The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

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What is Contact?

- Physically, contact stress is transmitted between two bodies when they touch.
- Numerically, contact is a severely discontinuous form of non-linearity.

Difficulties

- Complex non-linear behaviour = contact between two or more bodies.
- Relative sliding of the surfaces has to be evaluated iteratively.
- Deformable-to-deformable body contact generates non-linear time-dependent BC.
Contact Discretization

- Node-to-Surface (Implicit only)
- Surface-to-Surface (Implicit only)
- Node-to-Face (Explicit only)
- Edge-to-Edge
The Contact Problem

Node-to-Surface (strict master/slave formulation)
Node-to-Surface

Contact is enforced between a slave node and master surface facets local to the node:
- The opening/penetration distance is measured along the normal to the master surface
- A nodal area is assigned to each slave node to convert contact forces to contact stresses
- The more refined surface should act as the slave surface
- The stiffer body should be the master
The Contact Problem

Surface-to-Surface

More master surface nodes are involved in contact, reducing the likelihood of penetration.
Surface-to-Surface

Contact is enforced between the slave node and a larger number of master surface facets around it:

- The opening/penetration distance is measured along the slave surface facet normal
- Sliding is measured perpendicular to the slave normal
Contact Discretization comparison: Abaqus/Explicit

Undetected penetrations of master nodes into the slave surface do not occur with surface-to-surface discretization:

Source: Abaqus Analysis User’s Manual
The Contact Problem

Contact discretizations in Abaqus/Explicit

- **Node-to-Face**: No distinction is made between master/slave surfaces as in Node-to-Surface (i.e., contact is enforced everywhere).

- **Edge-Edge**: It is very effective in enforcing contact that cannot be detected as penetrations of nodes into faces.

Source: Abaqus Analysis User’s Manual
Surface Description

- Discrete Surface: Discontinuities in the surface normal direction at surface facet boundaries can contribute to convergence difficulties.

- Smooth Surface: Surface smoothing is used to reduce the discretization error associated with faceted representations of curved surfaces.
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Hard Contact Enforcement Methods

- Direct Enforcement Method: Strict enforcement of pressure-penetration relationship using Lagrange multiplier method (only implicit).

- Penalty method: approximate enforcement using penalty stiffness.
Direct Enforcement

Variational formulation for a steady-state analysis without contact:

$$\Pi = \frac{1}{2} U^T K U - U^T F$$

(1)

Contact Constraint:

$$U_i = U_i^*$$

(2)

Variational formulation for a steady-state analysis with contact enforcement using Lagrange multiplier method:

$$\Pi^* = \frac{1}{2} U^T K U - U^T F + \lambda (U_i - U_i^*)$$

(3)
The Contact Problem

Direct Enforcement

Equilibrium Condition $\delta \Pi^* = 0$:

$$\delta \Pi^* = \delta U^T K U - \delta U^T F + \lambda \delta U_i + \delta \lambda (U_i - U_i^*) = 0$$  \hspace{1cm} (4)

The above relationship can be written as:

$$K U + \lambda e_i = F$$ \hspace{1cm} (5)

$$e_i^T U = U_i^*$$ \hspace{1cm} (6)

In matrix form:

$$\begin{bmatrix} K & e_i \\ e_i^T & 0 \end{bmatrix} \times \begin{bmatrix} U \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ U_i^* \end{bmatrix}$$ \hspace{1cm} (7)

$\lambda$ is the vector of Lagrange multiplier degrees of freedom (constraint forces) *One per constraint.
The Contact Problem

Direct Enforcement

The contact virtual work contribution is:

\[ \delta \Pi^c = \lambda \delta U_i + \delta \lambda (U_i - U_i^*) \]  

This expression is written in Abaqus Theory Manual as:

\[ \delta \Pi^c = \delta ph + p \delta h \]  

where \( p \) is the Lagrangian multiplier, and \( h \) is the "overclosure".

Hard Contact

- \( p=0 \) for \( h < 0 \) contact is open
- \( h=0 \) for \( p = 0 \) contact is closed
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Direct Enforcement

ADVANTAGES:

- Accuracy: The constraints are satisfied exactly

DISADVANTAGES:

- Adds cost to the equation solver
- Potential convergence problems
Penalty Method

The right hand side of the potential function $\Pi$ is amended in the following manner:

$$\Pi^* = \frac{1}{2}U^T K U - U^T F + \frac{\alpha}{2}(U_i - U_i^*)^2$$

(10)

I use a large $\alpha$ in order to make $U_i = U_i^*$, i.e., $\alpha \gg max(k_{ii})$

