Chapter 12

Bayesian Updating Techniques in Structural Reliability [1]

4.1. Introduction

Computer simulation models such as finite element models are nowadays commonly used in various industrial fields in order to optimize the design of complex mechanical systems as well as civil engineering structures. In the latter case, the structures under consideration (*e.g.* cable-stayed bridges, dams, tunnels, etc.) are often one of a kind. Thus they are usually instrumented during their construction and after, so that experimental measurement data (displacement, strains, etc.) are collected all along their life time.

These measurements are traditionally used for detecting a possible unexpected behaviour of the system (*e.g.* a temporal drift of some indicator). In this case data is processed using classical statistical methods *without any physical modelling* of the structure. However this data could be used together with some computational model

---

[1] Chapter written by Bruno SUDRET

that would have been elaborated at the design stage in order to *update* the model predictions. Classical approaches in this field are purely deterministic: the analyst tries to select the set of model parameters that best fit the available data by minimizing the discrepancy between measurements and model prediction (*e.g.* using least-square minimization), without taking into account possible sources of error or uncertainty such as measurement error, model uncertainty, etc.

In this chapter Section 1 describes a probabilistic framework that allows one to combine a computational model (*e.g.* a finite element model), a prior knowledge on its input parameters and some experimental database. This framework makes use of *Bayesian statistics* which is rigorously presented in the sequel. Section 2 presents the retained probabilistic model for the interaction between measurements and model predictions. Section 3 recalls how to compute the probability of failure of a structure and how to *update* it using additional measurement data. Section 4 describes how to « invert » the reliability problem and compute *quantiles* of the mechanical response instead. Section 5 presents the *Markov Chain Monte Carlo* simulation method (MCMC) – that has been already introduced in the previous chapter – and how it may be used for Bayesian updating problems. Finally, Section 6 describes an application example related to the durability of concrete containment vessels of French nuclear power plants.

4.2. Problem statement: link between measurements and model prediction

Let us consider a mathematical model $\mathcal{M}(\boldsymbol{x},t)$ that represents the temporal evolution of the response $\boldsymbol{y}(t) \in \mathbb{R}^N$ of a mechanical system as a function of a vector of input parameters $\boldsymbol{x} \in \mathbb{R}^M$. These *basic* parameters are supposed to be uncertain or not well known. They are modelled by a random vector $\boldsymbol{X} = \{X_1, \ldots, X_M\}^\mathsf{T}$ whose joint probability density function (PDF) is prescribed. This PDF may be selected at the design stage from available data (see chapters 4 & 5) or from expert judgment. In this context the model response at time instant *t* is a random vector denoted by $\boldsymbol{Y}(t) = \mathcal{M}(\boldsymbol{X},t)$. The collection of random vectors $\{\boldsymbol{Y}(t), t \in [0,T]\}$ is a random process (see Chapter 10 for a rigorous definition).

Let $\tilde{\boldsymbol{y}}(t)$ be the « *true* » value of the system response at time instant t, *i.e.* the value that would be measured by a perfect, infinitely accurate device (no measurement error). This value is usually different from the *observed* value $\boldsymbol{y}_{obs}(t)$ which has been obtained by the measurement device at hand.

If a *perfect* model of the system behaviour was available, there would exist a vector of basic parameters denoted by $\tilde{\boldsymbol{x}}$ such that $\tilde{y}(t) = M(\tilde{\boldsymbol{x}},t)$. However models are always simplified representation of the real world that contains unavoidably approximations. Thus a so-called *measurement/model* error term is introduced in order to characterize the discrepancy between the model output and the corresponding observation. Considering various time instants $t^{(q)}$, $q = 1,\cdots,Q$, this assumption reads:

$$\boldsymbol{y}_{obs}^{(q)} = \mathcal{M}\left(\tilde{\boldsymbol{x}}, t^{(q)}\right) + \boldsymbol{e}^{(q)} \qquad [4.1]$$

In the latter equation the observed value $\boldsymbol{y}_{obs}(t)$ and the (implicit or explicit) definition of the model $\mathcal{M}$ are known, whereas $\tilde{\boldsymbol{x}}$ and $\boldsymbol{e}^{(q)}$ are unknown. If one assumes that the error $\boldsymbol{e}^{(q)}$ is a realization of a random vector $\boldsymbol{E}^{(q)}$ that characterize the measurement/model error (usually a Gaussian vector of zero mean value and covariance matrix $\boldsymbol{C}$) the above equation means that $\boldsymbol{y}_{obs}^{(q)}$ is a realization of a random vector $\boldsymbol{Y}_{obs}^{(q)}$ whose *conditional distribution* reads :

$$\boldsymbol{Y}_{obs}^{(q)} \big| \boldsymbol{X} = \boldsymbol{x} \ \sim \ \mathcal{N}(M(\boldsymbol{x}, t^{(q)}), \boldsymbol{C}) \qquad [4.2]$$

where $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{C})$ denotes a multinormal distribution with mean value $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{C}$.

In practice, depending on the problem under consideration, the error term $\boldsymbol{E}^{(q)}$ may represent either the measurement uncertainty, the model error or both. These two quantities are usually independent so that this error may be decomposed again as the sum of two terms. The total covariance may be split as $\boldsymbol{C} = \boldsymbol{C}_{mes} + \boldsymbol{C}_{\text{mod}}$, where $\boldsymbol{C}_{mes}$ represents the covariance matrix of the sole measurement error.

4.3. Computing and updating the failure probability

4.3.1. Structural reliability – problem statement

Structural reliability analysis aims at computing the probability of failure of a mechanical system whose parameters are uncertain and modelled within a probabilistic framework. Reliability methods that lead to compute a *probability of failure* with respect to some scenario are well documented in the books by Ditlevsen & Madsen [DIT 96] and Lemaire [LEM 09].

Let $\boldsymbol{X}$ denote the vector of input random variables describing the problem (that usually include the input parameters of some mechanical model $\mathcal{M}$), and let us denote by $\mathcal{D}_{\boldsymbol{X}} \subset \mathbb{R}^M$ its support. A failure criterion may be mathematically cast as a *limit state function* $\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} \mapsto g(\boldsymbol{x})$ such that $D_f = \{\boldsymbol{x} : g(\boldsymbol{x}) \leq 0\}$ is the *failure domain* and $D_s = \{\boldsymbol{x} : g(\boldsymbol{x}) > 0\}$ is the *safe domain*. The boundary between both domains is called the *limit state surface* ∂D. The *failure probability* is then defined by

$$P_f = \mathbb{P}\left(g\left(\boldsymbol{X}\right) \leq 0\right) = \int_{D_f} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \qquad [4.3]$$

where $f_{\boldsymbol{X}}$ is the joint PDF of $\boldsymbol{X}$.

