Chapter 10

Time-variant reliability problems

10.1. Introduction

Although more or less ignored so far in this book, the time dimension is often present in structural reliability problems and has to be properly taken into account. Let us come back to the most basic formulation known as "*R*-*S*", in which failure occurs when a demand *S* is greater than a capacity *R*. It is clear that for real structures both quantities may depend on time. Indeed:

- the resistance (or capacity) *R* of the structure (e.g. material properties) may be degrading in time. The degradation mechanisms usually present an initiation phase and a propagation phase. Examples of such mechanisms are the crack initiation and propagation in fracture mechanics, the corrosion of steel structures and reinforced concrete rebars, the steel toughness decrease under irradiation in nuclear components, the concrete shrinkage and creep, etc.;

- the load effect (or demand) *S* may be randomly varying in time due to the time variation of the loading, e.g. environmental loads (wind velocity, temperature, wave height, etc.) or service loads (traffic, occupancy loads, etc.).

Both types of time dependency may be present simultaneously or not and their nature is different: while degradation phenomena are usually monotonic and

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irreversible (corresponding to a decrease of the resistance), loads are usually « oscillating » in nature and should be modelled by *random processes.*

The aim of this chapter is not to fully cover the theory and tools of time-variant reliability problems, which is beyond the scope of this book. In contrast, it aims at defining the basic concepts and focuses on a specific approach known as "PHI2 method" which allows the analyst to solve time-variant reliability problems using time-invariant tools such as the FORM method. For a more complete treatment of time-variant reliability problems the reader is referred to the numerous publications by Rackwitz [e.g., RAC 01, RAC 04] and the books by Ditlevsen & Madsen [DIT 96, chapter 15] and Melchers [MEL 99, chapter 6].

10.2. Random processes

Random processes allow one to mathematically describe loads that are randomly varying in time [CRA 67, LIN 67]. In the sequel the basic notions are introduced without too much mathematical rigour.

10.2.1. *Definition and elementary properties*

A random process $X_t(\omega)$ is a set of random variables indexed by the time instant $t \in [0,T]$ with values in $\mathcal{D}_X \subset \mathbb{R}$. In this notation $\omega \in \Omega$ denotes the elementary events of an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. At each time instant the process reduces to a random variable $X_{t_0}(\omega)$ which is assigned some prescribed distribution. Conversely a *realization* or *trajectory* of the process corresponds to the usual function $t \rightarrow X_t(\omega_0)$ for a given ω_0. It will be simply denoted by small letters, say $x(t, \omega_0)$. In order to define completely a random process, the full set of joint probability distribution functions of any finite subset of random variables $\{X_{t_1}(\omega), \cdots, X_{t_N}(\omega)\}$ for any time instants $0 \le t_1 < \cdots < t_N \le T$ shall be prescribed. For structural reliability purposes however, specific types of processes are of common use, e.g. Poisson-, rectangular renewal wave- or Gaussian processes, whose description is much easier, as seen in the following paragraphs.

The usual definitions of marginal probability density functions (PDF), statistical moments (mean value $\mu_X(t)$, standard deviation $\sigma_X(t)$, etc.) that are well known for random variables naturally exist for random processes « at each time instant ». Of crucial importance is also the *autocorrelation function* defined as follows:

$$R_{XX}(t_1,t_2) = \mathrm{E}\left[X_{t_1} X_{t_2}\right] \tag{10.1}$$

where $\mathrm{E}[.]$ denotes the mathematical expectation. This function represents the statistical dependence of points of trajectories considered at time instants t_1, t_2. Equivalently the autocorrelation coefficient function is defined by:

$$\rho_{XX}(t_1,t_2) = \frac{\mathrm{E}\left[X_{t_1} X_{t_2}\right] - \mu_X(t_1)\mu_X(t_2)}{\sigma_X(t_1)\sigma_X(t_2)} \tag{10.2}$$

Loosely speaking a random process is said *stationary* if its “characteristics” are invariant in time. Various rigourous definitions may be given. We will limit here to second-order stationarity, which implies that the statistical moments $\mathrm{E}\left[X_t^k\right], k = 1,2$ do not depend on time and that the autocorrelation function is invariant under time shift: $R_{XX}(t_1+h, t_2+h) = R_{XX}(t_1,t_2)$. The latter equation implies that the autocorrelation function only depends on the time interval $\tau = t_2 - t_1$.

A random process is said *differentiable* if the following limit $\dfrac{X_{t+h}(\omega) - X_t(\omega)}{h}$ exists in the mean-square sense. The limit process is denoted by $\dot{X}_t$ and satisfies:

$$\lim_{h \to 0^+} \mathrm{E}\left[\left(\frac{X_{t+h} - X_t}{h} - \dot{X}_t\right)^2\right] = 0 \tag{10.3}$$

Due to linearity the mean value of the derivative process is equal to $\mu_{\dot{X}}(t) = \dfrac{d\mu_X(t)}{dt}$. It may easily be shown that its autocorrelation function reads:

$$R_{\dot{X}\dot{X}}(t_1,t_2) = \frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2} \tag{10.4}$$

In particular, for a stationary process, the following relationship holds: $R_{\dot{X}\dot{X}}(\tau) = -\dfrac{d^2 R_{XX}(\tau)}{d\tau^2}$.

