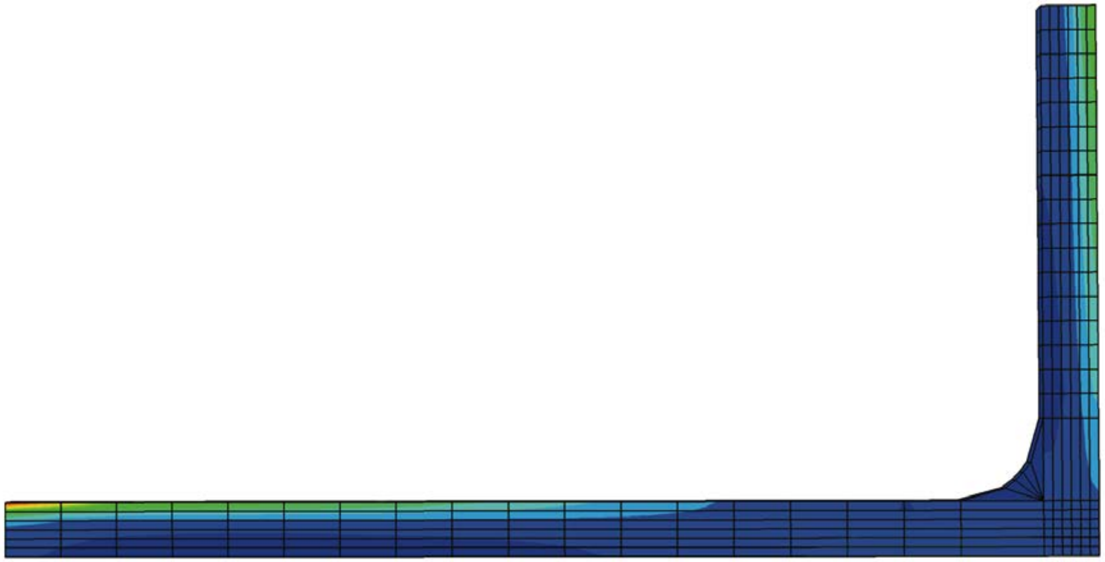
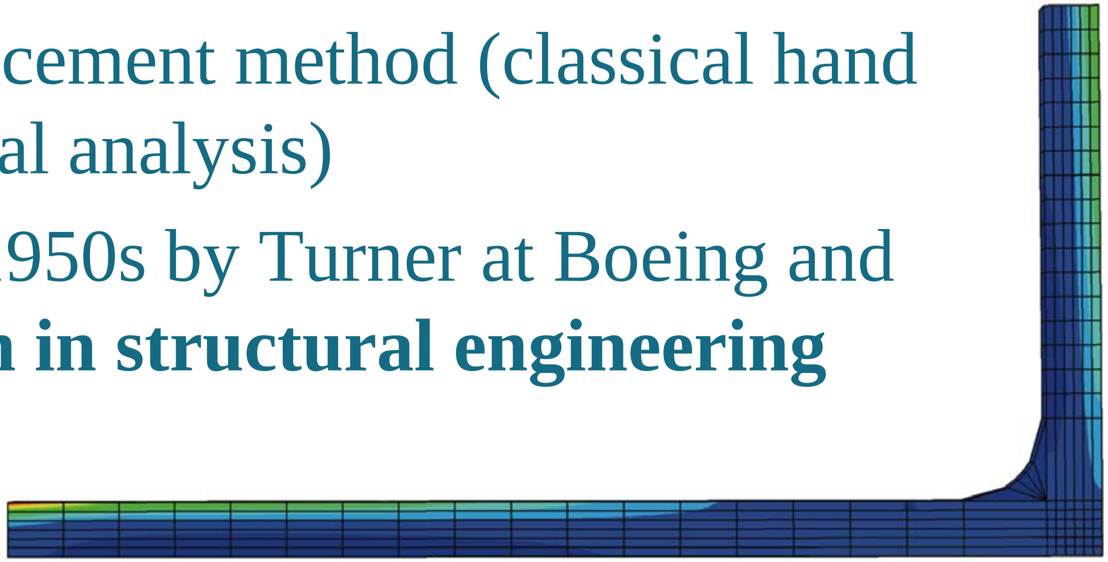
Chapter 2

The Direct Stiffness Method



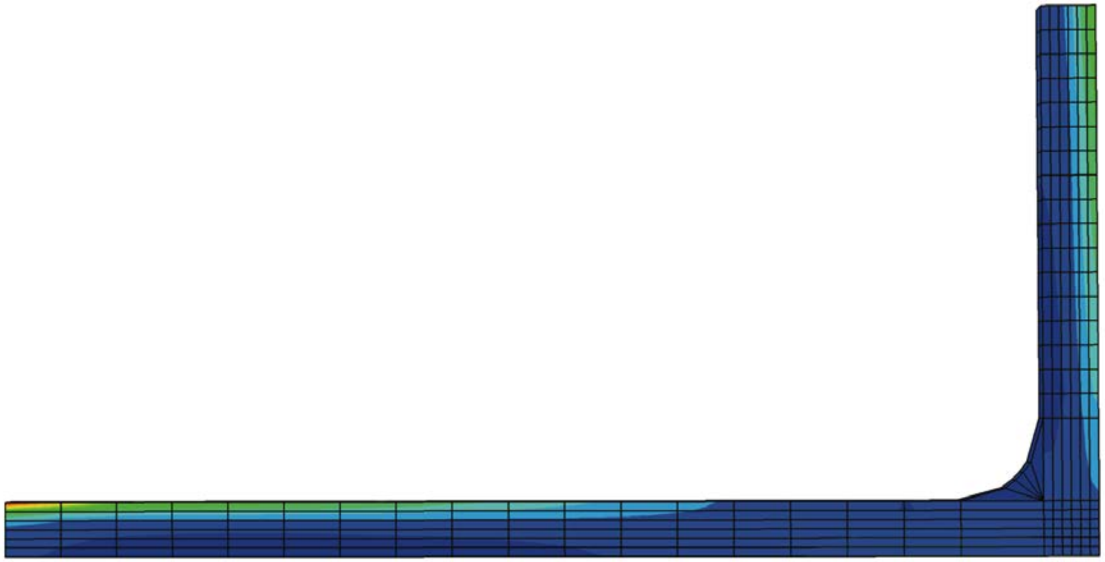
Direct Stiffness Method (DSM)

- **Computational** method for **structural analysis**
- **Matrix method** for computing the member forces and displacements in structures
- DSM implementation is the basis of most commercial and open-source finite element software
- Based on the displacement method (classical hand method for structural analysis)
- Formulated in the 1950s by Turner at Boeing and started a **revolution in structural engineering**