As the integration domain D_f is implicitly defined from the sign of the limit state function g and the latter is usually *not* analytical, a direct evaluation of the integral in Eq.[4.3] is rarely possible. It can be numerically estimated using *Monte Carlo simulation* (MCS): N_{sim} realizations of the input random vector $\boldsymbol{X}$ are drawn according to its joint PDF $f_{\boldsymbol{X}}$, and for each sample the *g*-function is computed. The probability of failure is estimated by the ratio N_f / N_{sim} where N_f is the number of samples (among N) that have lead to failure (*i.e.* a negative value of g).

This method, which is rather easy to implement, may be unaffordably costly in practice. Indeed, suppose that a probability of failure of the order of magnitude 10^{-k} is to be estimated with a relative accuracy of 5%: a number of $N_{\text{sim}} \approx 4.10^{k+2}$ simulations is then required. As failure probabilities usually range from 10^{-2} to 10^{-6} it is clear that MCS will not be directly applicable for industrial problems, for which a *single run* of the model $\mathcal{M}$ and the associated performance g may require hours of CPU. In order to bypass this difficulty, alternative *approximate* methods have been introduced such as the *First Order Reliability Method* (FORM).

FORM allows one to approximate the failure probability by recasting the integral in Eq.[4.3] in the *standard normal space*, *i.e.* a space in which all random variables ξ are normal with zero mean value and unit standard deviation. To this aim an *isoprobabilistic transform* $T : \boldsymbol{X} \to \xi\left(\boldsymbol{X}\right)$ is used.

If the basic random variables gathered in $\boldsymbol{X}$ are independent with respective marginal cumulative distribution function (CDF) $F_{X_i}(x_i)$, this transform reads: $\xi_i = \Phi^{-1}\left(F_{X_i}(x_i)\right)$, where Φ is the standard normal CDF. In the general case, the Nataf or Rosenblatt transforms may be used, see [LEM 09, chap. 4] for details.

After mapping the basic variables $\boldsymbol{X}$ into standard normal variables $\boldsymbol{\xi}$, Eq.[4.3] rewrites:

$$P_f = \int_{\left\{\xi\,:\,G(\xi)\equiv g\left(T^{-1}(\xi)\right)\le 0\right\}} \varphi_M\left(\xi\right) d\xi_1 \dots d\xi_M \qquad [4.4]$$

where $G\left(\xi\right) = g\left(T^{-1}\left(\xi\right)\right)$ is the limit state function in the standard normal space and φ_M is the multinormal (M-dimensional) PDF defined by $\varphi_M\left(\xi\right) = \left(2\pi\right)^{-\frac{M}{2}} \exp\left[-\frac{1}{2}\left(\xi_1^2 + \dots + \xi_M^2\right)\right]$. This PDF is maximal for $\boldsymbol{\xi} = \boldsymbol{0}$ and decreases exponentially with $\left\|\boldsymbol{\xi}^2\right\|$. Thus the points that contribute most to the integral in Eq.[4.4] are those points of the failure domain that are close to the origin of the space.

The next step of FORM consists in determining the so-called *design point* $\boldsymbol{\xi}^*$, *i.e.* the point of the failure domain D_f that is the closest to the origin. This point is solution of the following optimization problem:

$$\boldsymbol{\xi}^* = \operatorname{Arg}\min_{\xi\in\mathbb{R}^M}\left\{\frac{1}{2}\left\|\xi\right\|^2 \,/\, G\left(\xi\right) \le 0\right\} \qquad [4.5]$$

Dedicated constrained optimization algorithms may be used for solving it. The minimal (algebraic) distance from the limit state surface ∂D to the origin is called the *Hasofer-Lind reliability index:* $\beta = \operatorname{sign}\left(\mathrm{G}\left(0\right)\right)\left\|\xi^*\right\|$. Once $\boldsymbol{\xi}^*$ has been computed the limit state surface ∂D is linearized around this point and replaced by a tangent hyperplane. The failure domain is then substituted for by the half space defined by this hyperplane. The approximation of the integral in [4.4] by integrating over the half space leads to the FORM approximation $P_f \approx P_{f,FORM} = \Phi(-\beta)$.

The equation of the linearized limit state (*i.e.* the hyperplane) may be cast as $\tilde{G}\left(\xi\right) = \beta - \boldsymbol{\alpha}^{\mathsf{T}} \cdot \boldsymbol{\xi}$. In this expression the unit vector (which is orthogonal to the hyperplane) contains the cosines of the angles defining the direction of the design point. The square cosines α_i^2 are called *importance factors* since they allow one to decompose the variance of the (approximate) performance $\tilde{G}\left(\xi\right)$ into contributions of each variable ξ_i and by extension, to quantify the impact of each basic variable X_i onto the reliability.

The First Order Reliability Method allows the analyst to get an approximation of P_f at a reasonable computational cost (usually, from a few tens to a few hundreds of evaluations of g). The approximation is all the better since the reliability index β is large. Moreover, the approach yields importance factors. *Sensitivity measures* that quantify how the probability of failure changes when some assumption on the basic random variables is changed are also interesting quantitative indicators for the designer.

4.3.2. Updating the failure probability

The failure probability as defined in the previous paragraph is usually computed at the design stage. For an *already existing* system for which additional information is available (*e.g.* measurements of response quantities in time), it is possible to *update* the probability of failure by accounting for this data.