10.2.2. *Gaussian random processes*

In contrast to other fields (*e.g.* in quantitative finance), the random processes that are used in engineering in order to model time-varying loads (wind velocity, wave height, etc.) show some regularity that are related to the underlying physical phenomena. In practice the Gaussian random processes are of great importance in this field.

A scalar random process S_t is said Gaussian if the random vector $\left\{X_{t_1}(\omega), \cdots, X_{t_N}(\omega)\right\}$ is a Gaussian vector for any finite set of instants $0 \leq t_1 < \cdots < t_N \leq T$. It is completely defined by prescribing its mean value $\mu_S(t)$ and standard deviation $\sigma_S(t)$ at each time instant, as well as its autocorrelation coefficient function $\rho_S(t_1, t_2)$. Classical forms of autocorrelation coefficient functions are the *exponential* type ($\exp\left[-\left|t_1 - t_2\right| / \tau_S\right]$), the *square-exponential* type ($\exp\left[-\left((t_1 - t_2)/\tau_S\right)^2\right]$) and the cardinal sine type ($\sin\left[(t_1 - t_2)/\tau_S\right] / \left[(t_1 - t_2)/\tau_S\right]$). Once the process is defined through these properties trajectories may be simulated for computational purposes by various methods (Fourier decomposition, Karhunen-Loève expansion, EOLE decomposition, etc. [PRE 94, SUD 07]).

10.2.3. *Poisson and rectangular wave renewal processes*

Point processes appear in numerous situations when similar events occur randomly in time (computer connections to a server, customers arriving at a booth, etc.). In structural reliability problems, they allow one to count crossings of a limit state surface.

Let us denote by $T_i(\omega), i \geq 1$ the time of *i*-th occurrence of the event under consideration (with values in $]0, +\infty[$). The counting function $N_t(\omega)$ is defined by:

$$N_t(\omega) = \sup\left\{n : T_n(\omega) \leq t\right\} \qquad [10.5]$$

It is a random process whose trajectories are piecewise constant and take integer values, with discontinuities at the time instants where there is an occurrence of the observed phenomenon. Such a process is a *Poisson* process if it satisfies the following properties:

– for any finite set of instants $0 \le t_1 < \cdots < t_N$, random variables $N_{t_1}, N_{t_2} - N_{t_2}, \cdots, N_{t_N} - N_{t_{N-1}}$ are independent (assuming $N_0 = 0$);

– $\forall\, 0 \le s < t$, random variable $N_t - N_s$ follows a Poisson distribution with parameter $\lambda(t-s)$, where λ is called *process intensity*. Thus:

$$\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad [10.6]$$

For such processes it may be proven that the time to first occurrence has an exponential distribution with parameter λ, (*i.e.* $\mathbb{P}(T_1 \le t) = 1 - e^{-\lambda t}$). The time between two successive occurrences $T_{n+1} - T_n$ also follows an exponential distribution.

Poisson processes are useful for constructing rectangular renewal wave processes that are piecewise constant (*e.g.* exploitation or traffic loads) while changing their amplitude at random time instants. Such processes may be used to model traffic or exploitation loads.

Such a process is defined by a) the probability density function of the load amplitude (thus of the « jumps » in between) and b) the Poisson process intensity. A trajectory is depicted in Figure 10.1.

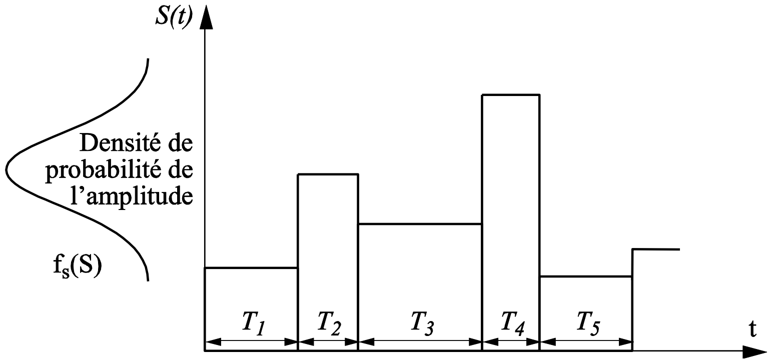

Figure 10.1. *Example of trajectory of a rectangular renewal wave process*

Rectangular renewal wave and Gaussian processes as well as those obtained by simple transforms (such as lognormal processes obtained by exponentiation of Gaussian processes) allow the analyst to model a large variety of loads for practical applications.

Finally, note that the parameters that define the processes (*e.g.* mean value) may be random variables as well. This happens for instance in offshore engineering when the environmental loads (wave height) are modelled for different sea state which also occur with some randomness in large time scales.

10.3. Time-variant reliability problems

10.3.1. *Problem statement*

As for time-invariant reliability problems one assumes now that the failure of the structure under consideration is characterized by a limit state function which may depend on time in two ways: either time may be an input parameter of the function or there are some random processes in its definition (the latter being stationary or non stationary with time-dependent hyperparameters). Let us denote this limit state function by $g\left(\boldsymbol{R}(\omega), \boldsymbol{S}_t(\omega), t\right)$, where $\boldsymbol{R}(\omega)=\left\{R_1(\omega), \cdots, R_p(\omega)\right\}^{\mathsf{T}}$ (resp. $\boldsymbol{S}_t(\omega)=\left\{S_1(\omega), \cdots, S_q(\omega)\right\}^{\mathsf{T}}$) is a random vector (resp. a set of scalar random processes) with prescribed joint probability density function.

