Introduction to fracture mechanics

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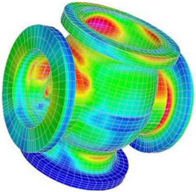

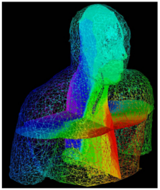

Outline

Learning goals

- Reviewing basic elasticity equations and solution methods
- Understanding some basic concepts of Linear Elastic Fracture Mechanics
- Introducing brittle and cohesive fracture
- Introducing some basic methods to be used in the development of numerical methods

Applications

- Fracture in brittle materials
- Fatigue fracture
- Fracture in concrete
- Development of numerical methods for fracture

Resources

Some useful resources:

- fracturemechanics.org
- Extended Finite Element Method for Fracture Analysis of Structures by Soheil Mohammadi, Blackwell Publishing, 2008

History

- Some research relevant to the field was conducted starting from the end of the 19th century
- Motivation was provided from experiments in brittle materials were the measured strength was much lower than the theoretical prediction
- In general very few developments were made in the field until World War II

Motivation

- Due to their local nature cracks were not considered a threat for large structures
- During the war several failures in ships and aircraft occurred as a result of crack propagation

Source:Wikipedia

- Research in the field was triggered as a result

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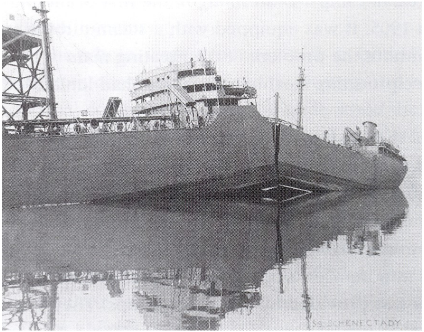

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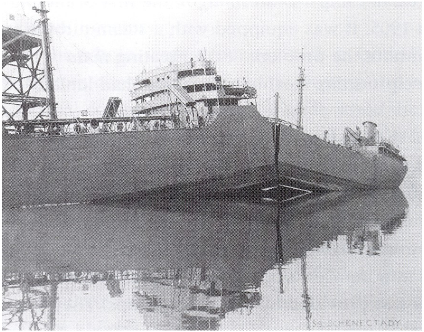

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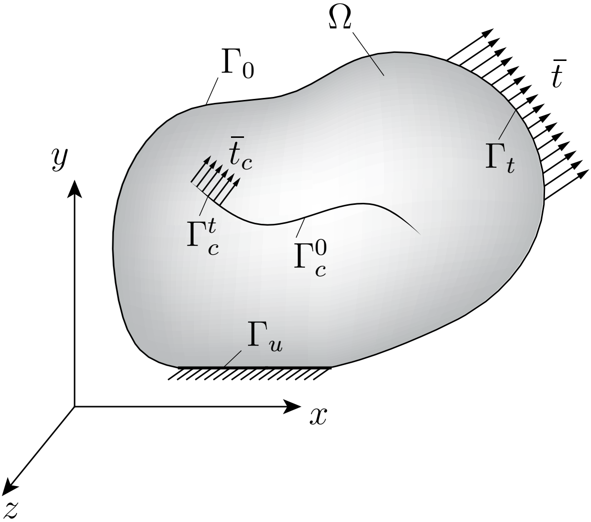
General problem (in the presence of a crack):



Governing equations:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} && \text{in} \quad \Omega \\
\boldsymbol{u} &= \bar{\boldsymbol{u}} && \text{on} \quad \Gamma_u \\
\boldsymbol{\sigma} \cdot \mathbf{n} &= \bar{\boldsymbol{t}} && \text{on} \quad \Gamma_t \\
\boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{0} && \text{on} \quad \Gamma_c^0 \\
\boldsymbol{\sigma} \cdot \mathbf{n} &= \bar{\boldsymbol{t}}^c && \text{on} \quad \Gamma_c^t
\end{aligned}
$$

where

- $\boldsymbol{\sigma}$ is the Cauchy stress tensor
- $\mathbf{n}$ is the unit outward normal
- $\mathbf{f}$ is the body force per unit volume
- $\boldsymbol{u}$ is the displacement field

General elasticity problem

Kinematic equations (small deformations):

$$\boldsymbol{\epsilon} = \boldsymbol{\nabla}_s \boldsymbol{u}$$

The constitutive equations are given by Hooke's law:

$$\boldsymbol{\sigma} = \boldsymbol{D} : \boldsymbol{\epsilon}$$

where $\boldsymbol{D}$ is the elasticity tensor.

