Method of Finite Elements II
Modeling oHysteresis

Prof. Dr. Eleni Chatzi, Dr. K. Agathos, Dr. G. Abbiati
Institute of Structural Engineering (IBK)
Department of Civil, Environmental, and Geomatic Engineering (DBAUG)
ETH Zürich
• Learn what **hysteresis** is
• Learn how to model it using the **Bouc-Wen** Model
• Modeling different effects including **strength deterioration**, **stiffness deterioration** and pinching
• Solve a nonlinear hysteretic problem under dynamic loads using MATLAB
• Overview further models for modeling hysteresis
The term hysteresis is used to designate the dependence of the state of a system on its history. We may discriminate between two types of hysteretic phenomena:

- **Rate-dependent hysteresis** usually occurs as a simple lag between input and output. If the input is reduced to zero, the output continues to respond for a finite time. When rate-dependent hysteresis is due to dissipative effects like friction, it is associated with power loss.

- Rate-independent hysteresis indicates a persistent memory of the system to its past (loading history/response) that remains after the transients have died out.
Modeling Hysteresis

Kechidi & Bourahla, 2016
stiffness degradation
strength deterioration
pinching

Kechidi & Bourahla, 2016

Modeling Hysteresis
Some Definitions

- **Softening**: Slope of hysteresis loop decreases with increasing displacement
- **Hardening**: Slope of hysteresis loop increases with increasing displacement
- **Pinching**: Pinching is a sudden loss of stiffness, primarily caused by damage and interaction of structural components under a large deformation. It is caused by closing (or unclosed) cracks and yielding of compression reinforcement before closing the cracks in reinforced concrete members, slipping at bolted joints (in steel construction) and loosening and slipping of the joints caused by previous cyclic loadings in timber structures with dowel-type fasteners (e.g. nails and bolts).
- **Stiffness degradation**: Progressive loss of stiffness in each loading cycle
- **Strength deterioration**: Degradation of strength when cyclically loaded to the same displacement level.

(source)
The Bouc-Wen model essentially occurs as a superposition of:

- a linear and
- a nonlinear (hysteretic) spring.
The Bouc-Wen hysteretic Model

\[ \sigma_1 = \alpha E \varepsilon \]
\[ \sigma_2 = (1 - \alpha) E z \]
\[ \sigma = \sigma_1 + \sigma_2 \]
\[ z = \varepsilon - x \]

\[ \sigma(x,t) = aE \varepsilon(x,t) + (1-a) E z(x,t) \quad \text{with} \quad a = \frac{E_P}{E} \]

Stress-Strain relationship

Modeling Hysteresis
The Bouc - Wen hysteretic Model

\[ \sigma_1 = \alpha E \varepsilon \]

\[ \sigma_2 = (1 - \alpha) E z \]

\[ \sigma = \sigma_1 + \sigma_2 \]

\[ z = \varepsilon - x \]

Stress-Strain relationship

\[ \sigma(x,t) = aE \varepsilon(x,t) + (1 - a) E z(x,t) \quad \text{with} \quad a = \frac{E_P}{E} \]
The Bouc - Wen hysteretic Model

$z$ is the so-called hysteretic parameter, whose evolution is governed by:

For the bilinear plasticity model:

- $\varepsilon < \varepsilon_y \rightarrow z = \varepsilon$
- $\varepsilon = \varepsilon_y \rightarrow z = \varepsilon_y$
- $\varepsilon > \varepsilon_y \rightarrow z = \varepsilon_y$

Stress-Strain relationship

$$\sigma(x,t) = aE\varepsilon(x,t) + (1-a)Ez(x,t) \quad \text{with} \quad a = \frac{E_p}{E}$$

Modeling Hysteresis
The Bouc - Wen hysteretic Model

\[ z \] is the so-called hysteretic parameter, whose evolution is governed by:

\[ \dot{z} = \dot{\varepsilon} \left[ 1 - \left| \frac{z}{z_Y} \right|^n \right] \left( \beta + \gamma \text{sgn}(\dot{z}) \right) \]