The main difference between a time-invariant and a time-variant problem lies in the fact that one does not know the time instant when failure occurs in the latter case. This instant is the smallest $\mathcal{T} \in [0,T]$ such that the limit state function takes a negative value. This leads to the definition of the *cumulative probability of failure*:

$$P_f(0,T) = \mathbb{P}\left(\exists\, t \in [0,T]\ :\ g\left(\boldsymbol{R}(\omega), \boldsymbol{S}_t(\omega), t\right) \le 0\right) \qquad [10.7]$$

In the general case this quantity shall not be confused with the *instantaneous probability of failure* denoted by $P_{f,i}(t)$ and defined as:

$$P_{f,i}(t) = \mathbb{P}\left(\ g\left(\boldsymbol{R}(\omega), \boldsymbol{S}_t(\omega), t\right) \le 0\right) \qquad [10.8]$$

The latter quantity which could be computed by “freezing” time in the limit state function and using classical methods (Monte Carlo simulation, FORM/SORM, importance sampling, etc.) does not have any particular interpretation, except for *right-boundary problems* that are defined in the next section. In particular the following inequality holds:

$$P_f(0,T) \ge \max_{t \in [0,T]} P_{f,i}(t) \qquad [10.9]$$

This lower bound is usually very poor and there is little interet in its computation.

10.3.2. *Right-boundary problems*

As remarked in the introduction the degradation of material properties introduces some time dependence into reliability problems. By definition however, this degradation tends to *decrease* the material resistance so that a limit state function of type « R - S » will be monotonically decreasing in time. Such a reliability problem in which all the trajectories of the limit state function are monotonically decreasing is called a *right-boundary problem.* In this specific case only one can prove that the cumulative failure probability is equal to the instantaneous failure probability computed at the right-boundary of the time interval (thus the name):

$$P_f(0,T) = P_{f,i}(T) \qquad [10.10]$$

Solving the time-variant reliability problem reduces to solving time-invariant problems, possibly for various values of *T*. Classical methods such as FORM/SORM and Monte Carlo simulation may be directly applied.

As an example, consider a steel rebar in a reinforced concrete structure which corrodes in time under the effect of concrete carbonation and/or chloride ingress. The uncorroded rebar cross section $\phi(t)$ may be modelled by:

$$\phi(t) = \begin{cases} \phi_0 & \text{si } t \leq T_{init} \\ \phi_0 \text{-} 2 i_{corr} \kappa \left(t - T_{init} \right) & \text{si } t > T_{init} \end{cases} \qquad [10.11]$$

where ϕ_0 is the initial rebar diameter, T_{init} is the initiation time for corrosion (*e.g.* the time required for the carbonated layer to attain the rebars), i_{corr} is the corrosion current density and κ is a constant.

The performance of the concrete structure may be related to the uncorroded rebar cross section $\phi(t)$: indeed the corroded external layer loses its mechanical resistance and the resulting rust tends to expand into the concrete pores and to crack and shatter the concrete surface (spalling phenomenon). Thus the failure w.r.t. to spalling may be defined by such an inequality: $\phi(t) \leq (1-\lambda)\phi_0$ where $\lambda = 0.05$ is a typical value for service limit state. In this setting the associated limit state function may be cast as $g(t) = \lambda\phi_0 \text{-} 2 i_{corr} \kappa \left(t - T_{init} \right)$, which is clearly decreasing monotonically in time for any realization of the (positive in nature) random variables $\left\{ \phi_0, i_{corr}, T_{init} \right\}$.

10.3.3. *General case*

As mentioned already, the unique feature of time-variant reliability analysis lies in the fact that the time-to-failure $\mathcal{T}(\omega)$ is random and not known in advance: depending on the realizations of the random processes $\boldsymbol{S}_t$, failure may happen more or less early. This time-to-failure $\mathcal{T}(\omega)$ satisfies:

$$\mathbb{P}\left(\mathcal{T} \leq t \right) = \mathbb{P}\left(\exists\, \tau \in [0,t] \ : \ g\left(\boldsymbol{R}(\omega), \boldsymbol{S}_\tau(\omega), \tau \right) \leq 0 \right) \overset{\text{def}}{=} P_f(0,t) \qquad [10.12]$$

From the above equation it is clear that the cumulative probability of failure $P_f\left(0,t\right)$ is nothing but the cumulative distribution function (CDF) of the time-to-failure $\mathcal{T}\left(\omega\right)$, *i.e.* the time required for the structure to "cross" the limit state surface. Computing this quantity relies upon the evaluation of the *mean outcrossing rate* which is defined in the next paragraph.