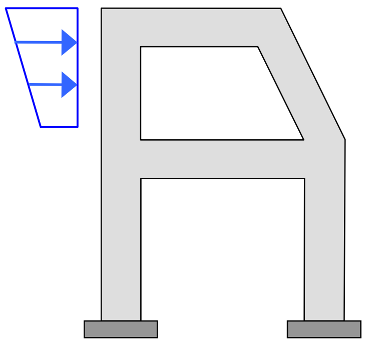


Goals of this Chapter

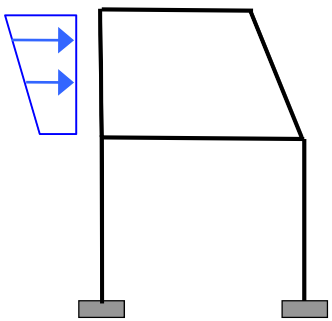
- DSM formulation
- DSM software workflow for …
  - linear static analysis (1st order)
  - 2nd order linear static analysis
  - linear stability analysis



Computational Structural Analysis

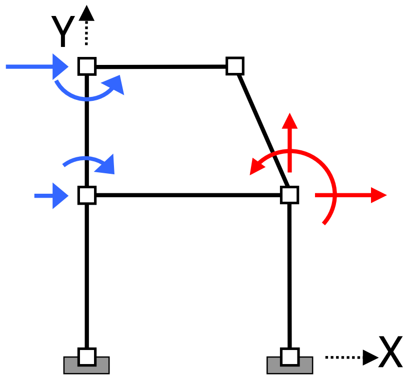


Physical problem



Continuous mathematical model

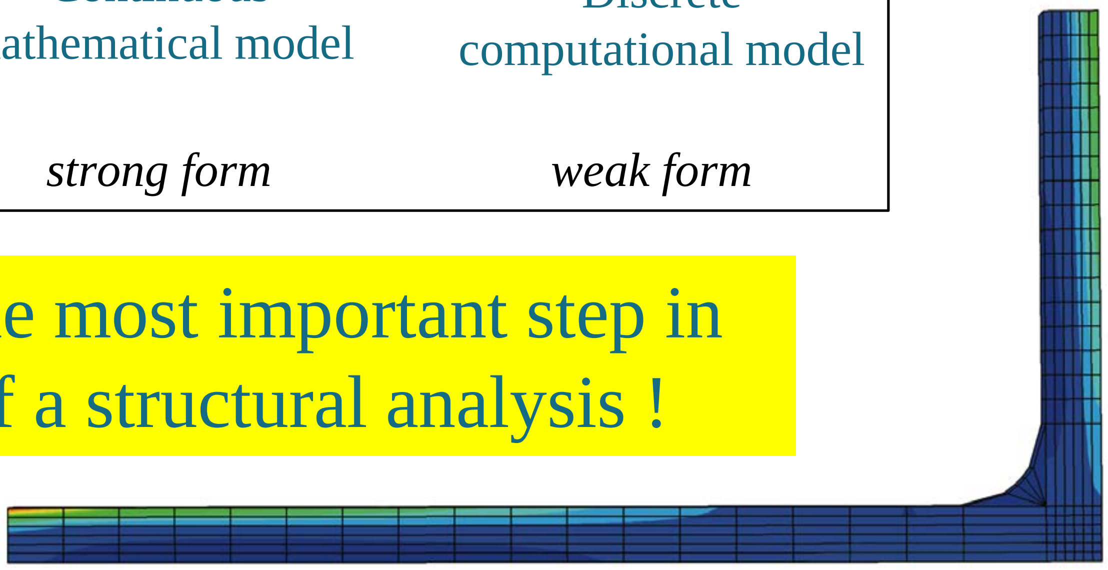
strong form



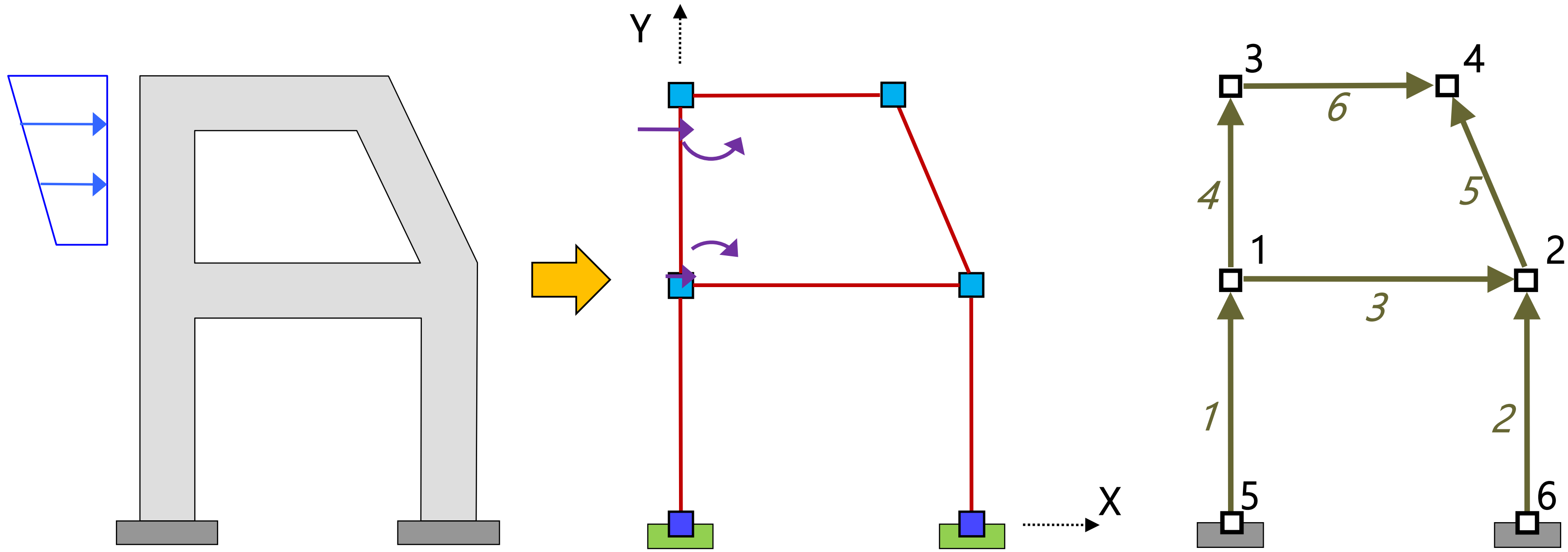
Discrete computational model

weak form

Modelling is the most important step in the process of a structural analysis !