Using index notation the above can be written as:

$$\begin{aligned} \sigma_{ji,j} + f_i &= 0 && \text{in} && \Omega \\ u_i &= \bar{u}_i && \text{on} && \Gamma_u \\ n_i \sigma_{ij} &= \bar{t}_j && \text{on} && \Gamma_t \\ n_i \sigma_{ij} &= 0 && \text{on} && \Gamma_c^0 \\ n_i \sigma_{ij} &= \bar{t}_j^c && \text{on} && \Gamma_c^t \end{aligned}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{x_j} + \frac{\partial u_j}{x_i}\right)$$

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}, \ \epsilon_{ij} = S_{ijkl}\sigma_{kl}$$

Displacement formulation

Displacement formulation :

- Kinematic equations are substituted into the constitutive equations
- Both of the above are substituted into the equilibrium equations
- The resulting system consists of three equations in three unknowns
- The problem is solved in terms of the displacements

Stress formulation

Stress formulation :

- The equilibrium equations are directly solved in terms of stresses
- Equilibrium provides three equations
- The (symmetric) stress tensor consists of six independent components
- Three more equations are needed!

Stress formulation

The Saint Venant compatibility equations can be employed as additional equations:

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} = \epsilon_{ik,jl} + \epsilon_{jl,ik}$$

In terms of stresses:

$$\left(S_{ijmn}\sigma_{mn}\right)_{,kl} + \left(S_{klmn}\sigma_{mn}\right)_{,ij} = \left(S_{klmn}\sigma_{mn}\right)_{,jl} + \left(S_{jlmn}\sigma_{mn}\right)_{,ik}$$

$\rightarrow$ of the above only three equations are independent

Stress formulation

The compatibility equations:

- Guarantee that the computed strains/stresses are the symmetric gradient of a vector field
- Guarantee that the resulting displacement field will not exhibit any gaps or overlaps
- In conjunction to the equilibrium equations form a system that can be solved to provide the stress field

Airy stress function

A function Φ is introduced for the 2D case such that:

$$\sigma_{11} = \sigma_{xx} = \frac{\partial^2 \Phi}{\partial x_2^2} = \frac{\partial^2 \Phi}{\partial y^2}$$

$$\sigma_{22} = \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x_1^2} = \frac{\partial^2 \Phi}{\partial x^2}$$

$$\sigma_{12} = \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

Φ is called a stress function.

In the absence of body forces $f_i = 0$, the equilibrium equations become:

$$\sigma_{xx,x} + \sigma_{xy,y} = \Phi_{,yyx} - \Phi_{,xyy} = 0$$

$$\sigma_{yx,x} + \sigma_{yy,y} = -\Phi_{,xyx} + \Phi_{,xxy} = 0$$

They are satisfied by default!

Airy stress function

Substituting the stress function in the compatibility equations the following can be obtained:

$$\Phi_{,xxxx} + 2\Phi_{,xyxy} + \Phi_{,yyyy} = 0$$

The above is called biharmonic equation and can also be written as:

$$\nabla^4 \Phi = 0$$

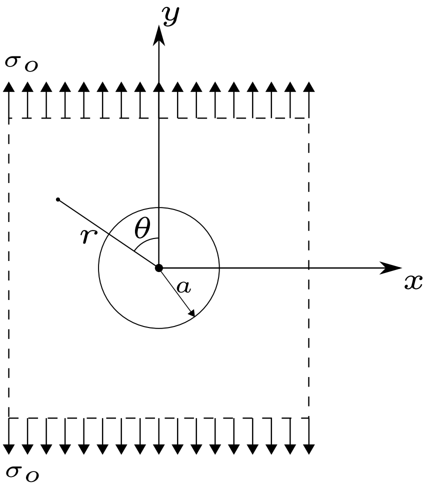
Airy stress function

Using stress functions:

- In the absence of body forces allows the reduction of the number of equations to one
- The problem reduces to the problem of determining an appropriate stress function for the problem at hand
- Solutions to several classical elasticity problems have been obtained

Circular hole in an infinite plate

We consider the problem of a circular hole in an infinite domain subjected to a far field stress:



Kirsch solved the problem in 1898 by using a stress function resulting in the stress field:

$$\sigma_{rr} = \frac{\sigma_o}{2}\left(1-\left(\frac{a}{r}\right)^2\right)+\frac{\sigma_o}{2}\left(1-4\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_o}{2}\left(1+\left(\frac{a}{r}\right)^2\right)-\frac{\sigma_o}{2}\left(1+3\left(\frac{a}{r}\right)^4\right)\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{\sigma_o}{2}\left(1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right)\sin 2\theta$$

Circular hole in an infinite plate

At the hole ($r = a$):

$$\sigma_{rr} = 0$$
$$\sigma_{\theta\theta} = \sigma_0 \left(1 - 2\cos 2\theta\right)$$
$$\sigma_{r\theta} = 0$$

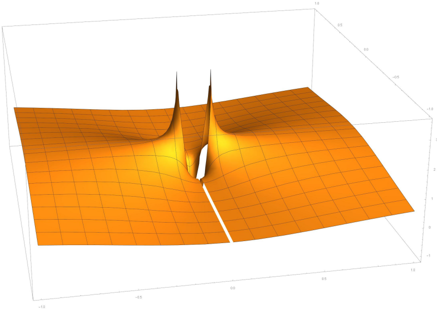
For $\theta = \pm\frac{\pi}{2}$:

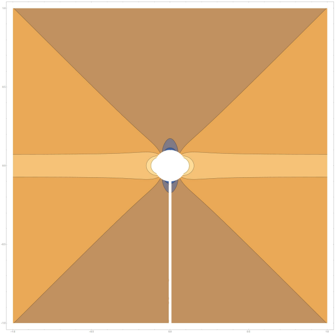
$$\sigma_{\theta\theta} = 3\sigma_0$$

A stress concentration factor of 3 occurs!

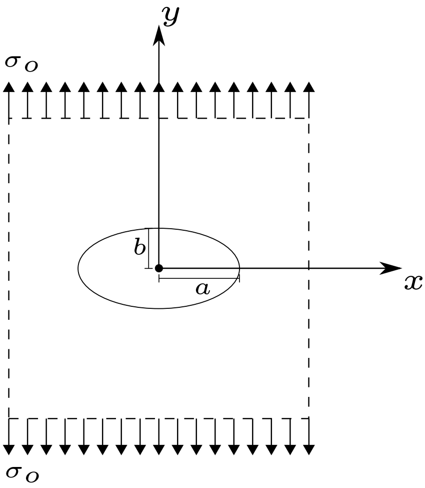
Circular hole in an infinite plate

3D and contour plot of the $\sigma_{\theta\theta}$ stress component:





Inglis (1913), obtained the solution for an elliptical hole in an infinite domain subjected to a far field stress:



Elliptical hole in an infinite plate

Maximum stress predicted by the solution, at $[a, 0]$, is:

$$\sigma_{\max} = \sigma_o \left(1 + 2\frac{a}{b}\right)$$

We observe that for $b = a$ (circular hole):

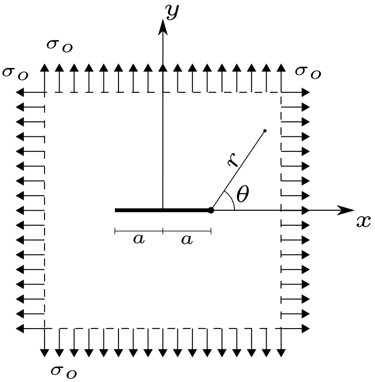
$$\sigma_{\max} = 3\sigma_o$$

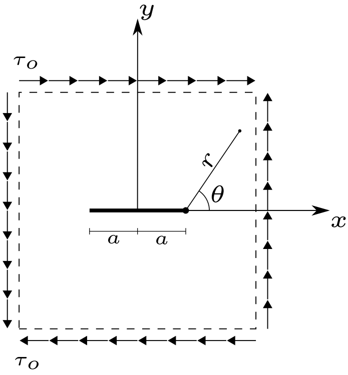
For $b = 0$ the hole turns into a sharp crack:

$$\sigma_{\max} = \infty$$

A singularity occurs!