10.3.3.1. *Outcrossing rate*

Let us denote by $N_t^+\left(\omega\right)$ the number of outcrossings of the zero-level by the limit state function (*i.e* the structure passes from the safe domain to the failure domain) within the time interval $\left[0,t\right]$. Failure occurs within this time interval either if it occurs at the initial instant $t=0$ or if there is *at least* one crossing of the zero-value by the limit state function before time instant *t*. Thus:

$$P_f(0,t) = \mathbb{P}\left(\left\{g\left(\boldsymbol{R}(\omega), \boldsymbol{S}_0(\omega), t=0\right) \le 0\right\} \cup \left\{N_t^+ > 0\right\}\right) \qquad [10.13]$$

After some derivations one can prove that the right-hand side expression may be upper bounded as follows [DIT 96, SUD 07]:

$$P_f(0,t) \le P_{f,i}(0) + \mathrm{E}\left[N_t^+\right] \qquad [10.14]$$

where $\mathrm{E}\left[N_t^+\right]$ stands for the *expected number of outcrossings* within $\left[0,t\right]$. The *outcrossing rate* $\nu^+(t)$ is defined by:

$$\nu^+(t) = \lim_{h\to 0^+} \frac{\mathbb{P}\left(N^+\left(t,t+h\right)=1\right)}{h} \quad \text{where} \quad N^+\left(t,t+h\right) = N_{t+h}^+ - N_t^+ \qquad [10.15]$$

This quantity corresponds to the probability of having *exactly one* outcrossing in the infinitesimal interval $\left]t,t+h\right]$, divided by *h*. One also considers that the stochastic processes involved in the calculation are *regular* so that $\lim_{h\to 0^+} \frac{\mathbb{P}\left(N^+\left(t,t+h\right)>1\right)}{h} = 0$. Under this regularity condition, and due to the additivity property of the counting variable in time, one proves that:

$$E\left[N_t^+\right] = \int_0^t \nu^+(\tau)\, d\tau \qquad [10.16]$$

By substituting for [10.16] into [10.14] and recalling [10.9], one finally obtains the following bounds on the cumulative failure probability:

$$\max_{t\in[0,T]} P_{f,i}\left(t\right) \leq P_f(0,T) \leq P_{f,i}(t=0) + \int_0^T \nu^+(\tau)d\tau \qquad [10.17]$$

Thus solving a time-variant reliability problem (or at least obtaining an upper bound to $P_f(0,T)$) "reduces" to computing the outcrossing rate. Some important analytical results related to simplified problems are now presented, which are used in the following as basic ingredients to solve general problems.

Stationary time-variant reliability problems correspond to cases when the limit state function does not depend explicitly on time and the input random processes (gathered in $\boldsymbol{S}_t\left(\omega\right)$) are stationary. The limit state function is formally denoted by $g\left(\boldsymbol{R}(\omega), \boldsymbol{S}_t\left(\omega\right)\right)$. In this specific case the outcrossing rate does not depend on time and may be evaluated at any time instant (*e.g.* $t=0$). Equation [10.17] reduces to:

$$P_{f,i}\left(t=0\right) \leq P_f(0,T) \leq P_{f,i}(t=0) + \nu^+\, T \qquad [10.18]$$

REMARK – In the case when the limit state function does not depend on random variables but only on stationary random processes (which is formally denoted by $g\left(\boldsymbol{S}_t\left(\omega\right)\right)$) the number of outcrossings $N_t^+\left(\omega\right)$ is a Poisson process of constant intensity ν^+. In this case a result that is better than the above upper bound is available, namely $P_f(0,T) \equiv F_T(T) \approx 1 - e^{-\nu^+ T}$. However this approximation is *not* valid anymore when g also depends on random variables $\boldsymbol{R}(\omega)$ since the outcrossings do not occur independently in time anymore ($N_t^+\left(\omega\right)$ is not a Poisson process). The correct estimation is $P_f(0,T) \approx \mathrm{E}_{\boldsymbol{R}}\left[1 - e^{-\left(\nu^+ | \boldsymbol{R}\right)T}\right]$ in this case and may be computed by specific methods. In the latter equation $\nu^+ \,|\, \boldsymbol{R}$ is the *conditional* outcrossing rate and $\mathrm{E}_{\boldsymbol{R}}\left[.\right]$ denotes the expectation with respect to these variables, see details in [SCH 91, RAC 98].

Computing the outcrossing rate of a scalar (resp. vector) random process through a given threshold (resp. a given hypersurface) is a complex matter and beyond the scope of this chapter. The reader is referred to [DIT 96, RAC 04] for a complete treatment. In this chapter one limits the presentation to the classical *Rice's formula*, [RIC 44] which serves as a basis of more advanced results.

Let $S_t(\omega)$ be a scalar differentiable random process and $\dot{S}_t(\omega)$ its derivative process. Let us denote by $f_{S\dot{S}}(s,\dot{s})$ their joint probability density function. Of interest is the outcrossing rate $\nu^+(t)$ of this process through a (possibly varying in time) threshold denoted by $a(t)$. Rice's formula reads:

$$\nu^+(t) = \int_{\dot{a}(t)}^{\infty} \left(\dot{s} - \dot{a}(t)\right) \; f_{S\dot{S}}\left(a(t), \dot{s}\right) d\dot{s} \qquad [10.19]$$

In case of a stationary random process and a constant threshold (say $a = 0$ in the case of a limit state function for reliability analysis), the above formula reduces to $\nu^+ = \int_0^{\infty} \dot{s} \; f_{S\dot{S}}\left(a, \dot{s}\right) d\dot{s}$. As an example, if S_t is a stationary Gaussian process with mean value μ_S and standard deviation σ_S, one proves that the outcrossing rate for a threshold a is $\nu_a^{+,\text{gaussien}} = \frac{1}{\sqrt{2\pi}} \frac{\sigma_{\dot{S}}}{\sigma_S} \varphi\left(\frac{a-\mu_S}{\sigma_S}\right)$ where $\varphi(x) = e^{-x^2/2} / \sqrt{2\pi}$ denotes the standard normal PDF. If the Gaussian process is not stationary and if the threshold $a(t)$ is time-dependent, then the outcrossing rate is equal to $\nu^{+,\text{gaussien}}(t) = \frac{\sigma_{\dot{S}}}{\sigma_S} \varphi\left(a(t)\right) \Psi\left(\frac{\dot{a}(t)}{\sigma_{\dot{S}} / \sigma_S}\right)$, where $\Psi(x) = \varphi(x) - x\int_{-\infty}^{-x} \varphi(u) du$ [CRA67].