Equilibrium condition $\delta \Pi^* = 0$:

$$\delta \Pi^* = \delta U^T K U - \delta U^T F + \alpha(U_i - U_i^*)\delta U_i = 0$$

(11)

$$(K + \alpha e_i e_i^T)U = F + \alpha U_i^* e_i$$

(12)
The Contact Problem

Penalty Method

The Penalty method corresponds to having a spring to bring back the penetrating node to the surface.
The Contact Problem

Penalty Method

ADVANTAGES:

- Convergence rates significantly improve
- Better equation solver performance

DISADVANTAGES:

- Small amount of penetration (typically insignificant)
- In some cases, the penalty stiffness needs to be adjusted
Strain-free adjustments of initial overclosures

- Within the penetration tolerance, all initial overclosures are treated with strain-free adjustments.

- Initial overclosures can be due to pre-processing errors or discretization of curved surfaces.

Source: Abaqus Analysis User’s Manual
The Contact Problem

Friction Models

Coulomb friction model

\[
\tau_{cr} = \mu p
\]

(13)

\[
\tau_{cr} = \min(\mu p, \tau_{max})
\]

(14)

Source: Abaqus Analysis User’s Manual
The Contact Problem

Friction Models

Additional features:

- Friction coefficient dependence on slip rate
- Friction coefficient dependence on contact pressure
- Anisotropic friction

\[ \mu = \mu_k + (\mu_s - \mu_k) e^{-a\tilde{\gamma}_{eq}} \]
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**Friction Enforcement**

- Lagrangian multiplier method
- Penalty method
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Exact stick formulation

Lagrange multipliers are used to enforce exact sticking conditions.

The virtual work due to friction is evaluated as:

\[ \delta \Pi = \int_S (\tau_i \delta \gamma_i + \Delta \gamma_i \delta q_i) dS \]  (15)

\( q_i \) are the Lagrangian multipliers used to enforce exact stick 
(\( \Delta \gamma_i = 0 \)).

- If \( \tau_{eq} > \tau_{crit} \) the element passes from sticking to slipping.
- If \( \Delta \gamma_i \tau_i(t) < 0 \) the element passes from slipping to sticking
Penalty method

- It approximates stick with stiff elastic behaviour. In Abaqus/Explicit by default the same penalty stiffness used in hard contact is used for frictional constraints. On the contrary in Abaqus/Standard (Implicit) it depends on the elastic slip.

- The elastic slip $\gamma_{\text{crit}}$ is calculated as: $\gamma_{\text{crit}} = F_f \bar{l}_i$, where $F_f$ is the slip tolerance, and $\bar{l}_i$ is the ”characteristic contact surface length”.

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Elastic behaviour ($\tau_{eq} < \tau_{crit}$):

$$\gamma_i^{el}(t + \Delta t) = \gamma_i^{el}(t) + \Delta \gamma_i$$ (16)

$$\tau_i = G\gamma_i^{el} = \frac{\tau_{crit}}{\gamma_{crit}} \gamma_i^{el} = \frac{\mu p}{\gamma_{crit}} \gamma_i^{el}$$ (17)

$$d\tau_i = Gd\gamma_i + \frac{\tau_i}{\tau_{crit}} (\mu p + \frac{\partial u}{\partial p} p) dp$$ (18)

The contributions from the contact pressure $p$ are non-symmetric!

Plastic behaviour ($\tau_{eq} > \tau_{crit}$):

$$\Delta \gamma_i = \gamma_i^{el}(t + \Delta t) - \gamma_i^{el}(t) + \Delta \gamma_i^{sl} = \gamma_i^{el} - \bar{\gamma}_i^{el} + \Delta \gamma_i^{sl}$$ (19)
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The shear stress at the end of the increment is evaluated with the elastic relationship:

\[ \tau_i(t + \Delta t) = \tau_i = G\gamma_i^{el} = \frac{\tau_{crit}}{\gamma_{crit}} \gamma_i^{el} \]  

(20)

Slip increment:

\[ \Delta \gamma_i^{sl} = \frac{\tau_i}{\tau_{crit}} \Delta \gamma_{eq} \]  

(21)

Replacing \( \gamma_i^{el} \) and \( \Delta \gamma_i^{sl} \) in eq. 19 yields:

\[ \Delta \gamma_i = \frac{\tau_i}{\tau_{crit}} \gamma_{crit} - \gamma_i^{el} + \frac{\tau_i}{\tau_{crit}} \Delta \gamma_{eq} \]  

(22)

\[ \tau_i = \frac{\gamma_i^{el} + \Delta \gamma_i}{\gamma_{crit} + \Delta \gamma_{eq}} \tau_{crit} \]  