We consider the problem of a sharp crack in an infinite domain subjected to a far field stress:





Westergaard solution

Westergaard solved the problem by considering the complex stress function:

$$\Phi = \mathrm{Re}\,\bar{\bar{Z}} + y\,\mathrm{Im}\,\bar{Z}$$

where:

Z is a complex function defined as: $Z = \dfrac{\sigma_o}{\sqrt{1 - \left(\dfrac{a}{z}\right)^2}}$

z is a complex number: $z = x + iy$

$\bar{Z}$ denotes the integral of Z

Westergaard solution

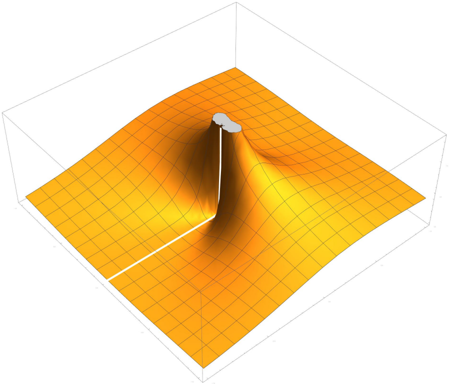
For $r \ll a$ the stress expression of the solution is:

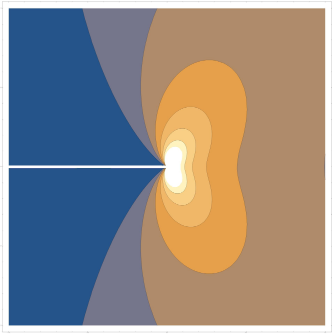
$$\sigma_{xx} = \sigma_o \sqrt{\frac{a}{2r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \ldots$$

$$\sigma_{yy} = \sigma_o \sqrt{\frac{a}{2r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \ldots$$

$$\sigma_{xy} = \sigma_o \sqrt{\frac{a}{2r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + \cdots$$

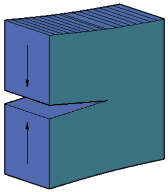
3D and contour plot of the σ_{yy} stress component:





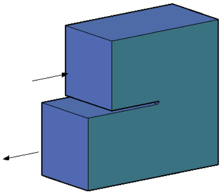
We define the stress intensity factors (SIFs) corresponding to the three modes of fracture as:

Mode I



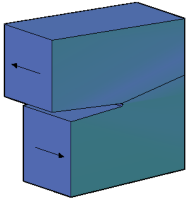
$$K_I = \lim_{\substack{r \to 0 \\ \theta = 0}} \sigma_{yy} \sqrt{2\pi r}$$

Mode II



$$K_{II} = \lim_{\substack{r \to 0 \\ \theta = 0}} \sigma_{xy} \sqrt{2\pi r}$$

Mode III



$$K_{III} = \lim_{\substack{r \to 0 \\ \theta = 0}} \sigma_{yz} \sqrt{2\pi r}$$

The mode I SIF for the Westergaard solution is:

$$K_I = \sigma_o \sqrt{\pi a}$$

Rewritting stresses w.r.t the SIFs gives:

$$\begin{aligned}
\sigma_{xx} = &\quad \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \ldots \\
\sigma_{yy} = &\quad \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \ldots \\
\sigma_{xy} = &\quad \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + \cdots
\end{aligned}$$

Energy release rate

It can be observed in the solutions above that:

- Stresses at the tip of a crack are infinite independently of the load applied
- Since no material can withstand infinite stress, structures should fail even in the presence of the smallest crack

Energy release rate

To circumvent the above situation, Griffith (1920) proposed an energy based failure criterion.

First the energy required to propagate the crack by a length a (in 2D) is considered:

$$U_\Gamma = 2\gamma_s a$$

where:

- γ_s is the required energy per unit length
- The factor 2 is used to account for both crack surfaces

Energy release rate

Next the total change of potential energy in the system is considered:

$$-\frac{d\Pi}{dt} = \frac{d\left(W - U_s\right)}{dt}$$

where:

U_s is the strain energy: $U_s = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\epsilon} = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$

W is the work produced by external forces

With respect to the crack length:

$$-\frac{d\Pi}{da} = \frac{d\left(W - U_s\right)}{da}$$

Energy release rate

The total change in potential energy should be equal to the energy dissipated to propagate the crack:

$$-\frac{d\Pi}{da} = \frac{d\left(W - U_s\right)}{da} = \frac{d\left(U_\gamma\right)}{da} = 2\gamma_s$$