The calculation of outcrossing rates of *vector processes* through hypersurfaces makes use of the so-called Belayev's formula which is presented in [DIT 96, RAC 04].

10.4. PHI2 method

In the previous section the basic concepts that are useful for posing and solving time-variant reliability problems have been introduced, namely the random processes, the outcrossing rate, the cumulative probability of failure and its associated bounds. In order to evaluate Equation [10.17] in practice, the outcrossing rate of the limit state function through the zero-level shall be computed. As already mentioned analytical results are available only in very specific cases. Otherwise the analyst has to resort to numerical methods.

Two classes of approaches are nowadays well established in order to solve time-variant reliability problems:

– the so-called *asymptotic* method developed by Rackwitz and co-authors, which estimates the outcrossing rate and its time integral from Rice's formula and various asymptotic approximations such as the Laplace integration (see [RAC 98, RAC 04] for details);

– the so-called *PHI2 method*, which is based on solving a system reliability problem and which has been developed in [AND 02, AND 04, SUD 08] based on a similar work by [HAG 92, LI 95]. As it will be explained in the sequel this approach allows one to solve time-variant problems using only tools available for solving *time-invariant* problems, namely the First Order Reliability Method (FORM) for systems. Thus it may be applied using classical reliability software such as PhimecaSoft [LEM 06] or Open TURNS (www.openturns.org).

By definition the outcrossing rate is computed from the probability of having one crossing of the limit state surface (zero-level of the limit state function) within two neighbour instants t and $t+h$ (Equation [10.15]). In the reliability context such a crossing means that the structure was in the safe domain at time instant t and in the failure domain at time instant $t+h$. Thus the outcrossing rate may be evaluated as follows (the notation $\boldsymbol{X}_t(\omega) = \left\{\boldsymbol{R}(\omega), \boldsymbol{S}_t(\omega)\right\}^{\mathsf{T}}$ is introduced for the sake of clarity):

$$\nu^+(t) = \lim_{h \to 0^+} \frac{\mathbb{P}\left(\left\{g\left(\boldsymbol{X}_t(\omega), t\right) > 0\right\} \cap \left\{g\left(\boldsymbol{X}_{t+h}(\omega), t+h\right) \leq 0\right\}\right)}{h} \qquad [10.20]$$

The numerator of the above equation is nothing but the probability of failure of a two-component parallel system which may be estimated by the FORM method for systems [LEM 09, chapter 9].

Each component-reliability problem (*i.e.* at time instants t and $t+h$) is first solved using FORM. Let us denote by $\beta(t)$ and $\boldsymbol{\alpha}(t)$ (resp. $\beta(t+h)$ and $\boldsymbol{\alpha}(t+h)$) the reliability index and the unit normal vector at the design point that are related to the limit state function $\left\{g\left(\boldsymbol{X}_t(\omega), t\right) \leq 0\right\}$ (resp. $\left\{g\left(\boldsymbol{X}_{t+h}(\omega), t+h\right) \leq 0\right\}$). The system probability of failure may be computed within the first order approximation by:

$$\begin{aligned}&\mathbb{P}_{\text{FORM}}\left(\left\{g\left(\boldsymbol{X}_t(\omega), t\right) > 0\right\} \cap \left\{g\left(\boldsymbol{X}_{t+h}(\omega), t+h\right) \leq 0\right\}\right) \\ &\quad = \Phi_2\left(\beta(t), -\beta(t+h), \boldsymbol{\alpha}(t) \cdot \boldsymbol{\alpha}(t+h)\right)\end{aligned} \qquad [10.21]$$

where $\Phi_2(x,y,\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x}\int_{-\infty}^{y}\exp\left[-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right]dx\,dy$ denotes the cumulative distribution function of the binormal distribution. By combining [10.20] and [10.21], one can prove that the outcrossing rate reads [SUD 08]:

$$\nu^+(t)=\|\dot{\boldsymbol{\alpha}}(t)\|\varphi\big(\beta(t)\big)\Psi\left(\frac{\dot{\beta}(t)}{\|\dot{\boldsymbol{\alpha}}(t)\|}\right)\quad \text{where}\quad \Psi(x)=\varphi(x)-x\int_{-\infty}^{-x}\varphi(u)du \qquad [10.22]$$

For stationary time-variant problems the outcrossing rate does not depend on time and thus simplifies into:

$$\nu^+=\frac{\varphi(\beta)}{\sqrt{2\pi}}\|\dot{\boldsymbol{\alpha}}(t)\| \qquad [10.23]$$

One can note the similarity between both above equations and those given at the end of Section 10.3 for the application of Rice's formula to Gaussian processes. In order to give an interpretation of [10.22] one may consider that the FORM method consists in "scalarizing" the outcrossing problem by considering the limit state function as an equivalent scalar process, whose outcrossing of the threshold $\beta(t)$ is of interest.

