Linearized weak form and total lagrangian formulation of a bar element

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1 Introduction

2 Linearization of the weak form

3 Finite element formulation

4 Total Lagrangian formulation of a bar element
learning goals

- Understanding how the weak form is linearized
- Understanding how geometrically nonlinear finite elements are formulated
- Gaining a basic understanding of how the above are implemented
Significance of the lecture

Applications:

- **Structures undergoing large displacements and/or rotations such as cables, arches and shells**
- Materials such as elastomers and biological/soft tissue
- Modeling of plastically deforming materials
Weak form

We recall:

Weak form: \[ \int_V \mathbf{S} : \delta \mathbf{E} dV = \int_V \mathbf{F}_b \cdot \delta \mathbf{u} dV + \int_A \mathbf{T} \cdot \delta \mathbf{u} dA \]

where:

\[ \mathbf{E} \text{ Green-Lagrange strain} \rightarrow \mathbf{E} = \frac{1}{2} \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u}^T \nabla \mathbf{u} \right) \]

\[ \mathbf{S} \text{ 2nd Piola-Kirchhoff stress tensor, for St. Venant materials} \rightarrow \mathbf{S} = D \mathbf{E} \]

\[ \mathbf{F}_b \text{ and } \mathbf{T} \text{ are applied bodyforces and tractions.} \]
Incremental solution

- Incremental solution procedures are used due to the nonlinear nature of the problem.

- Quantities are decomposed to their values at each step plus an increment.

- Two alternatives exist for the definition of the reference configuration.
Total/updated Lagrangian formulation

**Total Lagrangian (TL)**
- The initial configuration is the reference configuration.
- Stress & Strain measures at the target configuration (increment $i + 1$) are computed with respect to the initial configuration.
- Derivatives and Integrals are taken with respect to $V$ (initial conf.).

**Updated Lagrangian (UL)**
- The previous increment is the reference configuration.
- Stress & Strain measures at the target configuration (increment $i + 1$) are evaluated with respect to the previous configuration (increment $i$).
- Derivatives and Integrals are taken with respect to $v^i$. 
Total Lagrangian formulation

\[ \int_V S : \delta E dV = \int_V F_b \cdot \delta u dV + \int_A T \cdot \delta u dA \]

Displacements at increment \( i + 1 \):

\[ u^{i+1} = u^i + \Delta u \]

\( u^i \) is known from the previous step and should be considered constant

\( \Delta u \) is the incremental displacement to be solved for
Incremental decomposition of strains

By substituting in the definition of the Green-Lagrange strain:

\[ \mathbf{E}^{i+1} = \frac{1}{2} \left[ \nabla \mathbf{u}^{i+1} + \nabla \left( \mathbf{u}^{i+1} \right)^T + \nabla \left( \mathbf{u}^{i+1} \right)^T \nabla \mathbf{u}^{i+1} \right] \]

\[ = \frac{1}{2} \left[ \nabla \mathbf{u}^i + \nabla \left( \mathbf{u}^i \right)^T + \nabla \left( \mathbf{u}^i \right)^T \nabla \mathbf{u}^i \right] + \mathbf{E}^i \]

\[ + \frac{1}{2} \left[ \nabla \Delta \mathbf{u} + \nabla \Delta \mathbf{u}^T + \nabla \left( \mathbf{u}^i \right)^T \nabla \Delta \mathbf{u} + \nabla \Delta \mathbf{u}^T \nabla \mathbf{u}^i \right] + \mathbf{e}^i \]

\[ + \frac{1}{2} \left[ \nabla \Delta \mathbf{u}^T \nabla \Delta \mathbf{u} \right] \eta \]

Here \( \mathbf{e}^i \) is not to be confused with the Euler-Almansi strain.
Incremental decomposition of strains

In the above, the strain is decomposed as:

\[ E^{i+1} = E^i + \Delta E = E^i + e^i + \eta \]

where:

- \( E^i \) The strain at increment \( i \)
- \( \Delta E \) The strain increment
- \( e^i \) Linear with respect to \( \Delta u \)
- \( \eta \) Nonlinear with respect to \( \Delta u \)
We observe that the variation of $E^{i+1}$ is equal to the variation of the increment:

$$\delta E^i = 0 \Rightarrow \delta E^{i+1} = \delta \Delta E = \delta e^i + \delta \eta$$