We define the energy release rate as:

$$G = -\frac{d\Pi}{da}$$

and critical release rate as:

$$G_c = 2\gamma_s$$

Energy release rate

Griffith's criterion for crack propagation is written as:

$$G \geq G_c$$

where:

- The critical energy release rate (G_c) is a material property
- The crack will propagate if the energy release rate (change in potential energy) is greater than a critical value

Computing the total potential energy for the Westergaard solution yields:

$$\Pi = \frac{\pi \sigma_o^2 a^2}{2E'}, \; E' = \begin{cases} E & \text{plane stress} \\ \dfrac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

Differentiating with respect to a:

$$G = -\frac{d\Pi}{da} = \frac{\pi \sigma_o^2 a}{E'}$$

Energy release rate

Setting $G = G_c$:

$$\frac{\pi \sigma_c^2 a}{E'} = G_c$$

where σ_c is the critical/failure stress:

$$\sigma_c = \sqrt{\frac{E' Gc}{\pi a}}$$

Also:

$$K_{Ic} = \sigma_c \sqrt{\pi a}$$

is defined as the critical SIF.

Energy release rate

Griffith's criterion can be written in terms of the SIFs:

$$K_I \geq K_{Ic}$$

Also the mode I SIF can be related to the energy release rate by:

$$G = \frac{K_I^2}{E'}, \ G_c = \frac{K_{Ic}^2}{E'}$$

Mixed mode fracture

The above results are derived for pure mode I, for mixed mode fracture the following relation holds:

$$G = \frac{K_I^2 + K_{II}^2}{E'} + \frac{1+\nu}{E} K_{III}^2$$

Also appropriate criteria should be used which also yield the direction of propagation. Those assume the form:

$$f\left(K_I, K_{II}, K_{Ic}\right) = 0$$

Typically mode III is not taken into account and only K_{Ic} is experimentally measured.

Maximum circumferential stress criterion

Erdogan and Sih (1963) developed a criterion for mixed mode fracture according to which:

- The crack will propagate in a direction normal to the direction of the maximum circumferential stress
- The crack will propagate when the maximum circumferential stress reaches a critical value

Maximum circumferential stress criterion

By transforming the stresses from the Westergaard solution for modes I and II to a polar system of coordinates we obtain:

$$\begin{aligned}
\sigma_{rr} = & \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right)\\
\sigma_{\theta\theta} = & \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2}-\frac{3}{4}\sin\frac{3\theta}{2}\right)\\
\sigma_{r\theta} = & \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right)
\end{aligned}$$

Maximum circumferential stress criterion

The maximum circumferential stress is a principal stress, therefore the angle of crack propagation θ_c is obtained by setting $\sigma_{r\theta} = 0$ in the above equation which, after some manipulations, results in:

$$K_I \sin\theta_c + K_{II} \left(3\cos\theta_c - 1\right) = 0$$

Solving for θ_c gives:

$$\theta_c = 2 \arctan\left[\frac{1}{4}\left(\frac{K_I}{K_{II}} \pm \sqrt{\left(\frac{K_I}{K_{II}}\right)^2}\right) + 8\right]$$

Other criteria

Other criteria for mixed mode fracture have been proposed such as:

- The maximum energy release rate criterion
- The minimum strain energy criterion

SIF evaluation

- With the available criteria the direction of crack propagation can be determined provided the SIFs are known
- Several methods are available for computing the SIFs if the displacement, stress and strain fields are available
- The interaction integral method is one of the most widely used and will be presented in the following

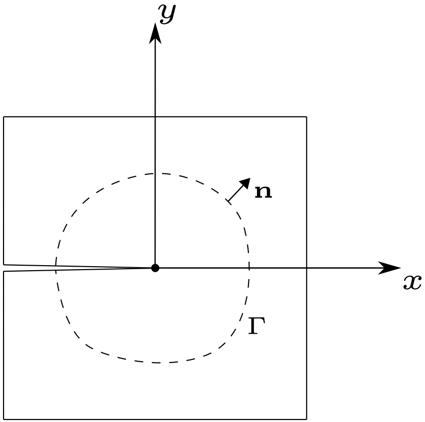
J integral

For the case of linear elasticity in the absence of body forces and crack tractions the J integral assumes the form:

$$J = \oint_{\Gamma} \left(\frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\epsilon} dy - \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$

where

- $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$
- $\mathbf{n}$ is the normal vector to the contour



