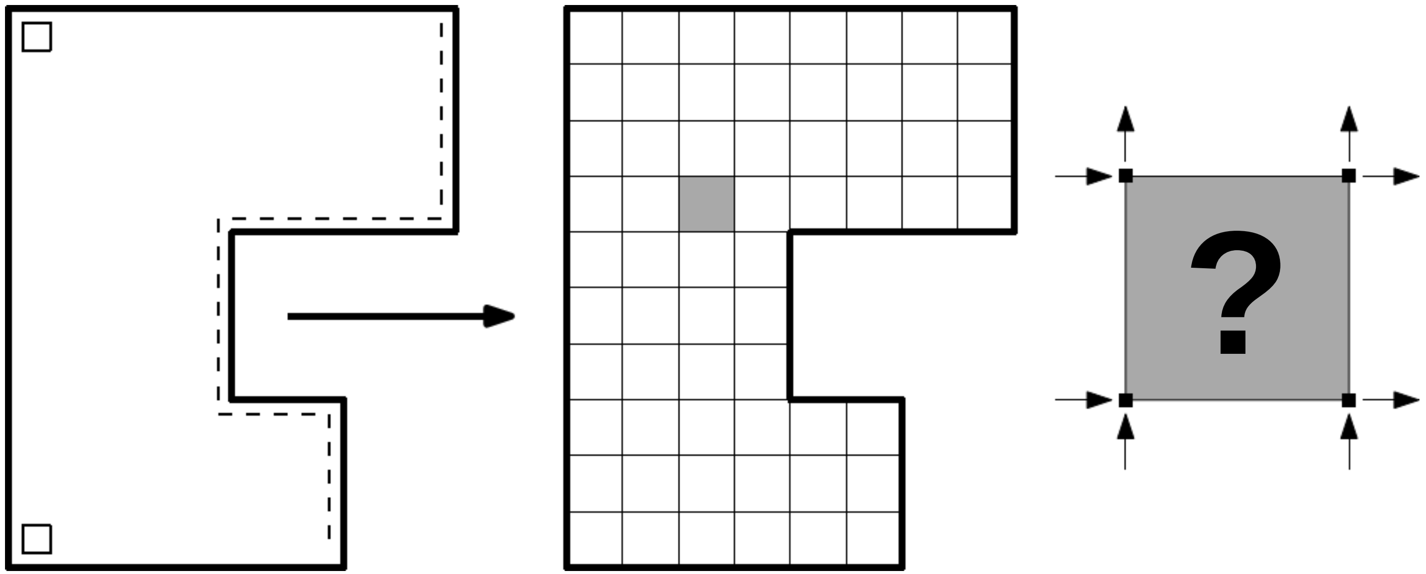
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Method of Finite Elements I: Shape Functions

Adrian Egger

Why shape functions?

- Discretization leads to solution in the nodes, but no information concerning the space in between
- Shape functions required to approximate quantities between nodes