Applying the incremental decomposition to stresses we obtain:

$$S^{i+1} = S^i + \Delta S$$
Applying the above decompositions to the weak form we obtain:

\[ \int_V \Delta S : \delta \Delta E dV + \int_V S^i : \delta \eta dV = f_{\text{ext}} - \int_V S^i : \delta e^i dV \]

\[ \int_V S^i : \delta e^i dV \] is known and has been moved to the RHS.
Applying the above decompositions to the weak form we obtain:

\[
\int_{V} \Delta S : \delta \Delta E dV + \int_{V} S^i : \delta \eta dV = f_{\text{ext}} - \int_{V} S^i : \delta e^i dV
\]

\[
\int_{V} S^i : \delta e^i dV \quad \text{is known and has been moved to the RHS (why?)}
\]
Applying the above decompositions to the weak form we obtain:

\[
\int_{V} \Delta S : \delta \Delta E dV + \int_{V} S^i : \delta \eta dV = f_{\text{ext}} - \int_{V} S^i : \delta e^i dV
\]

\[
\int_{V} S^i : \delta e^i dV
\]
is known and has been moved to the RHS (why?)

\[
\int_{V} S^i : \delta \eta dV
\]
is linear with respect to \( \Delta u \)
Applying the above decompositions to the weak form we obtain:

\[
\int_V \Delta \mathbf{S} : \delta \Delta \mathbf{E} dV + \int_V \mathbf{S}^i : \delta \eta dV = \mathbf{f}_{\text{ext}} - \int_V \mathbf{S}^i : \delta \mathbf{e}^i dV
\]

\[
\int_V \mathbf{S}^i : \delta \mathbf{e}^i dV
\]

is known and has been moved to the RHS (why?)

\[
\int_V \mathbf{S}^i : \delta \eta dV
\]

is linear with respect to \( \Delta \mathbf{u} \) (why?)
Applying the above decompositions to the weak form we obtain:

\[
\int_V \Delta S : \delta \Delta E dV + \int_V S^i : \delta \eta dV = f_{ext} - \int_V S^i : \delta e^i dV
\]

- \( \int_V S^i : \delta e^i dV \) is known and has been moved to the RHS (why?)
- \( \int_V S^i : \delta \eta dV \) is linear with respect to \( \Delta u \) (why?)
- \( \int_V \Delta S : \delta \Delta E dV \) is non linear with respect to \( \Delta u \)
Applying the above decompositions to the weak form we obtain:

\[ \int_V \Delta \mathbf{S} : \delta \Delta \mathbf{E} \, dV + \int_V \mathbf{S}^i : \delta \eta \, dV = \mathbf{f}_{\text{ext}} - \left( \int_V \mathbf{S}^i : \delta \mathbf{e}^i \, dV \right)_{f_{\text{int}}} \]

- \( \int_V \mathbf{S}^i : \delta \mathbf{e}^i \, dV \) is known and has been moved to the RHS (why?)
- \( \int_V \mathbf{S}^i : \delta \eta \, dV \) is linear with respect to \( \Delta \mathbf{u} \) (why?)
- \( \int_V \Delta \mathbf{S} : \delta \Delta \mathbf{E} \, dV \) is non linear with respect to \( \Delta \mathbf{u} \) (why?)
Linearization of stresses

For the variation of $\Delta E$ we assume:

$$\delta \eta = 0 \Rightarrow \delta \Delta E = \delta e^i$$

since $\delta \eta$ is higher order

For the stresses we employ the Taylor expansion:

$$\Delta S = \frac{\partial \Delta S}{\partial \Delta E} \cdot \Delta E = \frac{\partial \Delta S}{\partial \Delta E} \cdot \left( e^i + \eta \right)$$
Assuming linear elastic material response: \[ \frac{\partial \Delta \mathbf{S}}{\partial \Delta \mathbf{E}} = \mathbf{D} \]

