1373, 2007.

[JOI 02] JOINT COMMITTEE ON STRUCTURAL SAFETY, *JCSS Probabilistic Model Code*, JCSS 2002, http://www.jcss.ethz.ch.

[KOT 98] Z. A. KOTULSKI, "On efficiency of identification of a stochastic crack propagation model based on Virkler experimental data", *Arch. Mech.*, vol. 50, no. 5, pp. 829-847, 1998.

[KRE 83] P. KREE, C. SOIZE, *Mécanique Aléatoire*, Dunod, 1983.

[LAC 97] B. LACAZE, C. MAILHES, M. MAUBOURGUET, J.-Y. TOURNERET, *Probabilités et Statistique Appliquées*, Cépaduès Editions, 1997.

[LEM 05] M. LEMAIRE, *Fiabilité des Structures – Couplage Mécano-fiabiliste Statique*, Hermès, 2005.

[LEM 08] M. LEMAIRE, "Approche probabiliste du dimensionnement – Modélisation de l'incertain et méthode de Monte-Carlo", *Techniques de l'Ingénieur, BM 5003-5004*, 2008.

[LEM 09] M. LEMAIRE, *Structural reliability*, John Wiley & Sons, 2009.

[LOR 05] S. LORÉN, M. LUNDSTRÖM, "Modelling curved S–N curves", *Fatigue Fract. Engng. Mat. Struct.*, vol. 28, pp. 437-443, 2005.

[MEL 99] R.-E. MELCHERS, *Structural Reliability Analysis and Prediction*, John Wiley & Sons, 1999.

[MIN 45] M. MINER, "Cumulative damage in fatigue", *J. Appl. Mech.*, vol. 9, pp. 159-164, 1945.

[NES 06] L. NESPUREK, J.-M. BOURINET, A. GRAVOUIL, M. L., "Some approaches to improve the computational efficiency of the reliability analysis of complex crack propagation problems", *Proc. 3rd Int. ASRANet Colloquium*, Glasgow, 2006.

[NES 07] L. NESPUREK, J.-M. BOURINET, A. GRAVOUIL, M. L., "Probabilistic approach to crack propagation based on XFEM and FORM", *Proc. Fatigue Design 2007*, Senlis, France, 2007.

[OHA 04] A. O'HAGAN, J. FORSTER, *Kendall's advanced theory of statistics*, Arnold, 2004.

[PAR 63] P. PARIS, F. ERDOGAN, "A critical analysis of crack propagation laws", *Trans. ASME, J. Basic Eng.*, pp. 528-534, 1963.

[PAS 97] F. PASCUAL, W. MEEKER, "Analysis of fatigue data with runouts based on a model with non constant standard deviation and a fatigue limit parameter", *J. Testing Eval.*, vol. 25, pp. 292-301, 1997.

[PAS 99] F. PASCUAL, W. MEEKER, "Estimating fatigue curves with the random fatigue-limit model", *Technometrics*, vol. 41, no. 4, pp. 277-290, 1999.

[PER 05] F. PERRIN, B. SUDRET, M. PENDOLA, M. LEMAIRE, "Comparison of two statistical treatments of fatigue test data", *Proc. Fatigue Design 2005*, Senlis, France, 2005.

[PER 06] F. PERRIN, M. PENDOLA, T. MORO, G. MORIN, J.-M. CARDONA, "Complete probabilistic framework in fatigue design: application to exhaust manifolds", in: B. TOPPING, G. MONTERO, R. MONTENEGRO (Eds.), *Proc. 8th Int. Conf. Comput. Struct. Tech.*, Las Palmas de Gran Canaria, 2006.

[PER 07a] F. PERRIN, B. SUDRET, F. CURTIT, J.-M. STÉPHAN, M. PENDOLA, "Efficient Bayesian updating of structural fatigue design curve using experimental thermal fatigue data", *Proc. Fatigue Design 2007*, Senlis, France, 2007.

[PER 07b] F. PERRIN, B. SUDRET, M. PENDOLA, "Bayesian updating of mechanical models – Application in fracture mechanics", *Proc. 18^e Congrès Français de Mécanique (CFM'2007)*, Grenoble, 2007.

[PER 07c] F. PERRIN, B. SUDRET, M. PENDOLA, E. DE ROCQUIGNY, "Comparison of Markov chain Monte-Carlo simulation and a FORM-based approach for Bayesian updating of mechanical models", *Proc. 10th Int. Conf. on Applications of Stat. and Prob. in Civil Engineering (ICASP10)*, Tokyo, Japan, 2007.