- \( \varepsilon < \varepsilon_y \rightarrow z = \varepsilon \)
- \( \varepsilon = \varepsilon_y \rightarrow z = \varepsilon_y \)
- \( \varepsilon > \varepsilon_y \rightarrow z = \varepsilon_y \)

**Stress-Strain relationship**

\[
\sigma(x,t) = aE \varepsilon(x,t) + (1 - a)E z(x,t) \quad \text{with} \quad a = \frac{E_p}{E}
\]
The Bouc - Wen hysteretic Model

Dynamic evolution law

Stress-Strain relationship

\[ \sigma(x,t) = aE \varepsilon(x,t) + (1-a) Ez(x,t) \quad \text{with} \quad a = \frac{E_p}{E} \]

Bouc-Wen evolution equation

\[ \dot{z} = \dot{\varepsilon} \left[ 1 - \frac{z}{z_Y} \right]^n \left( \beta + \gamma \text{sgn}(\dot{\varepsilon}z) \right) \]

\( n \) controls the smoothness of the model

- \( n >> \) bilinear law
- \( n \approx 2 \) smooth

Modeling Hysteresis
The Bouc - Wen hysteretic Model

Let’s look at how this works:

Bouc-Wen evolution equation

\[
\dot{z} = \dot{\varepsilon} \left[ 1 - \frac{z}{z_Y} \right]^n (\beta + \gamma \text{sgn}(\dot{\varepsilon}z))
\]

\text{sgn} \text{ controls the branch we move in:}
\sigma, \varepsilon, z \text{ have the same sign } \text{WHY?}

\dot{\varepsilon} > 0, z > 0 \Rightarrow \text{sgn}(\dot{\varepsilon}z) = 1
\dot{\varepsilon} > 0, z < 0 \Rightarrow \text{sgn}(\dot{\varepsilon}z) = -1

\ldots

Modeling Hysteresis

Chair of Structural Mechanics
The Bouc - Wen hysteretic Model

Apart from its formulation on the material law level the Bouc-Wen model can also be used in a macroscopic sense, i.e., at the element level. Assume an SDOF system, e.g. a cantilever.

The following set of differential equations governs the motion of a SDOF oscillator with Bouc–Wen hysteresis:

\[ m\ddot{x} + c\dot{x} + a k x + (1 - a) k z = F(t) \]

and

\[ \dot{z} = A \dot{x} - \beta |\dot{x}| z |z|^{n-1} - \gamma \dot{x} |z|^n \]
The Bouc - Wen hysteretic Model

Apart from the material law level the Bouc-Wen model can also be used in a macroscopic sense, i.e., at the element level.

Assume an SDOF system, e.g. a cantilever.

The following set of differential equations governs the motion of a SDOF oscillator with Bouc–Wen hysteresis:

\[ m\ddot{x} + c\dot{x} + akx + (1-a)kz = F(t) \]
and
\[ \dot{z} = A\dot{x} - \beta |\dot{x}|z|z|^{n-1} - \gamma \dot{x}|z|^n \]
The Bouc - Wen hysteretic Model

Characteristics of the hysteresis

\[ \dot{z} = A\dot{x} - \beta |\dot{x}| z^{n-1} - \gamma \dot{z} z^n \]

where

\[ z_{\text{max}} = \pm \left( \frac{A}{\beta + \gamma} \right)^{\frac{1}{n}} \]

Parameter A simply controls the hysteresis amplitude. Along with \( n \), the hysteretic parameters \( \beta, \gamma \), determine the basic shape of the hysteresis. Their absolute value is not of interest, but rather their sum/difference which may define a hardening or softening relationship.