Then the first term of the weak form becomes:

\[ \int_V \Delta \mathbf{S} : \delta \mathbf{E} dV = \int_V \left( \mathbf{e}^i + \eta \right)^T \cdot \mathbf{D} \cdot \delta \mathbf{e}^i dV \]
Linearized Weak Form

Assuming linear elastic material response: \( \frac{\partial \Delta S}{\partial \Delta E} = D \)

Then the first term of the weak form becomes:

\[
\int_V \Delta S : \delta E dV = \int_V \left( e^i + \eta \right)^T \cdot D \cdot \delta e^i dV
\]
Linearized Weak Form

Assuming linear elastic material response: 
\[ \frac{\partial \Delta S}{\partial \Delta E} = D \]

Then the first term of the weak form becomes:
\[ \int_V \Delta S : \delta E dV = \int_V \left( e^i \right)^T \cdot D \cdot \delta e dV \]
Linearized Weak Form

Assuming linear elastic material response: \( \frac{\partial \Delta \mathbf{S}}{\partial \Delta \mathbf{E}} = \mathbf{D} \)

Then the first term of the weak form becomes:

\[
\int_V \Delta \mathbf{S} : \delta \mathbf{E} \, dV = \int_V \left( \mathbf{e}^i \right)^T \cdot \mathbf{D} \cdot \delta \mathbf{e} \, dV
\]

The linearized weak form is formulated as:

\[
\int_V \mathbf{e}^i \cdot \mathbf{D} \cdot \delta \mathbf{e}^i \, dV + \int_V \mathbf{S}^i : \delta \mathbf{\eta} \, dV = \mathbf{f}_{\text{ext}} - \int_V \mathbf{S}^i : \delta \mathbf{e} \, dV
\]

\( \text{linear w.r.t. } \Delta \mathbf{u} \)

\( \text{linear w.r.t. } \Delta \mathbf{u} \)

\( \mathbf{f}_{\text{int}}^i \rightarrow \text{known} \)
Summarizing:

\[
\int_{V} \mathbf{e}^i \cdot \mathbf{D} \cdot \delta \mathbf{e}^i dV + \int_{V} \mathbf{S}^i : \delta \mathbf{\eta} dV = f_{\text{ext}} - \int_{V} \mathbf{S}^i : \delta \mathbf{e} dV
\]

where:

\[
\mathbf{e}^i = \frac{1}{2} \left[ \nabla \Delta \mathbf{u} + \nabla \Delta \mathbf{u}^T + \nabla (\mathbf{u}^i)^T \nabla \Delta \mathbf{u} + \nabla \Delta \mathbf{u}^T \nabla \Delta \mathbf{u}^i \right]
\]

\[
\mathbf{\eta} = \frac{1}{2} \left[ \nabla \Delta \mathbf{u}^T \nabla \Delta \mathbf{u} \right] \rightarrow \delta \mathbf{\eta} = \delta \nabla \Delta \mathbf{u}^T \nabla \Delta \mathbf{u} = \nabla \delta \Delta \mathbf{u}^T \nabla \Delta \mathbf{u}
\]

\[
\mathbf{S}^i = \mathbf{D} \mathbf{E}^i
\]

\[
\mathbf{E}^i = \frac{1}{2} \left[ \nabla \mathbf{u}^i + \nabla (\mathbf{u}^i)^T + \nabla (\mathbf{u}^i)^T \nabla \mathbf{u}^i \right]
\]
The above definitions can be written using Voigt notation, for example:

\[
\nabla \Delta \mathbf{u} = \begin{bmatrix}
\Delta u_{1,1} & \Delta u_{1,2} \\
\Delta u_{2,1} & \Delta u_{2,2}
\end{bmatrix} \rightarrow \nabla \Delta \mathbf{u} = \begin{bmatrix}
\Delta u_{1,1} \\
\Delta u_{2,2} \\
\Delta u_{1,2} \\
\Delta u_{2,1}
\end{bmatrix}
\]