10.4.2. *Implementation of the PHI2 method – stationary case*

In case of a stationary problem the outcrossing rate is constant in time. It may be computed form Equation [10.23] by approximating it by a finite difference scheme:

$$\nu^+_{num}=\frac{\varphi(\beta)}{\sqrt{2\pi}}\left\|\frac{\boldsymbol{\alpha}(t+\Delta t)-\boldsymbol{\alpha}(t)}{\Delta t}\right\| \qquad [10.24]$$

To do so a sufficiently small time increment Δt shall be selected. The thumb rule $\Delta t\approx 10^{-3}\lambda_{\min}$ has proven efficiency and accuracy in applications. In this equation $\lambda_{\min}$ is the smallest correlation length among all the random processes $\boldsymbol{S}_t$ involved in the limit state function [SUD08]. The various steps for evaluating [10.24] are now summarized:

– The Gaussian vectors $\boldsymbol{S}^1(\omega)$ and $\boldsymbol{S}^2(\omega)$ corresponding to the Gaussian process $\boldsymbol{S}_t(\omega)$ at time instants t and $t+\Delta t$ are first defined. The components S_j^1 and

S_j^2 are pairwise correlated with correlation coefficient $\rho_{S_j}(t,t+\Delta t)$, where ρ_{S_j} is the autocorrelation coefficient function of $S_{t,j}$ (Equation [10.2]). Note that if the components S_j of $\boldsymbol{S}_t$ are correlated, this so-called *cross-correlation* has to be taken into account as well.

– The "instantaneous" limit state function $g_1\left(\boldsymbol{R}(\omega),\boldsymbol{S}^1(\omega)\right)$ is defined at time instant t by replacing the random processes $\boldsymbol{S}_t$ by vector $\boldsymbol{S}^1$ in $g\left(\boldsymbol{R}(\omega),\boldsymbol{S}_t(\omega)\right)$ and FORM is applied, which yields the reliability index $\beta^{(1)}$ and the unit normal vector $\boldsymbol{\alpha}^{(1)}$;

– The "instantaneous" limit state function $g_2\left(\boldsymbol{R}(\omega),\boldsymbol{S}^2(\omega)\right)$ is defined at time instant $t+\Delta t$ by replacing the random processes $\boldsymbol{S}_{t+\Delta t}$ by vector $\boldsymbol{S}^2$ in $g\left(\boldsymbol{R}(\omega),\boldsymbol{S}_t(\omega)\right)$ and FORM is applied, which yields the reliability index $\beta^{(2)}$ and the unit normal vector $\boldsymbol{\alpha}^{(2)}$;

– From these results the outcrossing rate [10.24] is evaluated then the cumulative failure probability:

$$\nu^+_{num} = \frac{\varphi\left(\beta^{(1)}\right)}{\sqrt{2\pi}}\left\|\frac{\boldsymbol{\alpha}^{(2)}-\boldsymbol{\alpha}^{(1)}}{\Delta t}\right\| \qquad P_f(0,T) \le \Phi\left(-\beta^{(1)}\right)+\nu^+_{num}\cdot T \qquad [10.25]$$

It is clear that the upper bound linearly increases with T. In order to interpret the result conveniently, the upper bound may be transformed into a "generalized reliability index" $\beta^{\inf}(0,T) = -\Phi^{-1}\left(\Phi\left(-\beta^{(1)}\right)+\nu^+\cdot T\right)$. From the relationship between probability of failure and reliability index the above value is a *lower* bound to the reliability index, thus the notation $\beta^{\inf}$. The upper bound reliability index associated to the lower bound in Equation [10.17] is simply equal to $\beta^{(1)}$.

Note that two different correlation coefficients are used in the analysis, which should not be confused: the first one is the autocorrelation coefficient of each input random process denoted by $\rho_{S_j}(t,t+\Delta t)$; the second one is the correlation between the linearized limit state surfaces at time instants t and $t+\Delta t$, which is given by the scalar product $\boldsymbol{\alpha}^{(1)}.\boldsymbol{\alpha}^{(2)}$.

10.4.3. *Implementation of the PHI2 method – non stationary case*

In this case the limit state function explicitly depends on time and / or the input random processes $\boldsymbol{S}_t$ show non stationarity. Thus the outcrossing rate is evolving in time and shall be computed at different time instants, then integrated over $\left[0,T\right]$ (Equation [10.17]) in order to get the upper bound to $P_f\left(0,t\right)$. In practice the time interval is discretized, say $\left\{t_i = iT/N, i = 0,\cdots,N\right\}$ and the procedure described in Section 10.4.2 is applied at each time instant. The upper bound to $P_f\left(0,t\right)$ may be computed using the trapezoidal rule:

$$P_f(0,T) \le P_{f,i}(0) + \frac{T}{N}\left(\frac{\nu^+(0)+\nu^+(T)}{2} + \sum_{i=1}^{N-1}\nu^+(t_i)\right) \qquad [10.26]$$

Note that the time increment T/N used for computing the integral is not of the the same order of magnitude as the time increment Δt used for evaluating the outcrossing rate.