System Identification (Modelling)



Global Coordinate System

Nodes

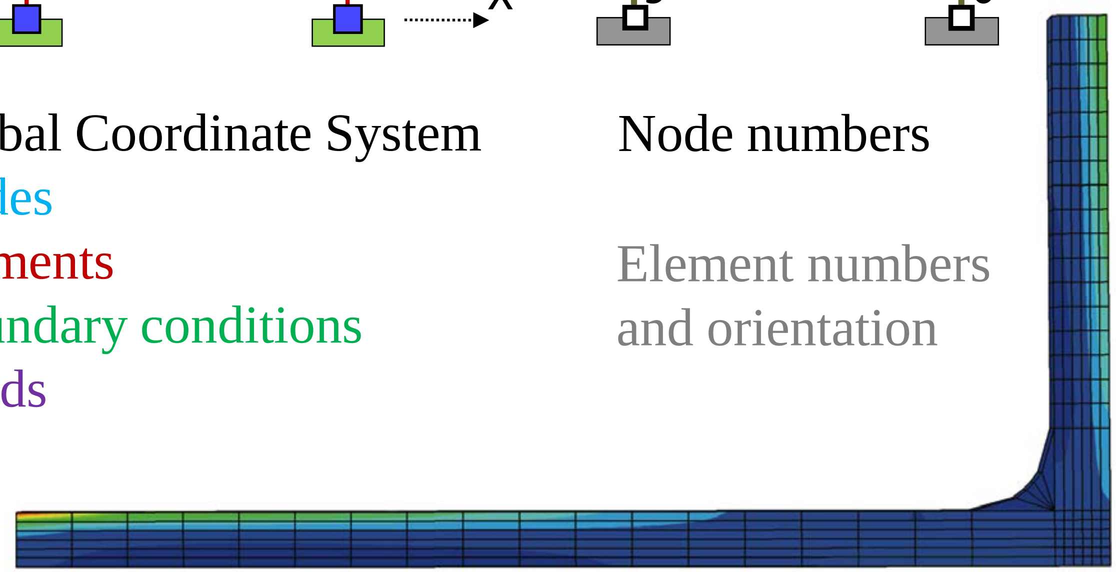
Elements

Boundary conditions

Loads

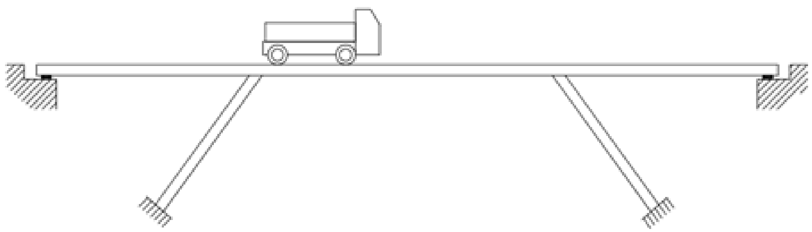
Node numbers

Element numbers and orientation

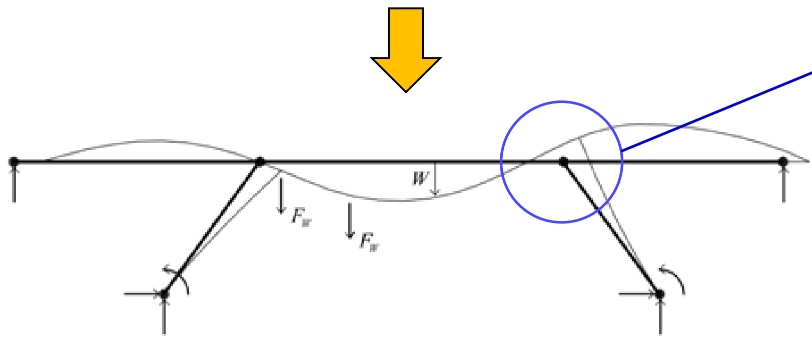


Deformations

System Deformations



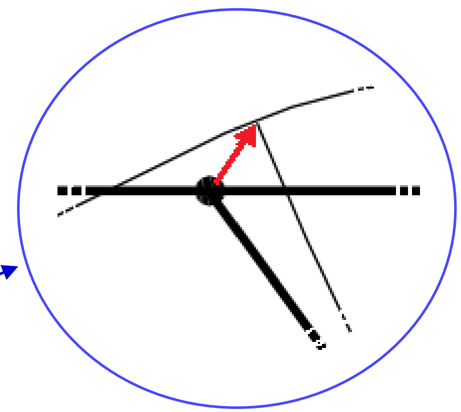
System identification

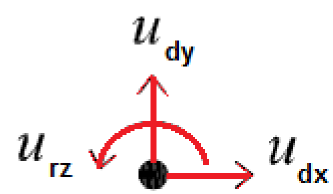


nodes, elements, loads and supports

deformed shape

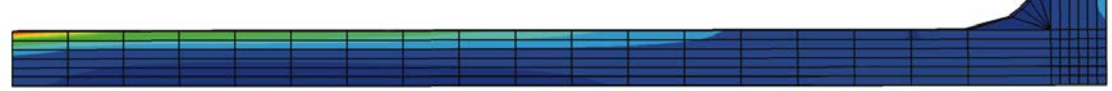
Nodal Displacements





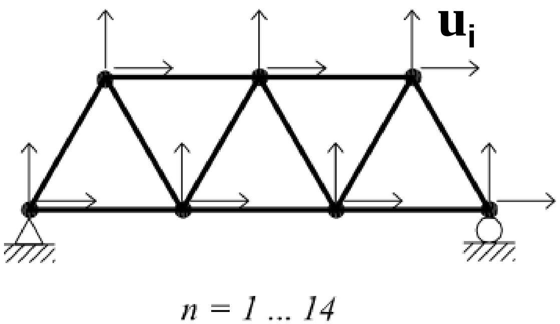
(deformational, nodal)