Let us consider a set of observations[2] $\mathcal{Y} = \left\{ y_{obs}^{(1)}, \ldots, y_{obs}^{(Q)} \right\}^{\mathsf{T}}$ collected at time instants $t^{(q)}, q = 1, \ldots, Q$ along the life time of the structure. Confronting this data to model simulation results leads to introduce the so-called *measurement events* $\left\{ H_q = 0 \right\}$ [DIT 96] using the following notation:

$$H_q = \mathcal{M}\left(\boldsymbol{X}, t^{(q)} \right) - y_{obs}^{q} + E^{(q)} \qquad [4.6]$$

In this equation $E^{(q)}$ denotes a Gaussian random variable that characterizes the *measurement/model error*. The *updated* failure probability $P_f^{upd}\left(t \right)$ is now defined as the following conditional probability:

$$P_f^{upd}\left(t \right) = \mathbb{P}\left(g\left(\boldsymbol{X}, t \right) \leq 0 \mid H_1 = 0 \cap \ldots \cap H_Q = 0 \right) \qquad [4.7]$$

When recasting the measurement events as the limit $\lim_{\theta \to 0} \left\{ -\theta < H_q \leq 0 \right\}$, one gets:

$$P_f^{upd}\left(t \right) = \lim_{\theta \to 0} \frac{\mathbb{P}\left(\left\{ g\left(\boldsymbol{X}, t \right) \leq 0 \right\} \cap \left\{ -\theta < H_1 \leq 0 \right\} \cap \ldots \cap \left\{ -\theta < H_Q \leq 0 \right\} \right)}{\mathbb{P}\left(\bigcap_{q=1}^{Q} \left\{ -\theta < H_q \leq 0 \right\} \right)} \qquad [4.8]$$

---

[2] From now on the response quantity under consideration and the associated measurements are supposed to be *scalar* quantities for the sake of simplicity.

In the above equation both the numerator and denominator are failure probabilities of *parallel systems* (intersections of events) that may be estimated by an extension of the FOR method to systems [LEM 09, chap. 9]. After some algebra, Eq.[4.8] reduces to [MAD87]:

$$P_f^{upd}(t) = \Phi\left(-\beta^{upd}(t)\right) \quad \text{avec} \quad \beta^{upd}(t) = \frac{\beta_0(t) - \boldsymbol{z}(t)^{\mathsf{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{\beta}}{\sqrt{1 - \left(\boldsymbol{z}(t)^{\mathsf{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{z}(t) \right)^2}} \qquad [4.9]$$

In this equation $\beta_0(t)$ is the *initial* reliability index associated with the event $\{g(\boldsymbol{X},t) \le 0\}$ and $\boldsymbol{\beta} = \{\beta_1,...,\beta_Q\}^{\mathsf{T}}$ gathers the reliability indices related to the events $\{H_q \le 0\}$. Moreover $\boldsymbol{z}(t) = \{z_1(t),...,z_Q(t)\}$ is the vector of correlations between the linearized margins $\{H_q = 0\}$ and $\{g(\boldsymbol{X},t) = 0\}$ whose components are $z_j(t) = \boldsymbol{\alpha}_0(t) \cdot \boldsymbol{\alpha}_j$. Finally $\boldsymbol{R}$ is the correlation matrix of the linearized measurement margins, whose generic entry reads $\boldsymbol{R}_{kl} = \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_l$. Thus the updated failure probability may be computed only from a set of FORM analyses.

4.4. Updating a confidence interval on response quantities

4.4.1 Quantiles as the solution of an inverse reliability problem

Suppose the random response of a mechanical model $Y(t) = \mathcal{M}(\boldsymbol{X},t)$ is of interest. Its variability may be fruitfully grasped through the computation of a confidence interval on the prediction, which means computing quantiles of $Y(t)$. Indeed a *e.g.* 95%-confidence interval (*i.e.* a range such that the probability of $Y(t)$ being in this range is 95%) is obtained by computing the 2.5% and 97.5% quantiles of $Y(t)$. As a consequence the computation of α-quantiles $y_\alpha(t)$ defined by:

$$\mathbb{P}\left(Y(t) \le y_\alpha(t)\right) = \alpha \quad ; \quad \alpha \in \left]0,1\right[\qquad [4.10]$$

is of interest. By introducing the mechanical model $\mathcal{M}$ in the previous equation one obtains $y_\alpha(t)$ as the solution of the following:

$$\mathbb{P}\left(M(\boldsymbol{X},t) - y_\alpha(t) \le 0\right) = \alpha \qquad [4.11]$$

Equation [4.11] may be considered at each time instant t as an *inverse reliability problem* [DER 94], in which the value of a parameter (here, y_α) is looked after so that a given "failure probability" is attained (here, α) for a given limit state function (here, $\mathcal{G}(\boldsymbol{X},t\,;y_\alpha) \equiv \mathcal{M}(\boldsymbol{X},t) - y_\alpha$). In order to solve this problem efficiently an extension of FORM has been proposed in [DER 94]. Within the FORM approximation the problem is recast as:

$$\text{find } y_\alpha : \quad P_{f,FORM}\left(g(\boldsymbol{X},t\,;y_\alpha) \leq 0\right) = \Phi(-\beta_c) \qquad [4.12]$$

where $\beta_c = -\Phi^{-1}(\alpha)$ is the target reliability index associated with the α-quantile of interest. The algorithm used for computing quantiles is presented in details in [PER 07, PER 08].

4.4.2 Updating quantiles of the response quantity

The "inverse FORM" approach may be elaborated one step further in order to compute *updated quantiles* that are defined as quantiles computed conditionally to observations. When combining [4.9] and [4.12] the "updated" version of the latter reads:

$$\text{find } y_\alpha : \quad P_{f,FORM}\left(g(\boldsymbol{X},t\,;y_\alpha) \leq 0 \,|\, H_1 = 0 \cap \ldots \cap H_Q = 0\right) = \Phi(-\beta_c) \qquad [4.13]$$

where the measurement events are defined in Eq.[4.6]. The "updated inverse FORM" algorithm as originally proposed in [SUD 06] couples the inverse FORM algorithm with Eq.[4.9] by modifying in each iteration the target reliability index $\beta_c^{(k+1)}$ which is equal at iteration k+1 to:

$$\beta_c^{(k+1)} = -\Phi^{-1}(\alpha)\sqrt{1 - \left(\boldsymbol{z}^{(k)}(t)^\mathsf{T} \cdot \boldsymbol{R} \cdot \boldsymbol{z}^{(k)}(t)\right)^2} + \boldsymbol{z}^{(k)}(t)^\mathsf{T} \cdot \boldsymbol{R} \cdot \boldsymbol{\beta}^{(k)} \qquad [4.14]$$

In the above equation, matrix $\boldsymbol{R}$ does not change from one iteration to the other in contrast to vectors $\boldsymbol{z}$ and $\boldsymbol{\beta}$. Note that the convergence of the algorithm is not proven although numerous application examples have shown the efficiency of the method.