10.4.4. *Semi-analytical example*

Let us consider a cantilever beam of length *L*, flexural modulus *EI* that is submitted to a pinpoint load *F* at its free extremity. The maximum deflection of the beam under quasi-static conditions is equal to $\delta = \dfrac{FL^3}{3EI}$ (the variation of the load in time is assumed slow enough so as to ignore dynamical effects). Of interest is the reliability of the beam with respect to an admissible threshold $\delta_{\max}$ for the maximal deflection. The flexural modulus is supposed to be lognormally distributed (parameters $\left(\lambda_{EI},\varsigma_{EI}\right)$). It is also supposed that the logarithm of the load is a stationary Gaussian process S_s of mean value λ_F, standard deviation ς_F and autocorrelation coefficient function $\rho_F\left(t\right) = e^{-(t/\tau_F)^2}$, where τ_F is the correlation length. So as to be able to perform analytical derivations, the limit state function associated with the criterion "the maximal deflection is below the admissible threshold" may be cast as:

$$g\left(EI,S_t\right) \overset{def}{=} \ln\delta_{\max} - \ln\delta = \ln\delta_{\max} - S_t - \ln\frac{L^3}{3} + \ln EI \qquad [10.27]$$

Let us select some particular instant t_0. Random variable $\ln EI$ is Gaussian by definition and may be cast as follows: $\ln EI = \lambda_{EI} + \varsigma_{EI} U_1$, where U_1 ~ *N(0,1)* is a standard normal variable. Similarly S_{t_0} is a Gaussian variable of parameters (λ_F, ς_F) that may be cast as $S_{t_0} = \lambda_F + \varsigma_F U_2$, where U_2 ~ *N(0,1)*. After substituting for these expressions in [10.27], the limit state function reveals linear in the reduced variables U_1, U_2. FORM is exact in this case and the associated reliability index reads:

$$\beta^{(1)} = \frac{\ln \delta_{\max} - \ln\left(L^3/3\right) - \lambda_F + \lambda_{EI}}{\sqrt{\varsigma_F^2 + \varsigma_{EI}^2}} \qquad [10.28]$$

The coordinates of the unit normal vector to the limit state surface at the design point reads: $\boldsymbol{\alpha}^{(1)} = \left\{-\varsigma_{EI} / \sqrt{\varsigma_F^2 + \varsigma_{EI}^2}\ ,\ \varsigma_F / \sqrt{\varsigma_F^2 + \varsigma_{EI}^2}\right\}^{\mathsf{T}}$.

In order to "freeze" the limit state function [10.27] at time instant $t_0 + \Delta t$, one should notice that S_{t_0} and $S_{t_0+\Delta t}$ are correlated Gaussian variates with correlation coefficient $\tilde{\rho} = e^{-(\Delta t/\tau_F)^2}$ (this number depends on the user choice of $\Delta t \ll \tau_F$, *e.g.* $\Delta t = 10^{-3}\tau_F$ as suggested above). The isoprobabilistic transform required by FORM in order to handle dependent Gaussian variates leads to introducing U_3 ~ *N(0,1)* and reads (after using the Cholesky decomposition of the correlation matrix):

$$S_{t_0} = \lambda_F + \varsigma_F U_2 \quad , \quad S_{t_0+\Delta t} = \lambda_F + \varsigma_F\left(\tilde{\rho} U_2 + \sqrt{1-\tilde{\rho}^2}\, U_3\right) \qquad [10.29]$$

The instantaneous limit state function at time instant $t_0 + \Delta t$ reveals linear in the three reduced variables U_1, U_2, U_3. The (exact) reliability index is identical to that obtained at time instant *t* (Equation [10.28]), which is logical since the problem is stationary. The unit normal vector now reads:

$$\boldsymbol{\alpha}^{(2)} = \left\{-\varsigma_{EI} / \sqrt{\varsigma_F^2 + \varsigma_{EI}^2}\ ,\ \tilde{\rho}\varsigma_F / \sqrt{\varsigma_F^2 + \varsigma_{EI}^2}\ ,\ \sqrt{1-\tilde{\rho}^2}\,\varsigma_F / \sqrt{\varsigma_F^2 + \varsigma_{EI}^2}\right\}^{\mathsf{T}}$$

In order to finish the computation numerical values shall be given to the various parameters. Then $\beta^{(1)} = \beta^{(2)}$ is computed from Equation [10.28]. The $\boldsymbol{\alpha}^{(i)}$-vectors are evaluated and the values are used to compute the outcrossing rate and the probability of failure using [10.25].

10.5. Industrial application: truss structure under time-varying loads

Consider the elastic 23 bar truss depicted in Figure 10.2 that has been already presented in Chapter 8.

Of interest is the time-variant reliability of such a truss structure under time-varying loads applied on the upper part.