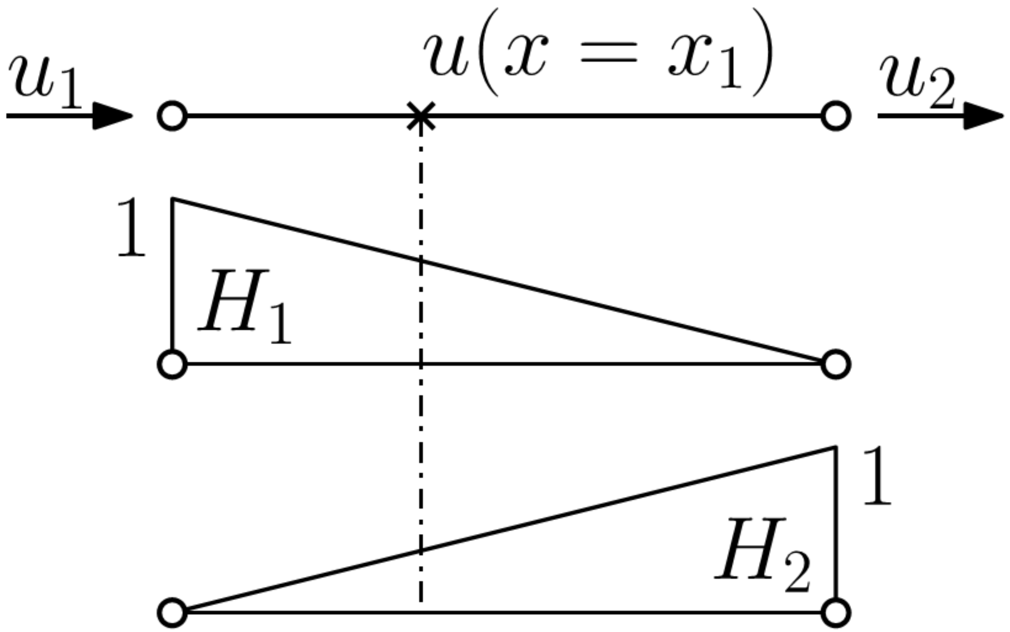


- Underlying assumption of how quantities are distributed in an element (stiffness, mass, element loads; displacements, strains, stress, internal forces, etc.)
- Geometry transformation

What can shape functions be used for?

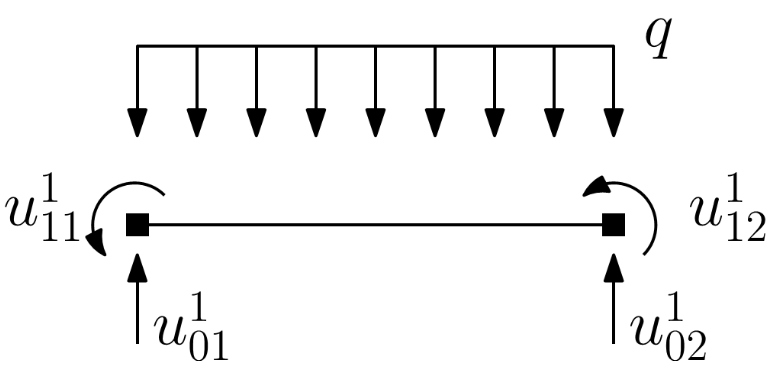
1. **Used to interpolate between nodes**
   i.e. discrete nodal quantities → continuous across element

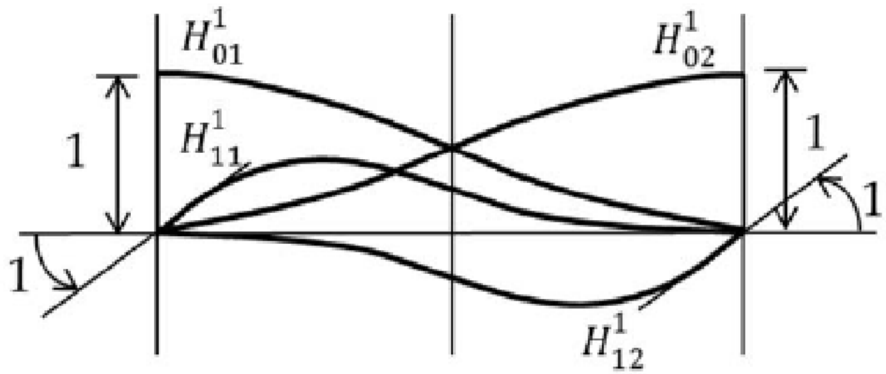
$$u(x) = \sum H_i(x) u_i = H_1(x) u_1 + H_2(x) u_2$$



What can shape functions be used for?