[PER 08] F. PERRIN, Intégration des données expérimentales dans les modèles probabilistes de prédiction de la durée de vie des structures, PhD Thesis, Université Blaise Pascal, Clermont-Ferrand, 2008.

[ROB 92] C. ROBERT, *L'analyse statistique bayésienne*, Economica, 1992.

[ROB 96] C. ROBERT, *Méthodes de Monte Carlo par chaînes de Markov*, Economica, 1996.

[RUB 81] R.-Y. RUBINSTEIN, *Simulation and the Monte Carlo Methods*, John Wiley & Sons, 1981.

[SAL 00] A. SALTELLI, K. CHAN, E. SCOTT, Eds., *Sensitivity Analysis*, J. Wiley & Sons, 2000.

[SAL 04] A. SALTELLI, S. TARENTOLA, F. CAMPOLONGO, M. RATTO, *Sensitivity Analysis in Practice – A Guide to Assessing Scientific Models*, J. Wiley & Sons, 2004.

[SAP 06] G. SAPORTA, *Probabilités, Analyse des Données et Statistique*, Technip Publications, 2nd edition, 2006.

[SCH 06] C. SCHWOB, L. CHAMBON, F. RONDE-OUSTAU, J. BERNADOU, "Probabilistic assessment of fatigue life using a stress-based fatigue criterion", in: B. TOPPING, G. MONTERO, R. MONTENEGRO (Eds.), *Proc. 8th Int. Conf. Comput. Struct. Tech.*, Las Palmas de Gran Canaria, 2006.

[SCH 07] C. SCHWOB, Approche non locale probabiliste pour l'analyse en fatigue des zones à gradient de contrainte, doctorate thesis, Ecole des Mines d'Albi Carmaux, 2007.

[SUD 03a] B. SUDRET, Z. GUÉDÉ, P. HORNET, J.-M. STÉPHAN, M. LEMAIRE, "Probabilistic assessment of fatigue life including statistical uncertainties in the S–N curve", *Trans. 17th Int. Conf. on Struct. Mech. in Reactor Tech. (SMiRT 17)* Paper M-232, Prague, Czech Republic, 2003.

[SUD 03b] B. SUDRET, Z. GUÉDÉ, M. LEMAIRE, "Probabilistic framework for fatigue analysis", M. MAES, L. HUYSE, Eds., *Proc. 11th IFIP WG7.5 Conference on Reliability and Optimization of Structural Systems, Banff, Canada*, Balkema, Rotterdam, pp. 251-257, 2003.

[SUD 05] B. SUDRET, Z. GUÉDÉ, "Probabilistic assessment of thermal fatigue in nuclear components", *Nuc. Eng. Des.*, vol. 235, pp. 1819-1835, 2005.

[SUD 06] B. SUDRET, F. PERRIN, M. BERVEILLER, M. PENDOLA, "Bayesian updating of the long-term creep deformations in concrete containment vessels", *Proc. 3rd Int. ASRANet Colloquium*, Glasgow, United Kingdom, 2006.

[SUD 07] B. SUDRET, Uncertainty propagation and sensitivity analysis in mechanical models – Contributions to structural reliability and stochastic spectral methods, Habilitation à diriger des recherches, Université Blaise Pascal, Clermont-Ferrand, France, 2007.

[THO 99] J.-J. THOMAS, G. PERROUD, A. BIGNONNET, D. MONNET, "Fatigue design and reliability in the automotive industry", *European Structural Integrity Society*, vol. 23, pp. 1-11, 1999.

[TOV 01] R. TOVO, "On the fatigue reliability evaluation of structural components under service loading", *Int. J. Fatigue*, vol. 23, pp. 587-598, 2001.

[VIR 78] D. VIRKLER, B. M. HILLBERRY, P. GOEL, "The statistical nature of fatigue crack propagation", *Trans. ASME, J. Eng. Mat. Tech.*, vol. 101, no. 2, pp. 148-153, 1978.

[YAN 96] J. YANG, S. MANNING, "A simple second order approximation for stochastic crack growth analysis", *Eng. Fract. Mech.*, vol. 53, no. 5, pp. 677-686, 1996.

[ZHE 98] R. ZHENG, B. ELLINGWOOD, *Stochastic Fatigue Crack Growth in Steel Structrues Subjected to Random Loading, Contract report ITL-98-1 for the US Army Corps of Engineers*, The Johns Hopkins University, 1998.