\[
\frac{1}{2} \nabla (\mathbf{u}^i)^T \nabla \Delta \mathbf{u} = \frac{1}{2} \begin{bmatrix}
\mathbf{u}_{1,1}^i \Delta u_{1,1} + \mathbf{u}_{2,1}^i \Delta u_{2,1} & \mathbf{u}_{1,1}^i \Delta u_{1,2} + \mathbf{u}_{2,1}^i \Delta u_{2,2} \\
\mathbf{u}_{1,2}^i \Delta u_{1,1} + \mathbf{u}_{2,2}^i \Delta u_{2,1} & \mathbf{u}_{1,2}^i \Delta u_{1,2} + \mathbf{u}_{2,2}^i \Delta u_{2,2}
\end{bmatrix}
\]

\[
\rightarrow \frac{1}{2} \nabla (\mathbf{u}^i)^T \nabla \Delta \mathbf{u} = \frac{1}{2} \begin{bmatrix}
\mathbf{u}_{1,1}^i \Delta u_{1,1} + \mathbf{u}_{2,1}^i \Delta u_{2,1} \\
\mathbf{u}_{1,2}^i \Delta u_{1,1} + \mathbf{u}_{2,2}^i \Delta u_{2,1} \\
\mathbf{u}_{1,1}^i \Delta u_{1,2} + \mathbf{u}_{2,1}^i \Delta u_{2,2} \\
\mathbf{u}_{1,2}^i \Delta u_{1,1} + \mathbf{u}_{2,2}^i \Delta u_{2,1}
\end{bmatrix}
\]
Finite element formulation

Coordinates and displacements are interpolated from the corresponding nodal quantities using FE interpolation:

\[ X_i = \sum_{J=1}^{N} N_J X_i^J, \]
\[ x_i = \sum_{J=1}^{N} N_J x_i^J, \]
\[ u_i = \sum_{J=1}^{N} N_J u_i^J, \]
\[ \Delta u_i = \sum_{J=1}^{N} N_J \Delta u_i^J \]

or in matrix form:

\[ X = NX^n, \]
\[ u^i = Nu^{ni}, \]
\[ x = Nx^n \]
\[ \Delta u = N\Delta u^n \]

Index \( J \) and superscript \( n \) refer to nodal quantities, \( N_J \) are the FE shape functions.
For example incremental displacements in 2D are approximated as:

\[
\begin{bmatrix}
\Delta u_1 \\
\Delta u_2 \\
\end{bmatrix}
= \begin{bmatrix}
N_1 & 0 & N_2 & 0 & \cdots & N_N & 0 \\
0 & N_1 & 0 & N_2 & \cdots & 0 & N_N \\
\end{bmatrix}
\begin{bmatrix}
\Delta u_1^1 \\
\Delta u_1^2 \\
\Delta u_1^N \\
\Delta u_2^1 \\
\Delta u_2^2 \\
\vdots \\
\Delta u_2^N \\
\end{bmatrix}
\]
Then, variations can also be computed simply as:

\[
\begin{bmatrix}
\delta \Delta u_1 \\
\delta \Delta u_2
\end{bmatrix} = 
\begin{bmatrix}
N_1 & 0 & N_2 & 0 & \ldots & N_N & 0 \\
0 & N_1 & 0 & N_2 & \ldots & 0 & N_N
\end{bmatrix}
\begin{bmatrix}
\delta \Delta u_1^1 \\
\delta \Delta u_1^2 \\
\vdots \\
\delta \Delta u_N^1 \\
\delta \Delta u_N^2
\end{bmatrix}
\]
Typically isoparametric elements are used:

\[ N_J = N_J(\xi_1, \xi_2, \xi_3) \]

where \( \xi_1, \xi_2, \xi_3 \) are the isoparametric coordinates. Then:

\[
\begin{bmatrix}
\frac{\partial}{\partial \xi_1} \\
\frac{\partial}{\partial \xi_2} \\
\frac{\partial}{\partial \xi_3}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\
\frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\
\frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial X_1} \\
\frac{\partial}{\partial X_2} \\
\frac{\partial}{\partial X_3}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\frac{\partial}{\partial X_1} \\
\frac{\partial}{\partial X_2} \\
\frac{\partial}{\partial X_3}
\end{bmatrix}
= J^{-1}
\begin{bmatrix}
\frac{\partial}{\partial \xi_1} \\
\frac{\partial}{\partial \xi_2} \\
\frac{\partial}{\partial \xi_3}
\end{bmatrix}
\]

The same transformation can be used for the current configuration.
Utilizing the above, the quantities $\nabla u, \nabla \Delta u$ can be computed.