2. **Used to discretize continuous quantities to nodal DOF**
   i.e. continuous across element → discrete nodal quantities





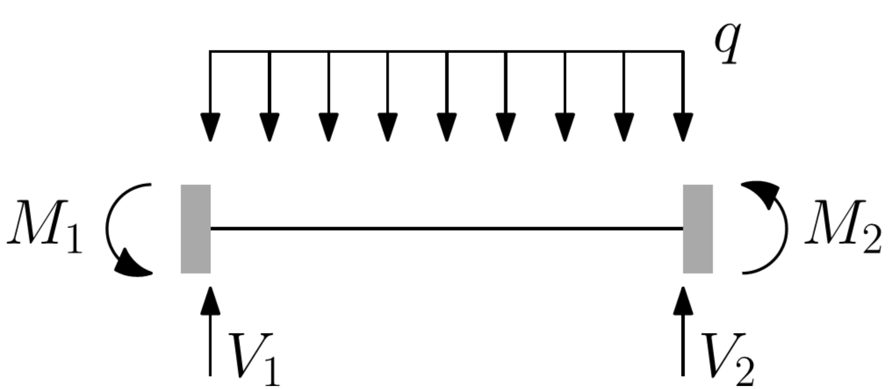
$$\boldsymbol{f}^{local}_{element} = \int_{x=0}^{L} \boldsymbol{H}^{\boldsymbol{T}} q(x)\, dx = \int_{x=0}^{L} \begin{bmatrix} H^1_{01} \\ H^1_{11} \\ H^1_{02} \\ H^1_{12} \end{bmatrix} q(x)\, dx = \begin{array}{c|c} \begin{bmatrix} \dfrac{qL}{2} \\ \dfrac{qL^2}{12} \\ \dfrac{qL}{2} \\ -\dfrac{qL^2}{12} \end{bmatrix} & \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \end{array}$$

Alternative way to derive loading vector

- Recap: We calculate the solution in the nodes
- "What is the influence of element loading in the nodes"
- We must fix the element such that reaction forces develop in the nodal DOF we are interested in!

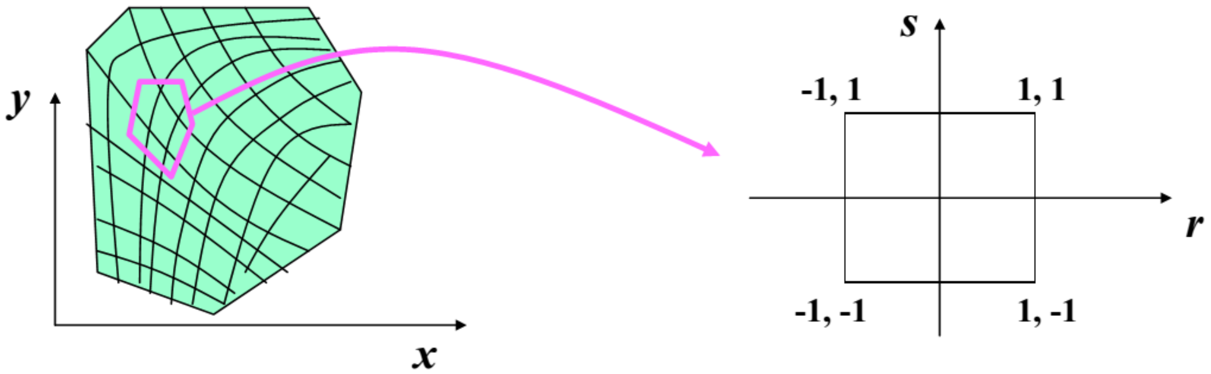
- Equivalent to solving differential equation

$$EIw^{IV} = -q$$



How to derive shape functions

- Interpolation functions are generally assumed!
  (within certain parameters and restrictions)
  - Minimal amount of continuity / differentiability
  - Etc.
- Wish to implement this repetitive task as easily as possible, i.e. computer implementation using highly optimized numerical schemes, and thus natural coordinates (r,s,t) are introduced ranging from -1 < r,s,t < 1.



Derivation of shape functions: Bar element (I)

1. Find a relationship for r(x). We choose -1 < r < 1.

$$r(x) = \left(\frac{2x}{l} - 1\right)$$

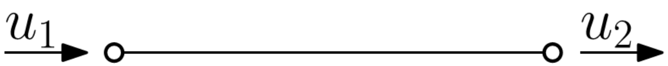
2. Choose an appropriate shape function polynomial

$$\begin{aligned}\hat{u}(r) &= \alpha_1 + \alpha_2 r \\ &= \begin{bmatrix} 1 & r \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\ &= \mathbf{A}\boldsymbol{\alpha}\end{aligned}$$