degrees of freedom = dofs

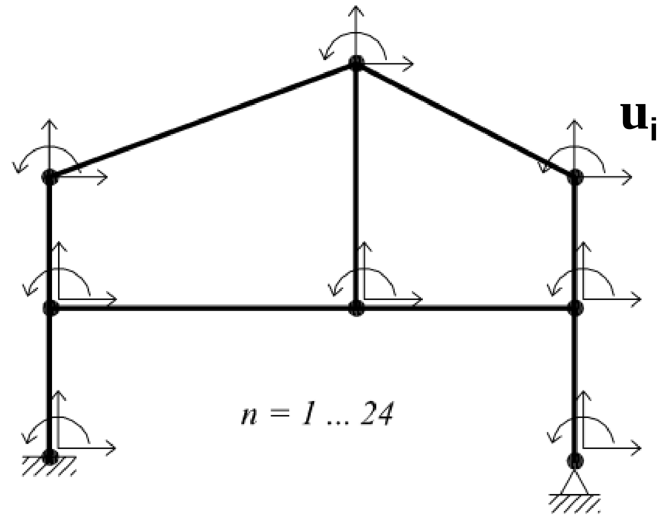


Degrees of Freedom

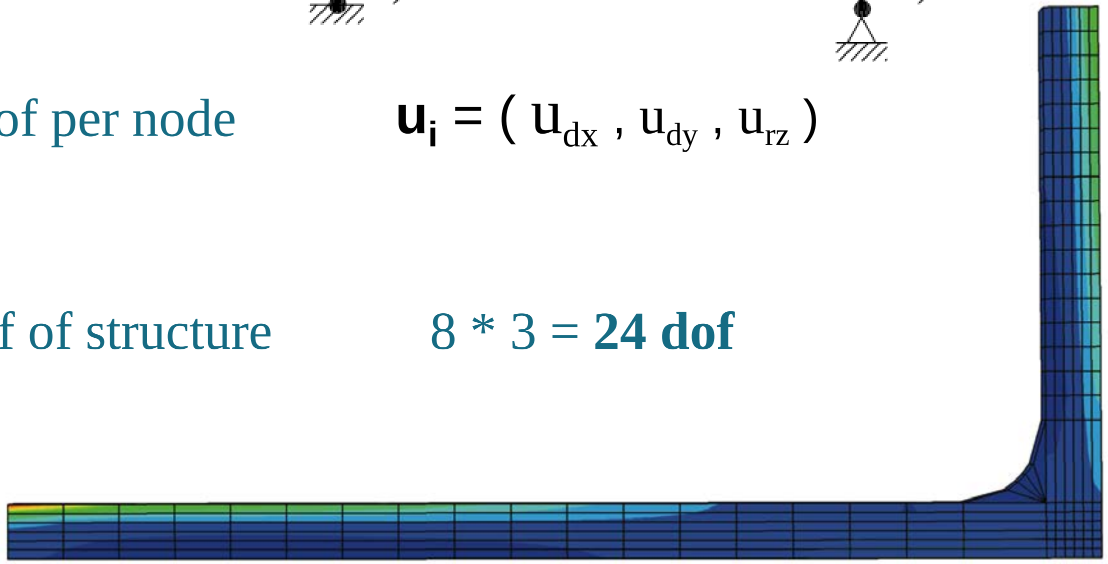
Truss Structure



Frame Structure

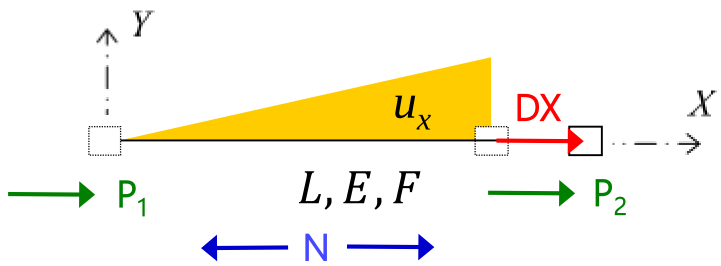


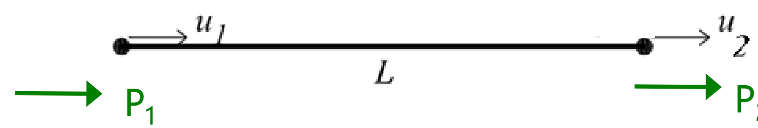
Truss Structure		Frame Structure
$\mathbf{u_i} = (u_{dx} , u_{dy})$	dof per node	$\mathbf{u_i} = (u_{dx} , u_{dy} , u_{rz})$
7 * 2 = **14 dof**	dof of structure	8 * 3 = **24 dof**



Elements: Truss

1 dof per node





X/Y = local coordinate system

u_x = displacement in direction of local axis *X*

DX = displacement of truss end

compatibility $\varepsilon = \dfrac{DX}{L}$

const. equation $\sigma = E\,\varepsilon$

equilibrum $P_2 = -P_1 = N$

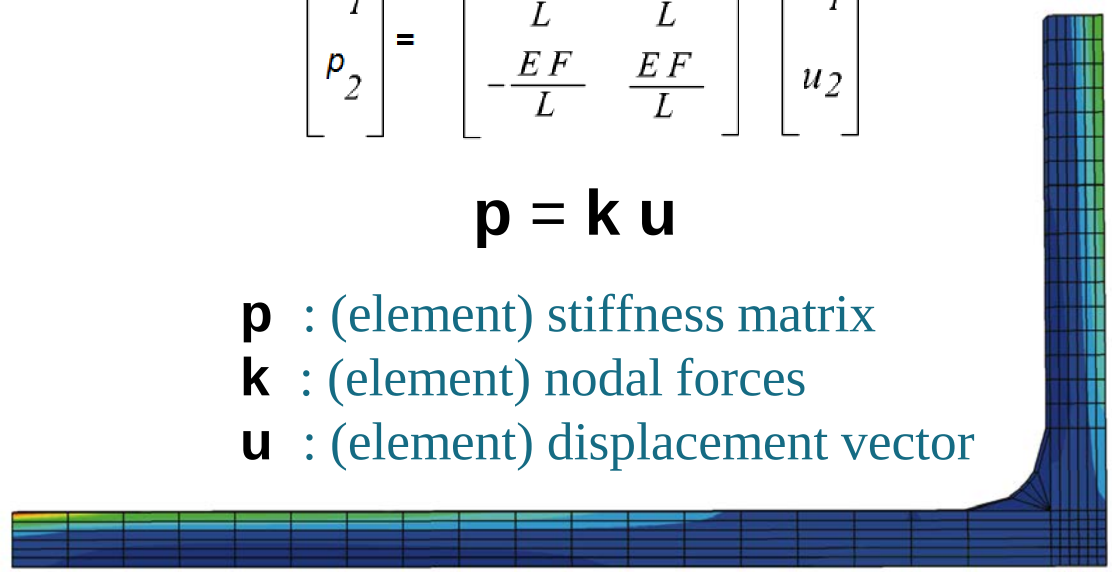
$$N = \int E\,\sigma = EF\,\sigma = \frac{EF}{L}DX$$

$$DX = (u_2 - u_1) \Rightarrow \quad P_1 = \frac{EF}{L}(u_1 - u_2), \quad P_2 = \frac{EF}{L}(-u_1 + u_2)$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{EF}{L} & -\frac{EF}{L} \\ -\frac{EF}{L} & \frac{EF}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

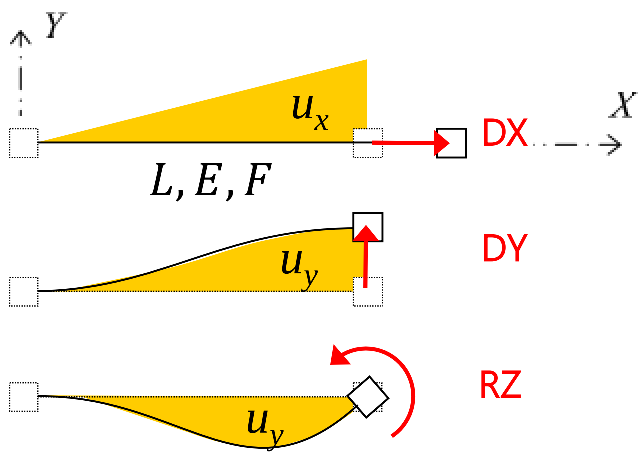
p = k u

p : (element) stiffness matrix
k : (element) nodal forces
u : (element) displacement vector