J integral

It can be shown that the J integral is:

- Path independent
- Zero for a closed path not containing any singularities
- Equal to the energy release rate: $J = G = \dfrac{K_I^2 + K_{II}^2}{E'}$

We consider two states of the cracked body:

- The actual stress state of the body due to the applied loads with the SIFs, displacement, stress and strain fields denoted as: $K_I, K_{II}, \mathbf{u}, \boldsymbol{\sigma}, \epsilon$ respectively
- An auxiliary state where the displacement, stress and strain fields are the asymptotic fields of the Westergaard solution denoted $\mathbf{u}^{\text{aux}}, \boldsymbol{\sigma}^{\text{aux}}, \boldsymbol{\epsilon}^{\text{aux}}$ respectively, SIFs for that state are denoted as: $K_I^{\text{aux}}, K_{II}^{\text{aux}}$

Interaction integral

The J integral evaluated for the sum of the two states (denoted as J^s) is:

$$\begin{aligned} J^s &= \oint_\Gamma \left[\frac{1}{2} \left(\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\text{aux}} \right) : \left(\epsilon + \epsilon^{\text{aux}} \right) dy - \left(\mathbf{t} + \mathbf{t}^{\text{aux}} \right) \cdot \frac{\partial \left(\mathbf{u} + \mathbf{u}^{\text{aux}} \right)}{\partial x} d\Gamma \right] \\ &= J + J^{\text{aux}} + I \end{aligned}$$

Where I is the interaction integral:

$$I = \oint_\Gamma \left[\boldsymbol{\sigma}^{\text{aux}} : \epsilon dy - \left(\mathbf{t} \cdot \frac{\partial \mathbf{u}^{\text{aux}}}{\partial x} + \mathbf{t}^{\text{aux}} \cdot \frac{\partial \mathbf{u}}{\partial x} \right) d\Gamma \right]$$

Interaction integral

Considering that the J integral is equal to the energy release rate, we obtain for the sum of the two states:

$$\begin{aligned}
J^s &= \frac{(K_I + K_I^{\text{aux}})^2 + (K_{II} + K_{II}^{\text{aux}})^2}{E'} \\
&= \frac{K_I^2 + 2K_I K_I^{\text{aux}} + (K_I^{\text{aux}})^2 + K_{II}^2 + 2K_{II} K_{II}^{\text{aux}} + (K_{II}^{\text{aux}})^2}{E'} \\
&= \frac{K_I^2 + K_{II}^2}{E'} + \frac{(K_I^{\text{aux}})^2 + (K_{II}^{\text{aux}})^2}{E'} + 2\frac{K_I K_I^{\text{aux}} + K_{II} K_{II}^{\text{aux}}}{E'} \\
&= J + J^{\text{aux}} + 2\frac{K_I K_I^{\text{aux}} + K_{II} K_{II}^{\text{aux}}}{E'}
\end{aligned}$$

Interaction integral

Combining the two definitions of J^s we obtain:

$$I = 2\frac{K_I K_I^{\text{aux}} + K_{II} K_{II}^{\text{aux}}}{E'}$$

Appropriate choice of the values of K_I^{aux} and K_{II}^{aux} yields the SIF values for the actual state.

For instance $K_I^{\text{aux}} = 1$ and $K_{II}^{\text{aux}} = 0$ yields:

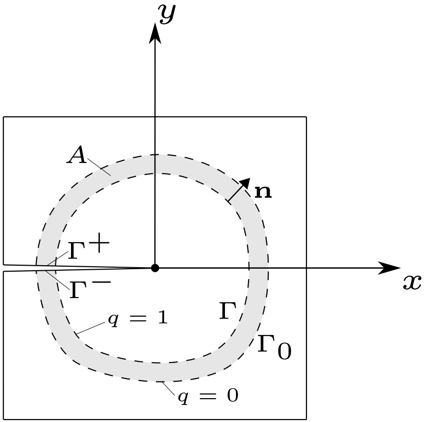
$$K_I = E' \frac{I}{2}$$

The current form of the interaction integral is a contour integral which is not very well suited for FE calculations.