For example:

$$\frac{\partial u_1}{\partial X_1} = \sum_{j=1}^{N} \frac{\partial N_j}{\partial X_1} u_i^j$$

$$\frac{\partial N_j}{\partial X_1} = J^{-1}_{11} \frac{\partial N_j}{\partial \xi_1} + J^{-1}_{12} \frac{\partial N_j}{\partial \xi_2} + J^{-1}_{13} \frac{\partial N_j}{\partial \xi_3}$$

where $J^{-1}_{ij}$ is the element $ij$ of matrix $J^{-1}_{ij}$. 
Then, in matrix form and using Voigt notation:

\[
\begin{bmatrix}
\Delta u_{1,1} \\
\Delta u_{2,2} \\
\Delta u_{1,2} \\
\Delta u_{2,1}
\end{bmatrix}
\n= 
\begin{bmatrix}
N_{1,1} & 0 & N_{2,1} & 0 & \cdots & N_{N,1} & 0 \\
0 & N_{1,2} & 0 & N_{2,2} & \cdots & 0 & N_{N,2} \\
N_{1,2} & 0 & N_{2,2} & 0 & \cdots & N_{N,2} & 0 \\
0 & N_{1,1} & 0 & N_{2,1} & \cdots & 0 & N_{N,1}
\end{bmatrix}
\begin{bmatrix}
\nabla \Delta u \\
\nabla \Delta u \\
\nabla \Delta u \\
\nabla \Delta u \\
\nabla \Delta u
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta u^1_1 \\
\Delta u^1_2 \\
\Delta u^2_1 \\
\Delta u^2_2 \\
\vdots \\
\Delta u^N_1 \\
\Delta u^N_2 \\
\Delta u^n
\end{bmatrix}
\]
Utilizing the above, the linear and non-linear part of the strain increment can be written in terms of the nodal displacement increments:

\[ e = B_L \Delta u^n \Rightarrow \delta e = B_L \delta \Delta u^n \]

\[ \eta = \frac{1}{2} \Delta u^n^T B_{NL}^T B_{NL} \Delta u^n \Rightarrow \delta \eta = \delta \Delta u^n^T B_{NL}^T B_{NL} \Delta u^n \]

Notice that \( B_{NL} = B_\nabla \)
Finite element formulation

By taking into account that:

\[ u = u_i + \Delta u \Rightarrow \delta u = \delta \Delta u \]

Substituting the above in the weak form we obtain:

\[
\delta \Delta u^n^T \int_V B_L^T D B_L dV \Delta u^n + \delta \Delta u^n^T \int_V B_{NL}^T S^i B_{NL} dV \Delta u^n = \\
\delta \Delta u^n^T \int_V N^T F_b dV + \delta \Delta u^n^T \int_A N^T T dA - \delta \Delta u^n^T \int_V B_L^T \hat{S}^i dV
\]
Finite element formulation

By taking into account that:

\[ u = u_i + \Delta u \Rightarrow \delta u = \delta \Delta u \]

Substituting the above in the weak form we obtain:

\[
\delta \Delta u^T \int_V B_L^T D B_L dV \Delta u^n + \delta \Delta u^T \int_V B_{NL}^T S_i B_{NL} dV \Delta u^n =
\]

\[
\delta \Delta u^T \int_V N^T F_b dV + \delta \Delta u^T \int_A N^T T dA - \delta \Delta u^T \int_V B_L^T \hat{S}_i dV
\]
Finite element formulation

By taking into account that:

\[ u = u_i + \Delta u \Rightarrow \delta u = \delta \Delta u \]

Substituting the above in the weak form we obtain:

\[
\int_V B_L^T D B_L dV \Delta u^n + \int_V B_{NL}^T S_i B_{NL} dV \Delta u^n = \\
\int_V N^T F_b dV + \int_A N^T T dA - \int_V B_L^T \hat{S}^i dV
\]
Finite element formulation

By taking into account that:

\[ u = u_i + \Delta u \Rightarrow \delta u = \delta \Delta u \]