Modeling Hysteresis
The Bouc - Wen hysteretic Model

Characteristics of the hysteresis

for $n = 1$:

$\beta + \gamma > 0$ \hspace{1cm} weak softening

$\beta - \gamma > 0$ \hspace{1cm} weak softening on loading,

$\beta + \gamma = 0$ \hspace{1cm} mostly linear unloading

$\beta + \gamma > \beta - \gamma$ \hspace{1cm} strong softening loading/unloading,

$\beta - \gamma < 0$ \hspace{1cm} narrow hysteresis

Heine 2001, Foliente 1993
The Bouc - Wen hysteretic Model

Characteristics of the hysteresis

for $n = 1$:

\[
\begin{align*}
\beta + \gamma &= 0 \\
\beta - \gamma &= 0 \\
\beta + \gamma &< 0 \\
\beta + \gamma &> \gamma - \beta
\end{align*}
\]

- weak hardening
- strong hardening

Heine 2001, Foliente 1993
The Bouc - Wen hysteretic Model

Characteristics of the hysteresis

\[ \begin{align*}
\beta + \gamma > 0, \gamma - \beta < 0 \\
\beta + \gamma > 0, \gamma - \beta = 0 \\
\beta + \gamma > \gamma - \beta > 0 \\
\beta + \gamma = 0, \gamma - \beta < 0 \\
0 > \beta + \gamma > \gamma - \beta
\end{align*} \]

Sengupta, Li 2013
The Bouc - Wen hysteretic Model

MATLAB DEMO

How to solve this nonlinear system of ODEs?

**Hint:** Use a state-space formulation and ode45

\[ m\ddot{x} + c\dot{x} + akx + (1-a)kz = F(t) \]

and

\[ \dot{z} = A\dot{x} - \beta |\dot{x}| z^{n-1} - \gamma \dot{x} |z|^{n} \]
The Bouc - Wen hysteretic Model

Hysteretic Energy

The hysteretic energy absorption is used by the BW model to simulate degradation. The energy absorbed by the hysteretic element is:

\[ \varepsilon(t) = (1 - a)k \int_{u(0)}^{u(T)} z(t)du = (1 - a)k \int_0^T z(t)\dot{u}(t)dt \]
The Bouc - Wen hysteretic Model

Modeling Deterioration Effects

Baber and Wen proposed a model with degradation:

\[
\dot{z} = \frac{A\dot{x} - \nu(t)\left(\beta |\dot{x}|z|z|^{n-1} + \gamma \dot{x}|z|^n\right)}{\eta(t)}
\]

where

\[
\nu(t) = 1.0 + \delta_{\nu}\varepsilon(t), \quad \eta(t) = 1.0 + \delta_{\eta}\varepsilon(t)
\]

strength deterioration

stiffness degradation

\[
\varepsilon(t) = \int_0^t z\dot{x}dt
\]

measure of the absorbed hysteretic energy
The Bouc - Wen hysteretic Model

Modeling Stiffness Degradation Effects

\[ \delta_\eta = 0.0 \]

\[ \delta_\eta = 0.1 \]

\[ \delta_\eta = 0.5 \]

Heine 2001

Modeling Hysteresis
The Bouc - Wen hysteretic Model

Modeling Strength Deterioration Effects

\[ \delta_v = 0.0 \quad \delta_v = 0.01 \quad \delta_v = 0.02 \]

\[ \begin{align*}
&f_h \quad u \\
&-8 \quad -6 \quad -4 \quad -2 \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \\
\end{align*} \]

Heine 2001

Modeling Hysteresis

Chair of Structural Mechanics
The Bouc - Wen hysteretic Model

Modeling Pinching Effects

Baber and Noori proposed a model with pinching:

\[
\dot{z} = A\dot{x}_1 - \beta |\dot{x}_1| z |z|^{n-1} + \gamma \dot{x}_1 |z|^n, \quad \dot{x}_2 = \sqrt{\frac{2}{\pi}} \frac{\delta_s \varepsilon(t)}{s_1} \dot{z} e^{-\left(\frac{z^2}{2s_2^2}\right)} , \quad x = x_1 + x_2
\]

Modeling Hysteresis
The Bouc - Wen hysteretic Model

Modeling Pinching Effects

Baber and Noori proposed a model with pinching:

\[ \dot{z} = A\dot{x}_1 - \beta |\dot{x}_1| z |z|^{n-1} + \gamma \dot{x}_1 |z|^n \]

\[ \dot{x}_2 = \sqrt{\frac{2}{\pi}} \frac{s_1 \varepsilon(t)}{s_1} \dot{z} e^{-z^2/2s_2^2}, \quad x = x_1 + x_2 \]