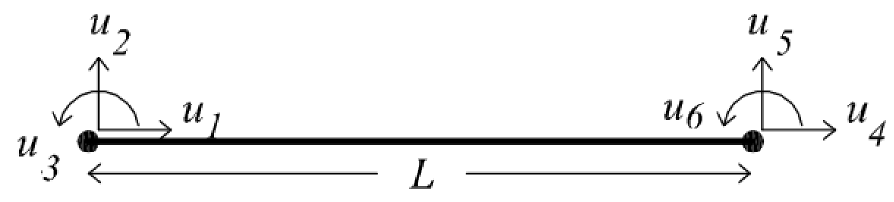
Elements: Beam

3 dof per node



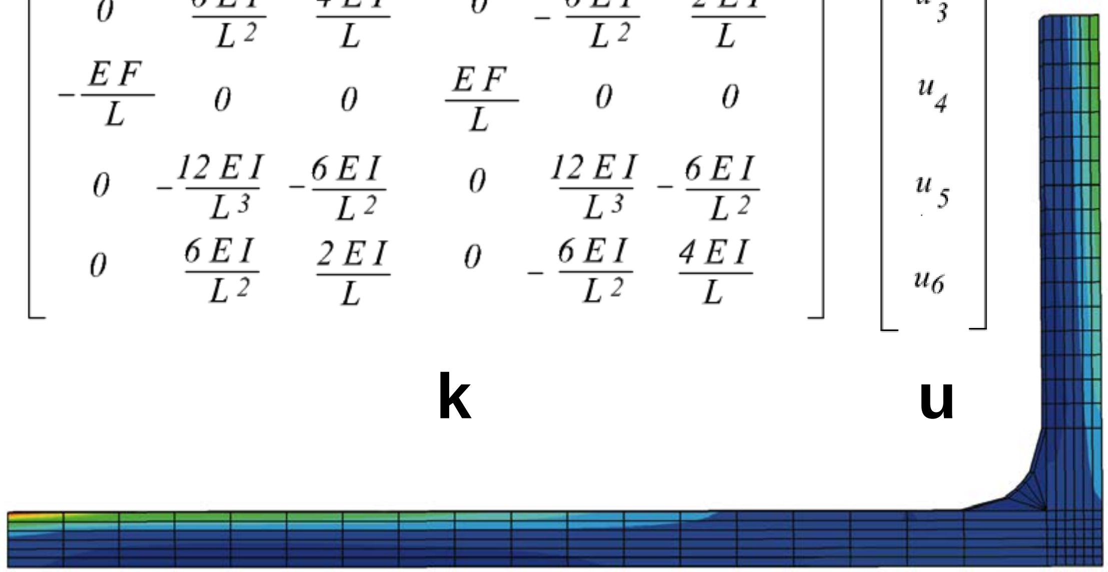
u_x = displacement in direction of local axis X

u_y = displacement in direction of local axis Y

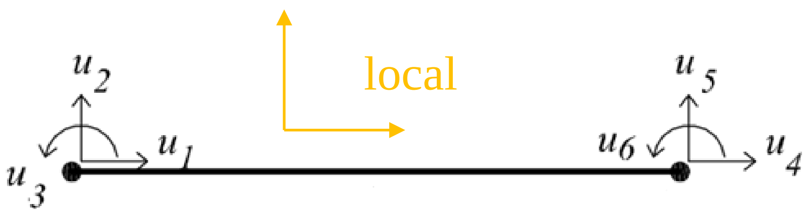


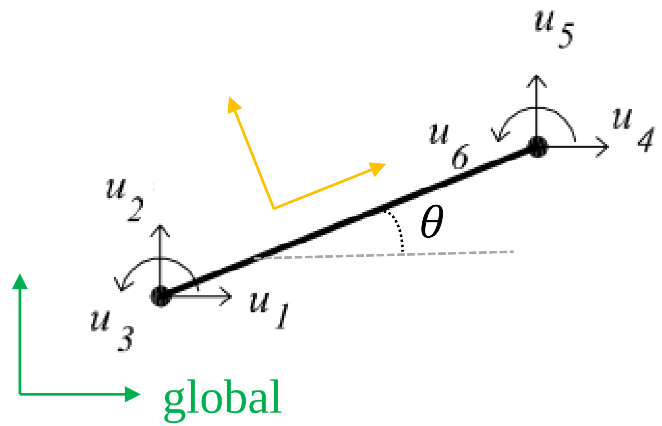
$$
\begin{bmatrix}
\frac{EF}{L} & 0 & 0 & -\frac{EF}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{EF}{L} & 0 & 0 & \frac{EF}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}
$$

k **u**



Elements: Global Orientation

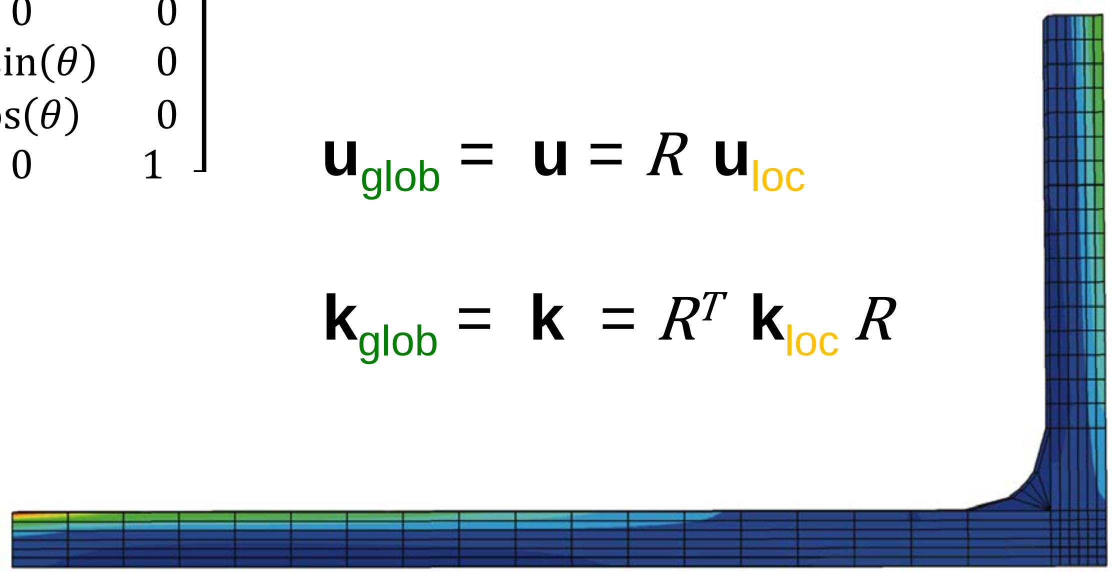




$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

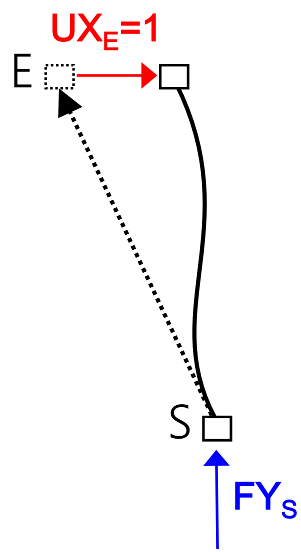
$$\mathbf{u}_{glob} = \mathbf{u} = R\ \mathbf{u}_{loc}$$