Substituting the above in the weak form we obtain:

\[
\begin{align*}
\Delta u^n &= \\
&= \left( K_T \int_V B_L^T D B_L dV + \int_V B_{NL}^T S^i B_{NL} dV \right) \\
&+ \left( K_L \int_V B_{NL}^T S^i B_{NL} dV - R_i \right) \\
&- \int_A N^T T dA - \int_V B_L^T \hat{S}^i dV + \int_V N^T F_b dV + \int_V N^T T dA - \int_V B_L^T \hat{S}^i dV
\end{align*}
\]
Finite element formulation

The tangent stiffness matrix, residual and force vectors are then written as:

\[
K_T = K_L + K_{NL}
\]

\[
K_L = \int_V B_L^T D B_L dV, \quad K_{NL} = \int_V B_{NL}^T S^i B_{NL} dV
\]

\[-R^i = f_{\text{ext}} - f_{\text{int}}\]

\[
f_{\text{ext}} = \int_V N^T F_b dV + \int_A N^T T dA, \quad f_{\text{int}}^i = \int_V B_L^T \hat{S}^i dV
\]
Substituting the above, the linearized equilibrium equation can be obtained in terms of the nodal unknowns:

\[ K_T \Delta u^n = -R^i \]

\[ K_T \Delta u^n = f_{\text{ext}} - f_{\text{int}} \]
Next the tangent stiffness matrix and force vector are derived for the two noded bar element.

It is convenient to write:

\[ u_1^2 = u_1^1 + l \cos \theta - L \]

\[ u_2^2 = u_2^1 + l \sin \theta \]

In what follows, displacements refer to the previous step \((u^i, u_1^1, \ldots)\), however the index \(i\) has been dropped to simplify notation.
Bar element

FE shape function for a two noded bar:

\[
N = \begin{bmatrix}
\frac{1 - \xi}{2} & \frac{1 + \xi}{2}
\end{bmatrix}
\]

Jacobian: \( J = \frac{L}{2} \)

Shape function derivative:

\[
N,1 = \frac{\partial N}{\partial X_1} = J^{-1} \frac{\partial N}{\partial \xi} = \frac{1}{L} \begin{bmatrix}
-1 & 1
\end{bmatrix}
\]
Bar element

By considering displacements along both directions:

\[
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}
= 
\begin{bmatrix}
    \frac{1 - \xi}{2} & 0 & \frac{1 + \xi}{2} & 0 \\
    0 & \frac{1 - \xi}{2} & 0 & \frac{1 + \xi}{2}
\end{bmatrix}
\cdot
\begin{bmatrix}
    u_1^1 \\
    u_1^2 \\
    u_2^1 \\
    u_2^2
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
    \frac{\partial u_1}{\partial X_1} \\
    \frac{\partial u_2}{\partial X_1}
\end{bmatrix}
= \frac{1}{L}
\begin{bmatrix}
    -1 & 0 & 1 & 0 \\
    0 & -1 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
    u_1^1 \\
    u_1^2 \\
    u_2^1 \\
    u_2^2
\end{bmatrix}
\]
Bar element

Displacement derivative:

\[
\frac{\partial u_i}{\partial X_1} = N_{,1} u_i^n = \frac{1}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u^1_i \\ u^2_i \end{bmatrix} = \frac{u^2_i - u^1_i}{L}
\]

For \(i = 1\):

\[
\frac{\partial u_1}{\partial X_1} = \frac{u^2_1 - u^1_1}{L} = \frac{u^1_1 + l \cos \theta - L - u^1_1}{L} = \frac{l \cos \theta - L}{L} = \frac{l \cos \theta}{L} - 1
\]