Parameters \( A, \beta, \gamma \) and \( n \) control the hysteresis shape. \( s_1 \) and \( s_2 \) reflect the degree of pinching and sharpness of hysteresis loops

Modeling Hysteresis
The Bouc - Wen hysteretic Model

Modeling combined Pinching & Deterioration effects

Baber and Noori proposed a generalized model with pinching & degradation:

\[
\dot{z} = \frac{h(z)}{\eta(t)} \left( \dot{x} - \nu(t) \left( \beta |\dot{x}|z^{n-1} + \gamma |\ddot{x}|z^{n} \right) \right)
\]

with pinching function

\[
h(z) = 1 - \zeta_1(\varepsilon) e^{-\frac{[z \text{sgn}(\dot{x}) - qz]}{\zeta_2(\varepsilon)}}
\]

and

\[
\zeta_1(\varepsilon) = \zeta_s \left(1 - e^{-p \varepsilon} \right)
\]

\[
\zeta_2(\varepsilon) = (\psi + \delta \varepsilon) \left(\lambda + \zeta_1(\varepsilon) \right)
\]
A BW Beam Element model

The model can be visualized as a linear and a nonlinear element in parallel:

\[ F = F_1 + F_2 \]

\[ F_1 = \alpha ku \]

\[ F_2 = (1 - \alpha)kz \]

The relation between generalized moments and curvatures is given by:

\[ M(t) = M_y \left[ \alpha \frac{\phi(t)}{\phi_y} + (1 - \alpha)z(t) \right] \]

where \( M_y \) is the yield moment; \( \phi_y \) is the yield curvature; \( \alpha \) is the ratio of the post-yield to the initial elastic stiffness and \( z(t) \) is the hysteretic component defined as:

\[ \dot{z}(t) = f(\phi(t), z(t)) \frac{1}{\phi_y} \] or alternatively \[ \frac{dz}{d\phi} = K_z \frac{1}{\phi_y} \] where \[ K_z = \left[ A - B \frac{1 + \text{sign}(d\phi)}{2} \left( \frac{|z(t)| + z(i)}{2} \right)^{n_g} \right] - C \frac{1 + \text{sign}(d\phi)}{2} \left( \frac{|z(t)| - z(i)}{2} \right)^{n_c} - D \frac{1 - \text{sign}(d\phi)}{2} \left( \frac{|z(t)| + z(i)}{2} \right)^{n_d} - E \frac{1 - \text{sign}(d\phi)}{2} \left( \frac{|z(t)| - z(i)}{2} \right)^{n_g} \]
A BW Beam Element model

In the above expression A, B, C, D & E are constants which control the shape of the hysteretic loop for each direction of loading, while the exponents \( n_B \), \( n_C \), \( n_D \) & \( n_E \) govern the transition from the elastic to the plastic state. Small values of \( n_i \) lead to a smooth transition, however as \( n_i \) increases the transition becomes sharper tending to a perfectly bilinear behavior in the limit \( (n_i \to \infty) \).

Finally, the flexural stiffness can be expressed as:

\[
K = EI = \frac{dM}{d\phi} = M_y \left[ \alpha \frac{1}{\phi_y} + (1 - \alpha) \frac{dz}{d\phi} \right] = M_y \left[ \alpha \frac{1}{\phi_y} + (1 - \alpha)K_z \frac{1}{\phi_y} \right] = EI_0 \left[ \alpha + (1 - \alpha)K_z \right]
\]
A BW Beam Element model

Stiffness Degradation

The stiffness degradation that occurs due to cyclic loading is taken into account by introducing the parameter $\eta$ into the differential equation:

$$\frac{dz}{d\phi} = \frac{K_z}{\eta} \frac{1}{\phi_y} \rightarrow K = EI_0 \left[ \alpha + (1-\alpha) \frac{K_z}{\eta} \right] \quad \text{where} \quad \eta = 1.0 + S_k \frac{\mu_{\text{max}} + \mu}{2}$$

The parameter $\eta$ depends on the current, $\mu = \phi/\phi_y$, and maximum achieved plasticity, $\mu_{\text{max}} = \phi_{\text{max}}/\phi_y$. $S_k$ is a constant which controls the rate of stiffness decay. Common values for $S_k$ are 0.1 and 0.05.