$$\mathbf{k}_{glob} = \mathbf{k} = R^T\ \mathbf{k}_{loc}\ R$$



Beam Stiffness Matrix

	UX_S	UY_S	UZ_S	UX_E	UY_E	UZ_E
FX_S =	k_{11}	k_{12}	k_{13}	k_{14}	k_{15}	k_{16}
FY_S =		k_{22}	k_{23}	k_{24}	k_{25}	k_{26}
MZ_S =			k_{33}	k_{34}	k_{35}	k_{36}
FX_S =				k_{44}	k_{45}	k_{46}
FY_S =		symm.			k_{55}	k_{56}
MZ_E =						k_{66}



e.g. k_{24} =

reaction
in global direction Y
at start node S

due to a

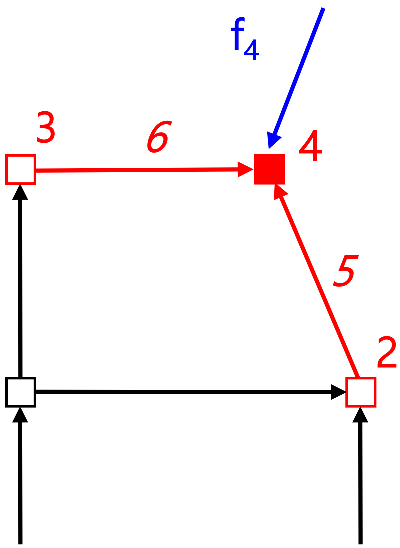
unit displacement
in global direction X
at end node E

$$\begin{Bmatrix} p_{iS} \\ p_{iE} \end{Bmatrix} = \begin{bmatrix} [\mathbf{k}_{iSS}] & [\mathbf{k}_{iSE}] \\ [\mathbf{k}_{iES}] & [\mathbf{k}_{iEE}] \end{bmatrix} \cdot \begin{Bmatrix} u_{iS} \\ u_{iE} \end{Bmatrix}$$

$$\mathbf{p} = \mathbf{k}\,\mathbf{u}$$

Element stiffness matrix
in global orientation

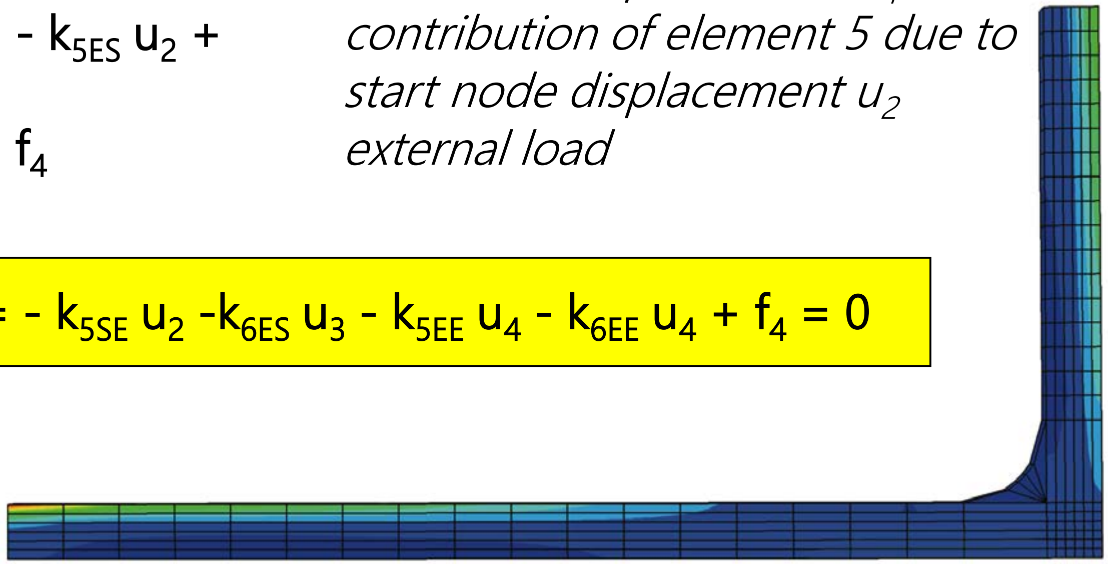
Nodal Equilibrum



r4: Vector of all forces acting at node 4

r4 =	$- k_{6ES}\, u_3 +$	*contribution of element 6 due to start node displacement u_3*
	$- k_{6EE}\, u_4 +$	*contribution of element 6 due to end node displacement u_4*
	$- k_{5EE}\, u_4 +$	*contribution of element 5 due to start node displacement u_4*
	$- k_{5ES}\, u_2 +$	*contribution of element 5 due to start node displacement u_2*
	f_4	*external load*

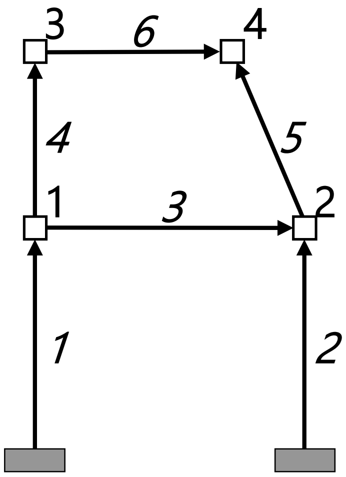
Equilibrum at node 4: $r_4 = - k_{5SE}\, u_2 - k_{6ES}\, u_3 - k_{5EE}\, u_4 - k_{6EE}\, u_4 + f_4 = 0$