3. Evaluate **A** at each DOF by substituting values of "r"

$$\begin{aligned}\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\ &= \mathbf{E}\boldsymbol{\alpha}\end{aligned}$$

Derivation of shape functions: Bar element (II)



4. Reorder the previous equation

$$\mathbf{u} = \mathbf{E}\boldsymbol{\alpha} \longrightarrow \boldsymbol{\alpha} = \mathbf{E}^{-1}\mathbf{u}$$

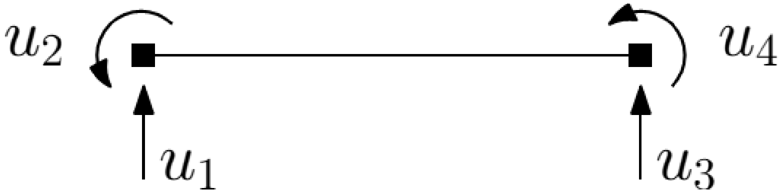
5. Substitute into previous equation

$$\begin{aligned} \hat{u} &= \mathbf{A}\mathbf{E}^{-1}\mathbf{u} \\ &= \mathbf{N}\mathbf{u} \end{aligned}$$

6. Extract shape functions (as a function of “r”)

$$\mathbf{N} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (1-r) & (1+r) \end{bmatrix}$$

Derivation of shape functions: Beam element (I)



1. Find a relationship for r(x). We choose 0 < r < 1.

$$r(x) = \frac{x}{l}$$

2. Choose an appropriate shape function polynomial

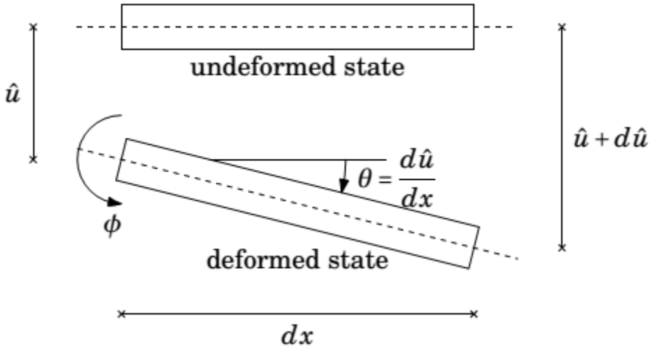
$$\hat{u}(r) = \alpha_1 + \alpha_2 r + \alpha_3 r^2 + \alpha_4 r^3$$

$$= \begin{bmatrix} 1 & r & r^2 & r^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$= \mathbf{A}\boldsymbol{\alpha}$$

Derivation of shape functions: Beam element (II)

3. Find an expression linking displacements and rotations



$$\phi = -\frac{d\hat{u}}{dx} = -\frac{d}{dr}\frac{dr}{dx}\hat{u} = -\frac{dr}{dx}\frac{d\hat{u}}{dr}$$
$$= -\frac{1}{l}\left(\alpha_2 + 2\alpha_3 r + 3\alpha_4 r^2\right)$$

4. Evaluate **A** at each DOF by substituting values of "r"

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{L} & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -\frac{1}{L} & -\frac{2}{L} & -\frac{3}{L} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$
$$= \mathbf{E}\boldsymbol{\alpha}$$

Derivation of shape functions: Beam element (III)

4. Reorder the previous equation

$$\mathbf{u} = \mathbf{E}\boldsymbol{\alpha} \longrightarrow \boldsymbol{\alpha} = \mathbf{E}^{-1}\mathbf{u}$$

5. Substitute into previous equation

$$\begin{aligned} \hat{u} &= \mathbf{A}\mathbf{E}^{-1}\mathbf{u} \\ &= \mathbf{N}\mathbf{u} \end{aligned}$$

6. Extract shape functions (as a function of "r")

$$\begin{aligned} \mathbf{N} &= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \\ &= \begin{bmatrix} (1-3r^2+2r^3) & (-r+2r^2-r^3)\,l & (3r^2-2r^3) & (r^2-r^3)\,l \end{bmatrix} \end{aligned}$$

Questions