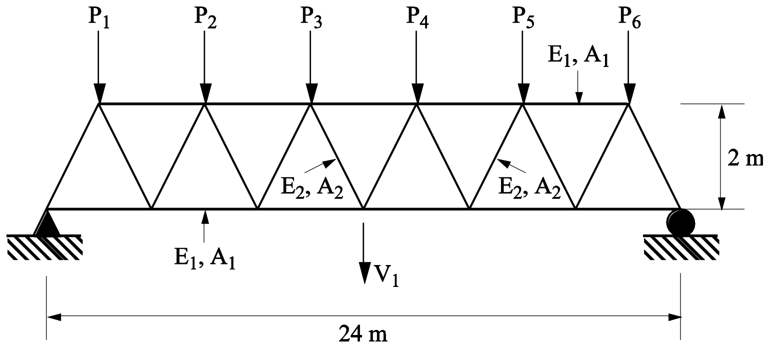

Figure 10.2. *23 bar- truss structure*

The input random variables are described in Table 10.1. The six vertical loads are modelled by a *single* stationary Gaussian process P_t with mean value 50 kN, standard deviation 7.5 kN, and Gaussian autocorrelation coefficient function $\rho_P(t) = e^{-(t/\tau_P)^2}$ where the correlation length is $\tau_P = 1\,\text{day}$. According to this value the time variation of the load is sufficiently slow so that inertial effects may be neglected: the quasi-static solution is thus valid. The time-variant reliability of the truss with respect to an admissible maximal deflection reads:

$$g(E_1, A_1, E_2, A_2, P_t) = v_{\max} - |\mathcal{M}(E_1, A_1, E_2, A_2, P_t)| \leq 0,\ v_{\max} = 16\text{ cm} \qquad [10.30]$$

where $\mathcal{M}(E_1, A_1, E_2, A_2, P_t)$ is the maximal deflection computed by finite element analysis. Due to stationarity a single evaluation of the outcrossing rate is necessary.

The initial problem has 4 basic random variables and a single random process. Using the PHI2 method it is transformed into two (time-invariant) FORM analysis which involve 4+2=6 random variables (including one for P_t and one for $P_{t+\Delta t}$). The time increment is $\Delta t = 10^{-3}$.

Random variable	**Distribution**	**Mean value**	**Standard deviation**
E_1, E_2 (MPa)	Lognormale	210 000	21 000
A_1 (cm²)	Lognormale	20	2
A_2 (cm²)	Lognormale	10	1
P_t (kN)	Gaussian process	50	7,5

Tableau 10.1. *23 bar-truss – description of the random variables*

The instantaneous reliability analysis yields β=4.032 and $\alpha^{(1)}$ = {-0.533447, -0.067651, -0.533447, -0.067651, 0.649397, 0.}^{T}. At time instant $t+\Delta t$ the same reliability index is obtained and the unit normal vector is $\alpha^{(2)}$ = {-0.533447, -0.067651, -0.533447, -0.067651, 0.649396 0.000918}^{T}.

It may be observed that only the last two components of the α-vector (*i.e.* the ones related to the random process) are changing between the two time instants. Using Equation [10.18] yields the outcrossing rate $\nu^+ = 4{,}3.10^{-5}$/ day. The upper bound to the cumulative failure probability is obtained from [10.25].

The evolution in time of this quantity is plotted in Figure 10.3. These results show that the cumulative failure probability may be greater than the instantaneous probability of failure by orders of magnitude. Note that the latter correspond to the time-invariant case when the loads are modelled by a single random variable.

10.6. Conclusion

Structural reliability methods are nowadays well established for time-invariant problems and they are used on a regular basis in industrial applications, as shown throughout this book. Time-variant reliability analysis is by far less mature. First handling random processes instead of random variables introduces some additional abstract concepts. Moreover the quantity of interest, namely the *cumulative* probability of failure is rather difficult to compute.

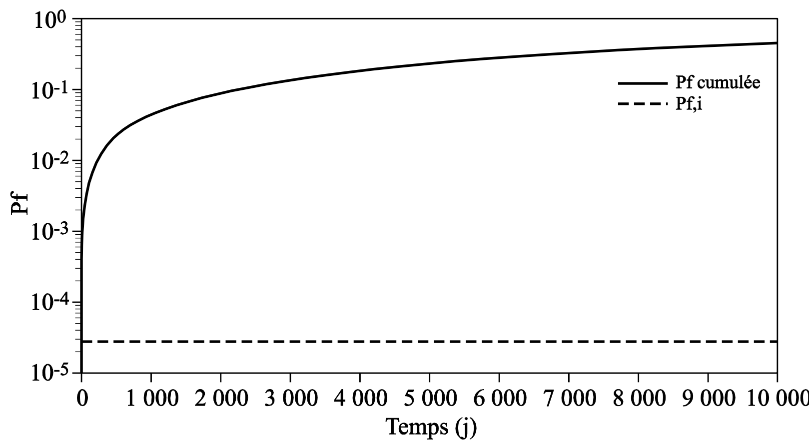

Figure 10.3. *23 bar-truss – cumulative probability of failure*

In this chapter only the basic concepts have been introduced. In particular the *stochastic dynamics* problems have not been addressed. Specific methods have been introduced for such problems, see *e.g.* [KRE 83, SOI 01, LUT 03]. Only the PHI2 method has been presented in details: it allows the analyst to compute the outcrossing rate using FORM for systems. This means that only classical *time-invariant* tools may be used for solving time-variant problems, which are available in many reliability softwares.

Finally note that the Monte Carlo method has not been presented here in the context of time-variant problems. Its use would require sampling trajectories of the random processes and then the solution of transient mechanical problems. This is obviously a very tedious and costly approach that shall be used only as a last resort, especially for nonlinear dynamics treated in the time domain (*e.g.* seismic analysis of structures).

10.7. References

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