4.4.3 **Conclusion**

The method proposed in the above paragraphs allows one to update the failure probability of a structure or indirectly, to update the confidence intervals of the prediction of a mechanical model by using measurement data gathered all along the life time of the structure.

This approach enables the reconciliation of the prior model predictions $Y(t)=\mathcal{M}(\boldsymbol{X},t)$ and the observed data $\mathcal{Y}=\left\{y_{obs}^{(1)},\ldots,y_{obs}^{(Q)}\right\}^{\mathsf{T}}$ in order to better estimate the probability of failure of the real structure ("as built") under consideration. However it does not bring any additional information onto the basic variables $\boldsymbol{X}$. An alternative approach based on Markov Chain Monte Carlo Simulation is presented in the next section for this purpose.

4.5. Bayesian updating of the model basic variables

4.5.1 Reminder on Bayesian statistics

Bayesian statistical methods [ROB 92, OHA 04] are usually used in order to combine some prior information on parameters of a random vector and data, *i.e.* realizations of this random vector. Let us denote by $\mathcal{X}=\left\{\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(Q)}\right\}$ a set of observations that shall be modelled by a PDF $f_X\left(\boldsymbol{x}\,;\boldsymbol{\theta}\right)$, where $\boldsymbol{\theta}$ is the vector of *hyperparameters* of size n_θ. Bayesian statistics assumes that some prior information on $\boldsymbol{\theta}$ exists that may be modelled by a *prior distribution* $p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)$ of support $D_\Theta \subset R^{n_\theta}$. Bayes' theorem in its continuous setting combines both sources of information in order to yield a *posterior distribution* $f_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)$:

$$f_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)=\frac{1}{c}\,p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)L\left(\boldsymbol{\theta}\,;\mathcal{X}\right) \qquad [4.15]$$

In this equation L is the *likelihood* of the observations which is defined in case of independent observations by:

$$L\left(\boldsymbol{\theta}\,;\mathcal{X}\right)=\prod_{q=1}^{Q} f_X\left(\boldsymbol{x}^{(q)};\boldsymbol{\theta}\right) \qquad [4.16]$$

and c is a normalizing constant defined by $c=\int_{D_\Theta} p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)L\left(\boldsymbol{\theta};\mathcal{X}\right)d\boldsymbol{\theta}$. From [4.15-4.16] one can furthermore obtain *the predictive distribution* of $\boldsymbol{X}$, namely:

$$f_X^p\left(\boldsymbol{x}\right)=\int_{D_\Theta} f_X\left(\boldsymbol{x},\boldsymbol{\theta}\right)f_\Theta\left(\boldsymbol{\theta}\right)d\boldsymbol{\theta} \qquad [4.17]$$

More directly the *point posterior distribution* of $\boldsymbol{X}$ reads $\hat{f}_{\boldsymbol{X}}(\boldsymbol{x}) = f_{\boldsymbol{X}}\left(\boldsymbol{x}, \hat{\boldsymbol{\theta}}\right)$, where $\hat{\boldsymbol{\theta}}$ is a characteristic value of the posterior distribution $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, *e.g.* the mean or median value.

4.5.2 Bayesian updating of the model basic variables

As observed from Eq.[4.2], each measurement data may be modelled by a random variable whose conditional distribution with respect to the vector of input variables reads:

$$f_{\boldsymbol{Y}_{obs}^{(q)}|\boldsymbol{X}}\left(\boldsymbol{y}\,;\,\boldsymbol{x},t^{(q)}\right) = \varphi_M\left(\boldsymbol{y} - \mathcal{M}\left(\boldsymbol{x},t^{(q)}\right);\boldsymbol{C}\right)$$
$$\equiv (2\pi)^{-M/2}(\det \boldsymbol{C})^{-1/2}\exp\left[-\frac{1}{2}\left(\boldsymbol{y} - \mathcal{M}\left(\boldsymbol{x},t^{(q)}\right)\right)^{\mathsf{T}}\cdot \boldsymbol{C}^{-1}\cdot\left(\boldsymbol{y} - \mathcal{M}\left(\boldsymbol{x},t^{(q)}\right)\right)\right] \qquad [4.18]$$

Let us denote by $p_{\boldsymbol{X}}(\boldsymbol{x})$ the *prior* distribution of the input random vector $\boldsymbol{X}$, *i.e.* the one used in reliability analysis *before* introducing measurement data. Using Bayes's theorem one can evaluate the *posterior* distribution denoted by $f_{\boldsymbol{X}}(\boldsymbol{x})$ through the likelihood of the measurement data gathered in $\mathcal{Y}$:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{c}\,p_{\boldsymbol{X}}(\boldsymbol{x})\,L(\boldsymbol{x}\,;\mathcal{Y}) = \frac{1}{c}\,p_{\boldsymbol{X}}(\boldsymbol{x})\prod_{q=1}^{Q}\varphi_M\left(\boldsymbol{y}_{obs}^{(q)} - \mathcal{M}\left(\boldsymbol{x},t^{(q)}\right);\boldsymbol{C}\right) \qquad [4.19]$$

The normalizing constant c in the above equation ensures that $f_{\boldsymbol{X}}(\boldsymbol{x})$ is a distribution (of integral 1). Its computation may be carried out using simulation methods (such as Monte Carlo simulation, Latin Hypercube Sampling, etc.) or numerical integration (*e.g.* Gauss quadrature method). However this is a rather complex computational task.

Another approach consists in sampling according to this posterior distribution by using a method that does *not* require the computation of the normalizing constant c. This is one feature of the so-called Markov Chain Monte Carlo simulation methods presented in the previous chapter.