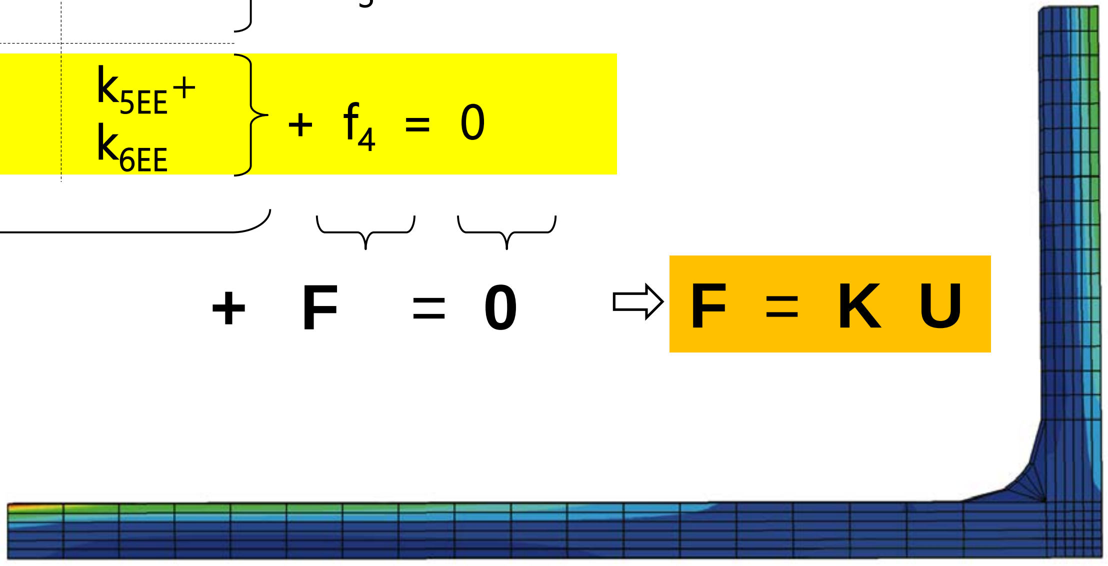
Global System of Equations

$$
\begin{aligned}
r_1 &= -\left\{ \left(k_{1EE}+k_{3SS}+k_{4SS}\right) u_1 + k_{3SE}\, u_2 + k_{4SE}\, u_3 \right\} + f_1 = 0 \\
r_2 &= -\left\{ k_{3ES}\, u_1 + \left(k_{2EE}+k_{3EE}+k_{5SS}\right) u_2 + k_{5SE}\, u_4 \right\} + f_2 = 0 \\
r_3 &= -\left\{ k_{4ES}\, u_1 + \left(k_{4EE}+k_{6SS}\right) u_3 + k_{6SE}\, u_4 \right\} + f_3 = 0 \\
r_4 &= -\left\{ k_{5ES}\, u_2 + k_{6ES}\, u_3 + \left(k_{5EE}+k_{6EE}\right) u_4 \right\} + f_4 = 0
\end{aligned}
$$

	u_1	u_2	u_3	u_4	
$r_1 = -$	$k_{1EE}+$ $k_{3SS}+$ k_{4SS}	k_{3SE}	k_{4SE}		$+ f_1 = 0$
$r_2 = -$	k_{3ES}	$k_{2EE}+$ $k_{3EE}+$ k_{5SS}		k_{5SE}	$+ f_2 = 0$
$r_3 = -$	k_{4ES}		$k_{4EE}+$ k_{6SS}	k_{6SE}	$+ f_3 = 0$
$r_4 = -$		k_{5ES}	k_{6ES}	$k_{5EE}+$ k_{6EE}	$+ f_4 = 0$



$$-\mathbf{K}\ \mathbf{U} + \mathbf{F} = \mathbf{0} \ \Rightarrow\ \mathbf{F} = \mathbf{K}\ \mathbf{U}$$