For \(i = 2\):

\[
\frac{\partial u_2}{\partial X_1} = \frac{u^2_2 - u^1_2}{L} = \frac{u^1_2 + l \sin \theta - u^1_2}{L} = \frac{l \sin \theta}{L}
\]
By taking into account that $\frac{\partial (\cdot)}{\partial X_2} = 0$ the Green-Lagrange strain at the previous step (only component 11 is of interest) assumes the form:

$$E_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2} \left( \frac{\partial u_1}{\partial X_1} \right)^2 + \frac{1}{2} \left( \frac{\partial u_2}{\partial X_1} \right)^2$$

$$= \left( \frac{l \cos \theta}{L} - 1 \right) + \frac{1}{2} \left( \frac{l \cos \theta}{L} - 1 \right)^2 + \frac{1}{2} \left( \frac{l \sin \theta}{L} \right)^2$$

$$= \frac{l^2 - L^2}{2L^2}$$
By taking into account that \( \frac{\partial (\cdot)}{\partial X_2} = 0 \) the linear part of the strain increment (only component 11 is of interest) assumes the form:

\[
e_{11} = \frac{\partial \Delta u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1} \frac{\partial \Delta u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_1} \frac{\partial \Delta u_2}{\partial X_1}
\]
$$e_{11} = \left[ \frac{1}{L} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial \Delta u_1}{\partial X_1} \\ \frac{\partial u_1}{\partial X_1} \end{bmatrix} \right] + \left( \frac{l \cos \theta}{L} - 1 \right) \frac{1}{L} \left[ \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial \Delta u_1}{\partial X_1} \\ \frac{\partial u_1}{\partial X_1} \end{bmatrix} \right] + \left( \frac{l \sin \theta}{L} \right) \frac{1}{L} \left[ \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial \Delta u_2}{\partial X_1} \\ \frac{\partial u_2}{\partial X_1} \end{bmatrix} \right] \right]$$
$e_{11} = \frac{l}{L^2} \begin{bmatrix} -\cos \theta & -\sin \theta & \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \Delta u_1^n \\ \Delta u_2^n \\ \Delta u_1^1 \\ \Delta u_2^1 \end{bmatrix}$
The nonlinear part of the strain increment is:

\[ \eta = \frac{1}{2} \left[ \left( \frac{\partial \Delta u_1}{\partial X_1} \right)^2 + \left( \frac{\partial \Delta u_2}{\partial X_1} \right)^2 \right] = \begin{bmatrix} \frac{\partial \Delta u_1}{\partial X_1} & \frac{\partial \Delta u_2}{\partial X_1} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \Delta u_1}{\partial X_1} \\ \frac{\partial \Delta u_2}{\partial X_1} \end{bmatrix} \]

\[ \begin{bmatrix} \frac{\partial \Delta u_1}{\partial X_1} \\ \frac{\partial \Delta u_2}{\partial X_1} \end{bmatrix} = \frac{1}{L} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \Delta u_1^1 \\ \Delta u_2^1 \\ \Delta u_1^2 \\ \Delta u_2^2 \end{bmatrix} \]

\[ B_{NL}=N,1 \]
If a linear stress strain relation is employed then:

\[ S_{11} = E \quad E_{11} = E \frac{l^2 - L^2}{2L^2} \]

where \( E \) is Young’s modulus

Then:

\[ \frac{\partial S_{11}}{\partial E_{11}} = E \]
Bar element

Linear part of the stiffness matrix:

\[ K_L = \int_V B_L^T E B_L dV \]

Employing that \( V = AL \), where \( A \) the cross section of the bar in the reference configuration, we obtain:

\[
K_L = EA \frac{l^2}{L^3} \begin{bmatrix}
\cos^2 \theta & \cos \theta \sin \theta & -\cos^2 \theta & -\cos \theta \sin \theta \\
\sin^2 \theta & -\sin \theta \cos \theta & \cos^2 \theta & -\sin^2 \theta \\
\text{Symm} & & & \\
\end{bmatrix}
\]
Similarly the nonlinear part of the tangent stiffness matrix is:

\[ K_{NL} = \int_{V} B_{NL}^{T} S_{11} B_{NL} dV \]

By carrying out the integration we obtain:

\[ K_{NL} = EA \frac{l^2 - L^2}{2L^3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \]
Finally the force vector can be obtained as:

\[ \mathbf{f}_{\text{int}} = \int_V B_L^T S_{11} dV \]

Performing the integration:

\[ \mathbf{f}_{\text{int}} = E A \frac{l^2 - L^2}{2L^2} \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ \cos\theta \\ \sin\theta \end{bmatrix} \]