Various algorithms such as the Gibbs sampler or the Metropolis-Hastings algorithm [HAS70] are available, see a review in [NTZ09]. The latter is a acceptation/rejection algorithm that works as follows. Suppose a random vector $\boldsymbol{X}$ of prescribed PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$ is to be sampled, and suppose the PDF has a complex expression that may be evaluated for any value $\boldsymbol{x}$ up to a constant. A Markov chain is initiated (value $\boldsymbol{x}^{(0)}$). At the current state of the chain $\boldsymbol{x}^{(k)}$ at iteration *k*, the next point $\boldsymbol{x}^{(k+1)}$ is evaluated as follows:

$$\boldsymbol{x}^{(k+1)} = \begin{cases} \tilde{\boldsymbol{x}} = q(\boldsymbol{x}|\boldsymbol{x}^{(k)}) & \text{with probability } \alpha(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}}), \\ \boldsymbol{x}^{(k)} & \text{otherwise.} \end{cases} \qquad [4.20]$$

In this equation $q(\boldsymbol{x}|\boldsymbol{x}^{(k)})$ is the *transition* (or *proposal*) distribution that is selected by the analyst and $\alpha(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}})$ is the *acceptance probability*. A common transition is obtained by generating the candidate $\tilde{\boldsymbol{x}}$ by adding to each component a random disturbance to $\boldsymbol{x}^{(k)}$ according to a prescribed (*e.g.* zero-mean Gaussian) distribution.

$$\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i^{(k)} + \zeta_i^{(k)} \; ; \; \zeta^{(k)} \sim \mathcal{N}(0,\sigma^2) \qquad [4.21]$$

This is the so-called *random walk algorithm*. In this case the acceptance probability reduced to:

$$\alpha(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}}) = \min\left\{1, \frac{f_X(\tilde{\boldsymbol{x}})}{f_X(\boldsymbol{x}^{(k)})}\right\} \qquad [4.22]$$

In order to decide if the candidate $\tilde{\boldsymbol{x}}$ is retained with the acceptance probability $\alpha(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}})$, a random number $u^{(k)}$ is uniformly sampled between 0 and 1. The candidate is accepted if $u^{(k)} < \alpha(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}})$ and rejected otherwise. Thus a sequence of points is simulated which is proven to asymptotically behave as realizations of the random vector $\boldsymbol{X}$. One must check if the Markov chain has attained its stationary state, *i.e.* that a sufficiently large number of points has been simulated. Various heuristic control methods have been proposed in the literature, see for instance a review in [ElA 06].

The Metropolis-Hastings algorithm may also be used in a cascade version in which the candidate point is first accepted or rejected with respect to the ratio of prior distributions, then with respect to the likelihood ratio. This algorithm proposed by Tarantola [TAR 05] for this purpose is now described.

[Initialization] $k=0$: The Markov chain is initialized by $\boldsymbol{x}^{(0)}$ that may be randomly selected or deterministic (*i.e.* the vector mean value).

While $k \leq N_{MCMC}$ (N_{MCMC} is the size of the MCMC sample set)

1. Generate a random increment $\zeta^{(k)} \sim N(0,\sigma^2)$ and a candidate $\tilde{\boldsymbol{x}} = \boldsymbol{x}^{(k)} + \zeta^{(k)}$.

2. Evaluate the prior acceptance probability: $\alpha_P(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}}) = \min\left\{1, \dfrac{f_X(\tilde{\boldsymbol{x}})}{f_X(\boldsymbol{x}^{(k)})}\right\}$

3. Randomly generate $u_P \sim \mathcal{U}[0,1]$. If $u_P < \alpha_p(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}})$ then $\tilde{\boldsymbol{x}}$ is accepted (*Go to 4.*) otherwise it is rejected (*Go back to 1.*)

4. Evaluate the likelihood acceptance probability $\alpha_L(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}}) = \min\left\{1, \dfrac{L(\tilde{\boldsymbol{x}}; \mathcal{Y})}{L(\boldsymbol{x}^{(k)}; \mathcal{Y})}\right\}$, where the likelihood function L has been defined in Eq. [4.19]. This step requires a run of the deterministic model $\mathcal{M}$.

5. Randomly generate $u_L \sim \mathcal{U}[0,1]$. If $u_L < \alpha_L(\boldsymbol{x}^{(k)}, \tilde{\boldsymbol{x}})$ then $\tilde{\boldsymbol{x}}$ is accepted: $\boldsymbol{x}^{(k+1)} \leftarrow \tilde{\boldsymbol{x}}$ and $k \leftarrow k+1$. Otherwise $\tilde{\boldsymbol{x}}$ is rejected.

Coming back to the initial problem of updating the predictions of a model by using observation data, the MCMC algorithm is applied in cascade to the *posterior* distribution of the random vector $\boldsymbol{X}$, as defined in Eq.[4.19]. The sample set of points that is obtained, say $\mathcal{X}' = \left\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(N_{MCMC})}\right\}$, is then used as input of a Monte Carlo simulation of model $\mathcal{M}$. In practice the evaluations of $\mathcal{M}$ onto the sample set $\mathcal{X}'$ have already being carried out during the process of generating $\mathcal{X}'$. Computing the updated confidence intervals of the model prediction reduces to estimating the related empirical quantiles on the already available response sample set $\left\{\mathcal{M}\left(\boldsymbol{x}^{(1)}\right), \cdots, \mathcal{M}\left(\boldsymbol{x}^{(N_{MCMC})}\right)\right\}$.

In conclusion the Bayesian approach based on Markov Chain Monte Carlo simulation allows one to update the distribution of the input random vector by incorporating the observations made on the system response. From this updated (*i.e.* posterior) distribution updated confidence intervals may be computed that compare with those obtained by the "updated inverse FORM" algorithm. Both approaches are now benchmarked on an industrial example.

4.6. Updating the prediction of creep strains in containment vessels of nuclear power plants

4.6.1 Industrial problem statement

The containment vessel of a nuclear power plant contains the reactor pressure vessel and the components of the primary circuit, namely pumps, steam generators and pipes. The leak tightness of this vessel shall be guaranteed in case of an hypothetical accident such as LOCA (loss of coolant accident) that could happen when a pipe is ruptured, thus generating a rapid pressure increase within the vessel while possible releasing radioactive products from the primary circuit.

The containment vessels of French PWR (pressurized water reactors) are made of one or two walls made of reinforced and pre-stressed concrete. The so-called *concrete creep* phenomenon, which corresponds to delayed strains in concrete due to ageing leads to the decrease of the tension of the pre-stressing cables in time. In order to assess the safety of the containment vessel all along the life time of the plant in the context of hypothetical LOCA accidents, it is necessary to predict accurately the evolution in time of the delayed stresses and associated loss of cable prestress.

However the creep phenomenon is very complex in nature. Its physical origins are fully understood, especially when its kinetics on long-term time scales is concerned. In order to bypass the lack of detailed modelling, a detailed monitoring of the containment vessels has been installed. Thus measurements of the delayed strains in standard conditions are carried out on a regular basis. The Bayesian framework that has been presented in the previous sections is well adapted to exploit this experimental feedback together with physical models of creep.