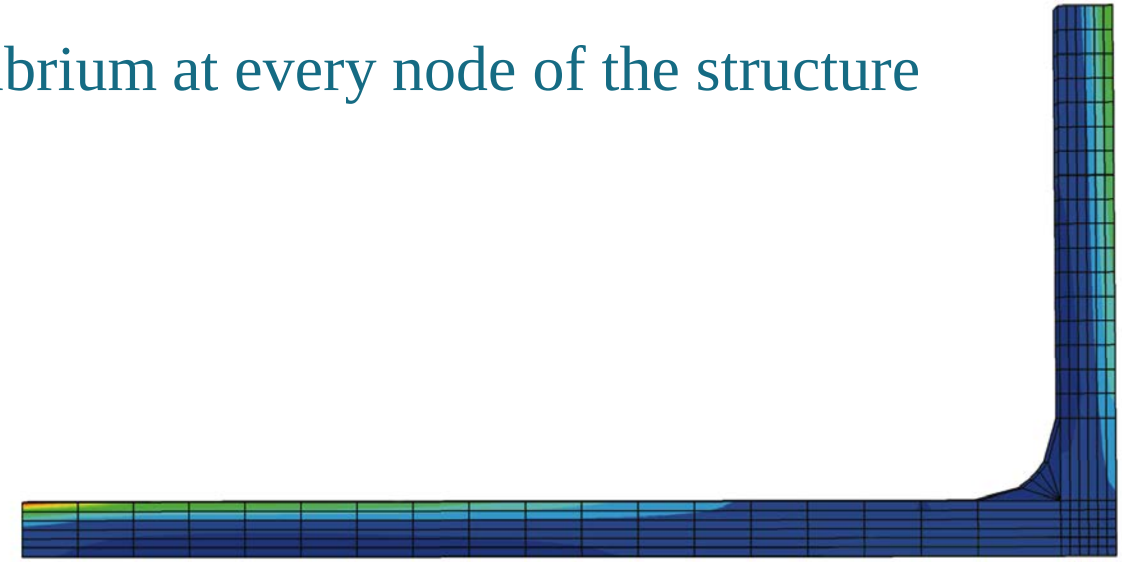
Global System of Equations

F = global load vector = Assembly of all **f**e

K = global stiffness matrix = Assembly of all **k**e

U = global displacement vector = unknown

F = K U = equilibrium at every node of the structure



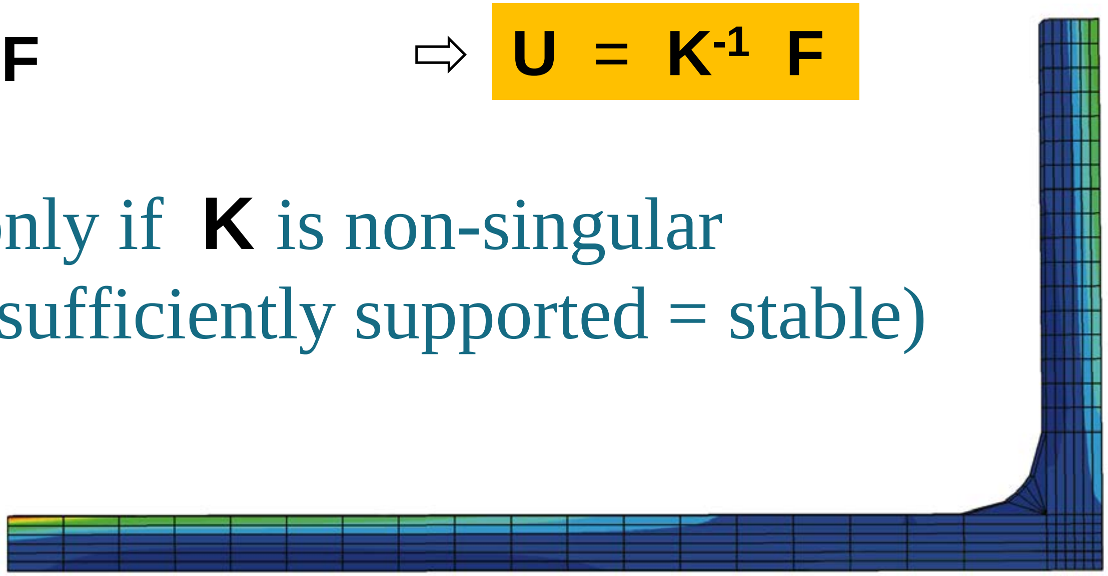
Solving the Equation System

What are the nodal displacements for
a given structure (= stiffness matrix **K**)
due to a given load (= load vector **F**) ?

$$\mathbf{K}\ \mathbf{U} = \mathbf{F} \qquad \text{left multiply } \mathbf{K}^{-1}$$

$$\Rightarrow \mathbf{K}^{-1}\mathbf{K}\ \mathbf{U} = \mathbf{K}^{-1}\mathbf{F} \qquad \Rightarrow \mathbf{U} = \mathbf{K}^{-1}\ \mathbf{F}$$

Inversion possible only if **K** is non-singular
(i.e. the structure is sufficiently supported = stable)



Beam Element Results

1. Element nodal displacements

Disassemble **u** from resulting global displacements **U**

2. Element end forces

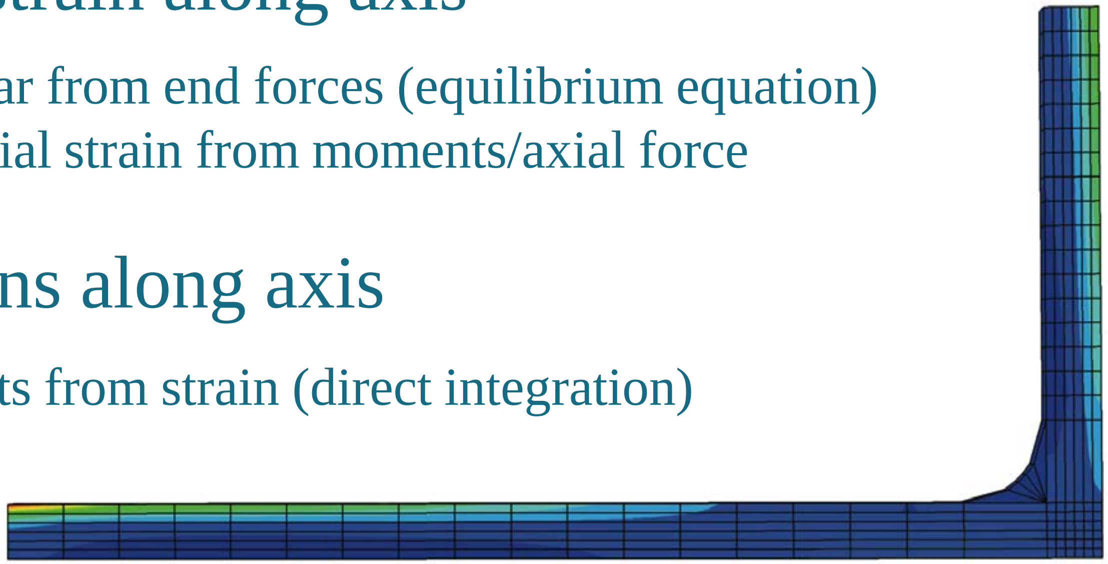
Calculate element end forces = **p = k u**

3. Element stress and strain along axis

Calculate moment/shear from end forces (equilibrium equation)
Calculate curvature/axial strain from moments/axial force

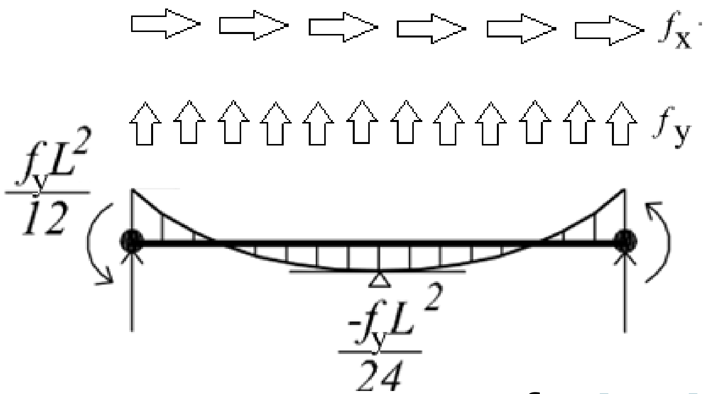
4. Element deformations along axis

Calculate displacements from strain (direct integration)



Lateral Load

1. Adjust global load vector

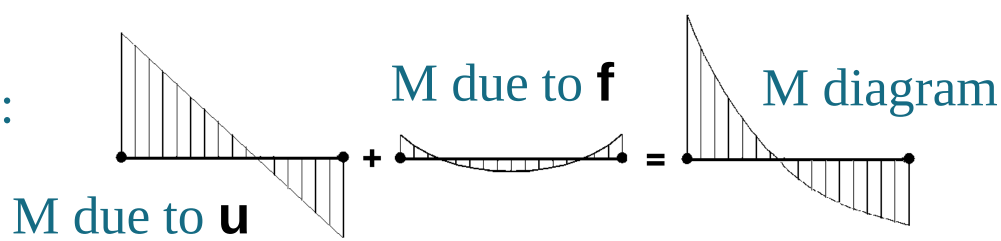


$$\mathbf{f} = \begin{Bmatrix} \frac{L}{2} \\ 0 \\ 0 \\ \frac{L}{2} \\ 0 \\ 0 \end{Bmatrix} f_x + \begin{Bmatrix} 0 \\ \frac{L}{2} \\ \frac{L^2}{12} \\ 0 \\ \frac{L}{2} \\ -\frac{L^2}{12} \end{Bmatrix} f_y$$

f = local load vector => add to global load vector **F**

2. Adjust element stresses

e.g. bending moment M:



Linear Static Analysis (1^{st} order)

Workflow of computer program

1. System identification: Elements, nodes, support and loads
2. Build element stiffness matrices and load vectors
3. Assemble global stiffness matrix and load vector
4. Solve global system of equations (=> displacements)
5. Calculate element results

Exact solution for displacements and stresses