(23)
The critical stress equality yields:

\[ \tau_{crit} = G(\gamma_{eq}^{pr} - \Delta \gamma_{eq}^{sl}) = \frac{\tau_{crit}}{\gamma_{crit}}(\gamma_{eq}^{pr} - \Delta \gamma_{eq}^{sl}) \]  

(24)

where \( \gamma_{eq}^{pr} \) is the "equivalent elastic predictor strain"
The Contact Problem

\[ \tau_{\text{crit}} = G(\gamma_{eq}^{pr} - \Delta \gamma_{eq}^{sl}) = \frac{\tau_{\text{crit}}}{\gamma_{\text{crit}}} (\gamma_{eq}^{pr} - \Delta \gamma_{eq}^{sl}) \]  

(25)

\[ \Delta \gamma_{eq}^{sl} = \gamma_{eq}^{pr} - \gamma_{\text{crit}} \]  

(26)

Replacing eq. 26 into eq. 23 yields:

\[ \tau_i = \frac{\gamma_i^{pr}}{\gamma_{\text{crit}} + \Delta \gamma_{eq}^{sl}} \tau_{\text{crit}} = \frac{\gamma_i^{pr}}{\gamma_{eq}} \tau_{\text{crit}} = n_i \tau_{\text{crit}} \]  

(27)

where \( n_i \) is the normalized slip direction.

The iterative solution scheme for \( \tau_{\text{crit}} \) as a function of the slip rate \( (\dot{\gamma}_{eq}^{sl} = \Delta \gamma_{eq}^{sl}/\Delta t) \) yields:

\[ \Delta \tau_i = (\delta_{ij} - n_in_j) \frac{\tau_{\text{crit}}}{\gamma_{eq}^{pr}} d\gamma_j + n_i(\mu + p \frac{\partial \mu}{\partial p}) dp + n_in_j \frac{p}{\Delta t} \frac{\partial \mu}{\partial \dot{\gamma}_{eq}} d\gamma_j \]  

(28)

The unsymmetric terms may have a strong effect on the speed of convergence of the Newton scheme!
The Contact Problem

Contact Algorithm (Implicit calculations)

Newton-Raphson iterative scheme

\[
(K_T)^i_{n+1} \Delta U^n_{i+1} = (F_{int})^i_{n+1} + (F_{ext})_{n+1} + (R_c(U^n_{i+1}))^{i+1}_{n+1}
\]

\[
U^{i+1}_{n+1} = U^i_{n+1} + \Delta U^{i+1}_{n+1}
\]

where \(K_T\) is the tangent stiffness matrix, \(i\) refers to the ongoing iteration with the Newton-Raphson process and \(n\) refers to the loading increment.

The contact forces vector \(R_c\) depends on \(U\) which influences both the contact surface shape and the magnitude of the contact reaction.
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Inverting the tangent stiffness matrix yields:

\[ \Delta U_{n+1}^{i+1} = ((K_T)^{i}_{n+1})^{-1})(F_{int}^{i}_{n+1} + (F_{ext})_{n+1}) \]

... + 

\[ ... + ((K_T)^{i}_{n+1})^{-1})(R_c(U_{n+1}^{i+1}))^{i+1}_{n+1} \]

which may be written in a simpler way:

\[ \Delta U_{n+1}^{i+1} = (\Delta U_{lib})^{i+1}_{n+1} + (\Delta U_{c})^{i+1}_{n+1} \]

with:

\[ (\Delta U_{lib})^{i+1}_{n+1} = ((K_T)^{i}_{n+1})^{-1})(F_{int}^{i}_{n+1} + (F_{ext})_{n+1}) \]

\[ (\Delta U_{c})^{i+1}_{n+1} = ((K_T)^{i}_{n+1})^{-1})(R_c(U^{i+1}_{n+1}))^{i+1}_{n+1} \]

The displacement is split into two parts, one independent from the contact problem (prediction), and a term depending exclusively on contact (correction).
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Institute of Structural Engineering
Method of Finite Elements II
The Contact Problem
Contact in Explicit Codes

- Calculation is advanced explicitly element-by-element (No need to assemble a global stiffness matrix and no Newton iterations are performed).

- No convergence problems related to faceted representation of curves (smoothing is not relevant).

- Contact forces do not depend on the displacements (no iterative process is carried out).

- Easier to deal with sliding friction because calculations are advanced explicitly element-by-element.

- Time step is very small and therefore is suitable to analyse short contact dynamic problems where friction plays an important role, i.e., impact.