4.6.2 Deterministic models

Let us consider a cylindrical portion of the containment vessel that is sufficiently far away from local geometrical details (reinforcements, material hatch, etc.) so that it is relevant to consider that the concrete stress tensor under cables pre-stress is bi-axial (the pre-stress cables are vertical and circumferential in this zone). The mechanical model used in the sequel for delayed stresses is defined in the French standard BAEL [BAE 99] although it takes into account specific modifications as investigated by Granger for containment walls [GRA95]. Accordingly the total strain tensor may be decomposed into five contributions:

The total strain tensor $\boldsymbol{\varepsilon}$ can be decomposed into the elastic, creep and shrinkage components:

$$\boldsymbol{\varepsilon}(t,t_d,t_l) = \boldsymbol{\varepsilon}^{el}(t) + \boldsymbol{\varepsilon}^{as}(t,t_d) + \boldsymbol{\varepsilon}^{ds}(t,t_d) + \boldsymbol{\varepsilon}^{bc}(t,t_l) + \boldsymbol{\varepsilon}^{dc}(t,t_d,t_l) \qquad [4.23]$$

where

- t is the time spent starting from the concrete casting, t_d (resp. t_l) denotes the time when drying starts (resp. the time of loading, *i.e.* cable tensioning in the present case);
- $\boldsymbol{\varepsilon}^{el}(t)$ is the elastic strain;
- $\boldsymbol{\varepsilon}^{as}(t,t_d)$ is the autogeneous shrinkage, corresponding to the shrinkage of concrete when insulated from humidity changes;
- $\boldsymbol{\varepsilon}^{ds}(t,t_d)$ is the drying shrinkage ;
- $\boldsymbol{\varepsilon}^{bc}(t,t_l)$ is the basic creep corresponding to the creep of concrete when insulated from humidity changes;
- $\boldsymbol{\varepsilon}^{dc}(t,t_d,t_l)$ is the drying creep.

The following models are used for each component. The elastic strains are related to the stress tensor $\boldsymbol{\sigma}$ by Hooke's law:

$$\boldsymbol{\varepsilon}^{el} = \frac{1+\nu^{el}}{E_i}\boldsymbol{\sigma} - \frac{\nu^{el}}{E_i}(\mathrm{tr}\,\boldsymbol{\sigma})\,\boldsymbol{1} \qquad [4.24]$$

where E_i is the elastic Young's modulus (measured at $t = t_l$) and ν^{el} is the Poisson's ratio. The autogeneous and drying shrinkage are modelled by (time unit is one day in the sequel):

$$\boldsymbol{\varepsilon}^{as}(t,t_d) = \boldsymbol{\varepsilon}^{as}_{\infty}\frac{t-t_d}{50+t-t_d}\boldsymbol{1} \qquad \boldsymbol{\varepsilon}^{ds}(t,t_d) = \boldsymbol{\varepsilon}^{ds}_{\infty}\frac{100-RH}{50}\frac{t-t_d}{45R_m^2/4+t-t_d}\boldsymbol{1} \qquad [4.25]$$

In these equations $\boldsymbol{\varepsilon}^{as}_{\infty}$ (resp. $\boldsymbol{\varepsilon}^{ds}_{\infty}$) is the asymptotic autogeneous shrinkage (resp. the asymptotic drying shrinkage), RH is the relative humidity in %, R_m is the drying radius (half of the containment wall thickness, in cm) and ***1*** is the unit strain tensor, meaning that these strains are isotropic. The basic creep is modelled by:

$$\boldsymbol{\varepsilon}^{bc}(t,t_1) = 3500\left(\frac{1+\nu^c}{E_i}\boldsymbol{\sigma} - \frac{\nu^c}{E_i}(tr\boldsymbol{\sigma})\mathbf{1}\right)\left(\frac{2{,}04}{0{,}1+(t_i-t_d)^{0{,}2}}\right)\left(\frac{\sqrt{t-t_1}}{22{,}4+\sqrt{t-t_1}}\right) \qquad [4.26]$$

where ν^c is the creep Poisson's ratio. The drying creep is modelled by :

$$\boldsymbol{\varepsilon}^{dc}(t,t_d,t_l) = 3200\frac{\text{tr}\,\boldsymbol{\sigma}/2}{E_i}\left(\boldsymbol{\varepsilon}^{ds}(t,t_d)-\boldsymbol{\varepsilon}^{ds}(t_l,t_d)\right)\mathbf{1} \qquad [4.27]$$

In a prestressed concrete containment vessel, the stress tensor may be regarded as bi-axial in the current zone, i.e. having a vertical component $\sigma_{zz}^0 = 9{,}3$ MPa and an orthoradial component $\sigma_{\theta\theta}^0 = 13{,}3$ MPa. The drying radius, which is equal to half of the wall thickness, is 0.6 m. The cable tensioning is supposed to occur two years after the casting of concrete $(t_l - t_d = 2 \text{ ans})$.

Due to the presence of reinforcing bars and prestressed cables, the above equations for creep and shrinkage (initially obtained for unreinforced concrete) are corrected by a multiplicative factor λ = 0.82 obtained from the design code and experimental results[GRA 95]. The other parameters are modelled by independent random variables, whose parameters are gathered in Table 4.1.

Parameter	**Notation**	**Distribution**	**Mean value**	**Coef. of variation**
Concrete Young's modulus	E_i	Lognormal	33 700 MPa	7.4 %
Poisson's ratio	ν^{el}	Truncated normal [0;0,5]	0.2	50 %
Creep Poisson's ratio	ν^c	Truncated normal [0;0,5]	0.2	50 %
Relative humidity	RH	Truncated normal [0;100%]	40 %	20 %
Max. autogeneous shrinkage strain	ε_∞^{as}	Lognormal	90×10^{-6}	10 %
Max. drying shrinkage strain	ε_∞^{ds}	Lognormal	526×10^{-6}	10 %

Table 4.1 *Concrete creep model – probabilistic description of the parameters*

A fictitious containment vessel is considered for which it is supposed that experimental measurements of the axial strain ε_{zz} are available. Measurements are supposed to have been carried out approximately every 150 days from 1,500 and 2,500 days after the concrete structure has been loaded. They are reported in Table 4.2.