Strength Deterioration

The strength deterioration is simulated by multiplying the yield moment $M_y$ with a degrading parameter $S_\beta$:

$$M(t) = S_\beta M_y \left[ \alpha \frac{\phi(t)}{\phi_y} + (1-\alpha) z(t) \right]$$

The parameter $S_\beta$ depends on the damage of the section which is quantified by the Damage Index DI:

$$S_\beta = 1 - S_d DI \quad \text{where} \quad DI = \frac{\mu_{\text{max}} - 1}{\mu_c - 1} \left[ 1 - \frac{1}{S_p 1 \int dE_{\text{diss}} / 4E_{\text{mon}}} \right]^{S_{p2}}$$
A BW Beam Element model

Strength Deterioration

In the above expression $S_d, S_{p1}, S_{p2}$ are constants controlling the amount of strength deterioration; $\mu_c$ is the maximum plasticity that can be reached, $\mu_c = \phi_u / \phi_y$; $\int dE_{diss}$ is the energy dissipated before unloading occurs and finally $E_{mon}$ is the amount of energy absorbed during a monotonic loading until failure as shown in Figure 4.

\[ E_{el} = \frac{M^2}{2K} \]

**Figure 4.** Dissipated Energy ($E_{diss}$) and Monotonic Energy ($E_{mon}$).

The model can also be appropriately modified to simulate pinching.
A BW Beam Element model – Solution Procedure

Global Level

\[ M\ddot{u} + C\dot{u} + F_h(u) = R^{\text{Newmark}} t Du = \ldots \]

Local Level

\[
\begin{bmatrix}
\Delta F_A \\
\Delta M_A \\
\Delta F_B \\
\Delta M_B
\end{bmatrix}
= K_e \begin{pmatrix} EI_A, EI_B \end{pmatrix}
\begin{bmatrix}
\Delta w_A \\
\Delta \theta_A \\
\Delta w_B \\
\Delta \theta_B
\end{bmatrix}
\]

\[ \Delta \phi_A = -\frac{\Delta M_A}{EI_A} \Rightarrow \phi_A = \phi_A^- + \Delta \phi_A \]

\[ \Delta \phi_B = \frac{\Delta M_B}{EI_B} \Rightarrow \phi_B = \phi_B^- + \Delta \phi_B \]

\[ \text{derived } EI, M, F_h^e (\text{actual}) \]

Calculated curvature from previous step

Chair of Structural Mechanics
A BW Beam Element model – Solution Procedure

Use the Newton-Raphson (iterative) method until convergence, within the Newmark step

Calculating \( tK \) can be computationally costly

Iterate until convergence for load step \( t \)
Alternative Models of Hysteresis

Clough Bilinear Stiffness Degrading Model
*simulates dominant bending behaviour*

Bilinear Origin-Oriented Model
*simulates shear behaviour*

Source: eqsols.com (Bispec)
Alternative Models of Hysteresis

- Effective yield strength and rotation ($M_y$ and $\theta_y$)
- Effective stiffness $K_e = M_y/\theta_y$
- Capping strength and associated rotation for monotonic loading ($M_c$ and $\theta_c$)
- Pre-capping rotation capacity for monotonic loading $\theta_p$
- Post-capping rotation capacity $\theta_{pc}$
- Residual strength $M_r = kM_y$
- Ultimate rotation capacity $\theta_u$

Modified Ibarra-Medina-Krawinkler Deterioration Model

calibrated on steel beam-to-column connections

Takeda Bi-linear Degrading Stiffness

simulates dominant bending behaviour

Source: berkeley.edu (Opensees)  
Source: civil.canterbury.ac.nz
Alternative Models of Hysteresis

Fukada Trilinear Degrading Stiffness

modeling plastic hinges in RC beams

Stewart Degrading Stiffness

Initially used for representation of timber framed structural walls sheathed in plywood nailed to the framework, but has been further successfully applied to RC columns with plain round reinforcement bars

Source: civil.canterbury.ac.nz