In order to obtain a domain integral form, we introduce a function q such that:

$$\begin{aligned} q &= 1 \qquad \text{in} \qquad \Gamma \\ q &= 0 \qquad \text{in} \qquad \Gamma^0 \end{aligned}$$

Definition of the contour Γ^0:



we define $C = \Gamma + \Gamma_0 + \Gamma^+ + \Gamma^-$

Interaction integral - Domain form

Then the interaction integral can be substituted with:

$$I = \oint_C \left[\boldsymbol{\sigma}^{\text{aux}} : \boldsymbol{\epsilon} dy - \left(\mathbf{t} \cdot \frac{\partial \mathbf{u}^{\text{aux}}}{\partial x} + \mathbf{t}^{\text{aux}} \cdot \frac{\partial \mathbf{u}}{\partial x} \right) d\Gamma \right] q$$

By applying the divergence theorem, the domain form of the interaction integral is obtained:

$$I = \int_A \left[\left(\boldsymbol{\sigma}^{\text{aux}} : \boldsymbol{\epsilon} \right) \mathbf{e_1} - \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}^{\text{aux}}}{\partial x} + \boldsymbol{\sigma}^{\text{aux}} \cdot \frac{\partial \mathbf{u}}{\partial x} \right) \right] \nabla q dA$$

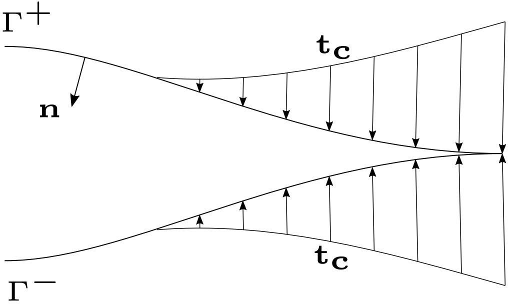
where $\mathbf{e_1}$ is the unit vector along the x direction.

Cohesive zone models

- Methods mentioned so far assume linear elastic material behavior and apply mostly to brittle materials
- When nonlinear effects are present in an area of considerable size compared to the length scale of the problem these methods will fail
- Next a simple method will be briefly presented which:
  - Successfully predicts the response of structures subjected to size effects
  - Is easy to incorporate into numerical models

Cohesive zone models

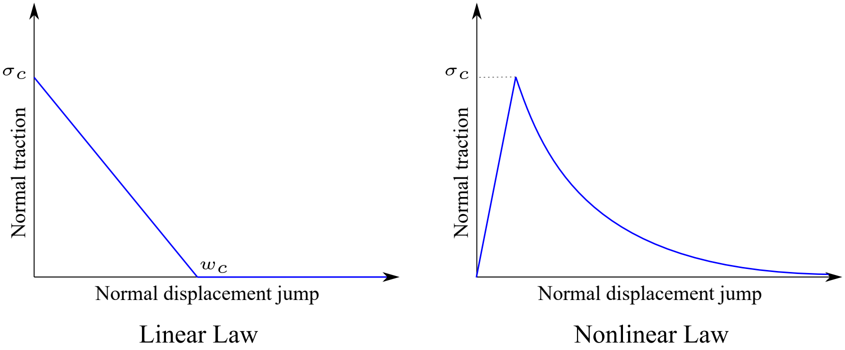
In this model some forces are introduced which resist separation of the surfaces:



Those forces gradually reduce to a value of zero which corresponds to full separation

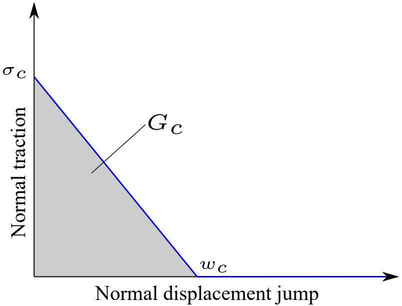
Cohesive zone models

Cohesive forces are related to the crack opening displacements through a traction-separation law:

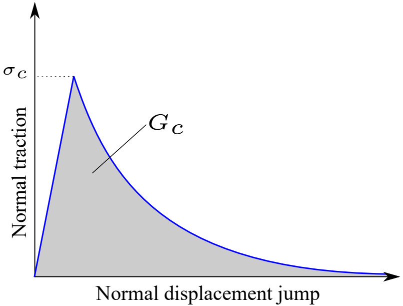


Cohesive zone models

The area under the curve is equal to the critical energy release rate G_c:



Linear Law



Nonlinear Law