Measurement	Date (days)	Value (10^{-6})
#1	1152	497
#2	1303	523
#3	1451	590
#4	1601	652
#5	1750	685
#6	1900	756
#7	2054	777
#8	2201	822
#9	2153	858
#10	2501	925

Table 4.2 *Concrete creep model – fictitious strain measures*

The measurement/model error is supposed to be normally distributed with zero mean value and standard deviation 15.10^{-6}. The various errors at different time instants are supposed independent.

4.6.3 Prior and posterior estimations of the delayed strains

All the simulation results have been obtained using the probabilistic model reported in Table 4.1. The *prior* 95% confidence interval on the vertical strain ε_{zz} is computed by the "inverse FORM" approach. Results are gathered in Figure 4.1, in which the measurement values from Table 4.2 have been plotted as well. These results have been validated by brute force Monte Carlo simulation [PER08].

It may be observed that using the prior estimation of the parameters' distribution (Table 4.1) leads to a large underestimation of the vertical delayed strains of circa 40%. This may be explained by the fact that the values of the prior model parameters have been taken from a codified building code (BAEL) and thus are not well adapted to the specific concrete used for containment vessels.

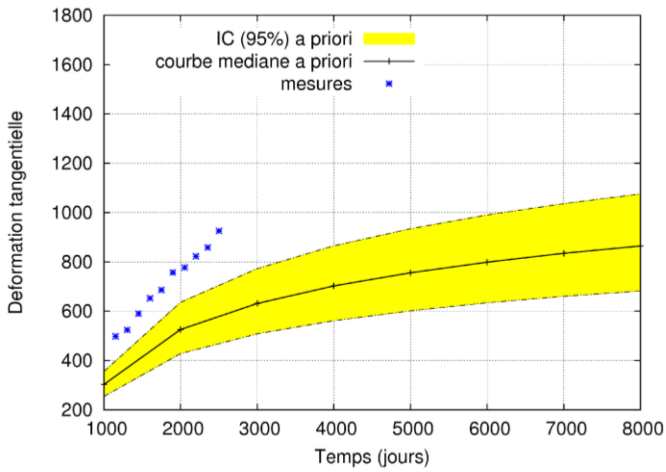


Figure 4.1 *Prior predictions of the vertical total strain* ε_{zz} *and fictitious experimental results*

The « updated inverse FORM » approach is then applied using the measurement values in Table 4.2. Results are plotted in Figure 4.2. It can be observed that the *posterior* 95% confidence interval now covers the experimental data and that it is much smaller than the prior interval. The Bayesian framework has allowed one to reconcile the experimental data with the model and to reduce the uncertainty on the prediction of long-term behaviour of the structure. It has been shown in [SUD06] that the posterior result is not much sensitive to the number of data used for the updating process since the time variation of creep is rather slow and smooth.

The MCMC approach for updating the distributions of the input parameters has been applied as well. The results are plotted in Figure 4.3 and corroborate those obtained by the "inverse FORM" approach, the maximal discrepancy between the updated quantiles obtained by each approach being less than 4%. The obtained updated confidence interval is slightly tighter than that obtained by inverse FORM (Figure 4.2) and 4 times smaller than the prior interval.

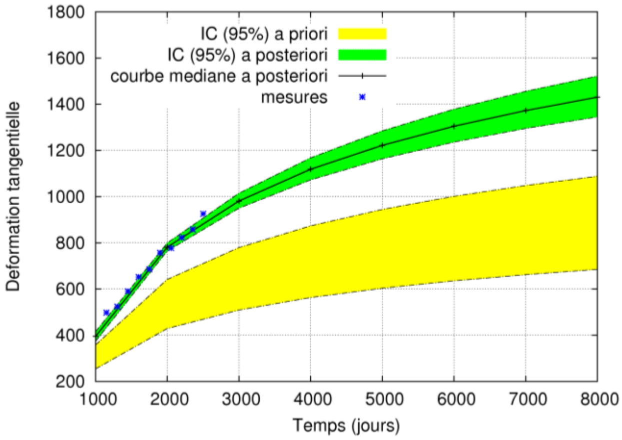


Figure 4.2 *Prior/posterior predictions of the vertical total strain* ε_{zz} *obtained by the « updated inverse FORM » approach and experimental results*

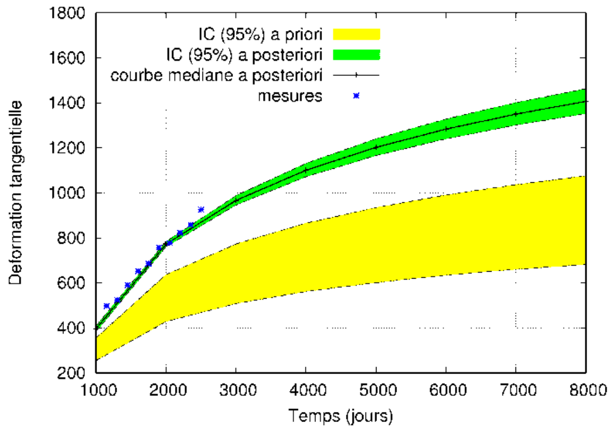
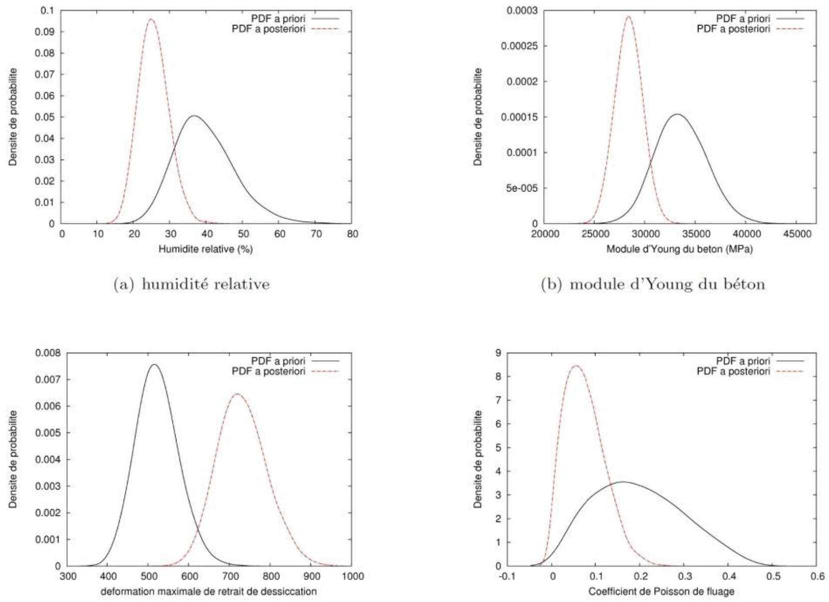


Figure 4.3 *Prior/posterior predictions of the vertical total strain* ε_{zz} *obtained by MCMC and experimental results*

As indicated in Section 4.5 the MCMC method yields the posterior distribution of the various input random variables of the problem. Some of these distributions are plotted in Figure 4.4. One can observe that all the posterior distributions are less scattered than the corresponding priors: adding information within the computational model has reduced the uncertainty.



(a) humidité relative

(b) module d'Young du béton

(c) déformation maximale de retrait de dessiccation

(d) coefficient de Poisson de fluage

Figure 4.4. *Prior and posterior distributions of selected random variables*

4.7. Conclusion

Structural reliability methods are usually used when designing a complex structure so as to guarantee that the failure probability associated with various criteria is sufficiently low. For exceptional civil engineering structures such as cable-stayed bridges or nuclear concrete containment vessels, monitoring is usually established from the very construction of the system. Thus a large amount of data is collected all along the life time of the system. In this chapter it has been shown that this data may be used in order to refine the long-term evolution of the structure.

The Bayesian updating techniques presented in this chapter allow the analyst to address efficiently this question. The "inverse FORM" approach only updates the quantiles of the model response, without yielding any additional information on the model input variables. In contrast the Markov chain Monte Carlo simulation allows one to update the distributions of the input variables from a prior to a posterior estimate. The latter posterior distributions may be re-propagated through the mechanical model in order to obtain updated quantiles.

The same methods have been used successfully for predicting the crack propagations in steel structures [SUD 07] and the delayed strains of a concrete containment vessel by using a detailed (finite element) model for creep and shrinkage [BER11]. In the latter case the computational cost of a single run of the model is rather large. Thus a surrogate model of the finite element model has been built first, namely a *polynomial chaos expansion* (see chapter 8 for details). This surrogate which is essentially a polynomial function of the input variables may then be used straightforwardly within the MCMC algorithm.

As a conclusion it shall not be forgotten that the Bayesian framework, although it is an elegant approach for integrating experimental feedback into computational models, cannot replace completely a proper physical modelling: in particular the physical model $\mathcal{M}$ shall at least describe the general trend of the time evolution of the structural behaviour in order to obtain relevant results.

ACKNOWLEDGMENTS. — The application of Bayesian updating methods to the monitoring of civil engineering structures is one of the research topic of the MIRADOR project (*Modélisation Interactive et Recalage par l'Auscultation pour le Développement d'Ouvrages Robustes*) that has been supported by the French National Research Agency (*Agence Nationale de la Recherche)* under Grant ANR-06-RGCU-008. This support is gratefully acknowledged.

4.8. Bibliography

[BAE 99] BAEL, Règles BAEL 91 modifiées 99 (Règles techniques de conception et de calcul des ouvrages et constructions en béton armé suivant la méthode des états-limites), Eyrolles, France, 2000.

[BER 11] BERVEILLER M., LE PAPE Y., SUDRET B., PERRIN F., « Updating the long-term creep strains in concrete containment vessels by using MCMC simulation and polynomial chaos expansions », *Struc. Infrastruc. Eng.*, vol. 7, 2011.

[DER 94] DER KIUREGHIAN A., ZHANG Y., LI C., « Inverse Reliability Problem », J. Eng. Mech., vol. 120, p. 1154-1159, 1994.

[DIT 96] DITLEVSEN O., MADSEN H., Structural reliability methods, J. Wiley and Sons, Chichester, 1996.

[ElA 06] EL ADLOUNI S., FAVRE A., BOBEE B., "Comparison of methodologies to assess the convergence of Markov chain Monte Carlo methods", *Comput. Stat. Data Anal.* vol. 50 (10), p. 2685-2701, 2006.

[GRA 95] GRANGER L., Comportement différé du béton dans les enceintes de centrales nucléaires, PhD thesis, Ecole Nationale des Ponts et Chaussées, 1995.

[HAS 70] HASTINGS W., « Monte Carlo sampling methods using Markov chains and their application », *Biometrika*, vol. 57, n°1, p. 97-109, 1970.

[LEM09] LEMAIRE M., Structural reliability, ISTE/Wiley, 2009.

[MAD87] MADSEN H., « Model updating in reliability theory », LIND N., Ed., Proc. 5th Int. Conf. on Applications of Stat. and Prob. In Civil Engineering (ICASP5), Vancouver, Canada, vol. 1, p. 564-577, 1987.

[NTZ09] NTZOUFRAS I. *Bayesian modeling using Winbugs,* John Wiley & Sons, 2009.

[OHA04] O'HAGAN A., FORSTER J., *Kendall's advanced theory of statistics*, Vol. 2B Bayesian inference, Arnold, 2004.

[PER07] PERRIN F., SUDRET B., PENDOLA M., DE ROCQUIGNY E., « Comparison of Markov chain Monte Carlo simulation and a FORM-based approach for Bayesian updating of mechanical models », Proc. 10th Int. Conf. on Appl. of Stat. and Prob. In Civil Engineering (ICASP10), Tokyo, Japan, 2007.

[PER08] PERRIN F., Intégration des données expérimentales dans les modèles probabilistes de prévision de la durée de vie des structures, Thèse de doctorat, Université Blaise Pascal, Clermont-Ferrand, 2008.

[ROB92] ROBERT C., *L'analyse statistique bayésienne*, Economica, 1992.

[SUD06] SUDRET B., PERRIN F., BERVEILLER M., PENDOLA M., « Bayesian updating of the long-term creep deformations in concrete containment vessels », Proc. 3rd Int. ASRANet Colloquium, Glasgow, United Kingdom, 2006.

[SUD07] SUDRET B., Uncertainty propagation and sensitivity analysis in mechanical models – Contributions to structural reliability and stochastic spectral methods, Habilitation à diriger des recherches, Université Blaise Pascal, Clermont-Ferrand, France, 2007.

[TAR05] TARANTOLA A., *Inverse problem theory and methods for model parameter estimation*. Society for Industrial and Applied Mathematics (SIAM), 2005.