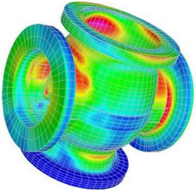
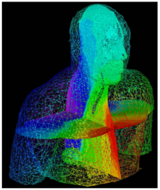
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

Prof. Dr. Eleni Chatzi

Lecture 1 - 17 September, 2015





Course Information

Instructor
Prof. Dr. Eleni Chatzi, email: chatzi@ibk.baug.ethz.ch
Office Hours: HIL E14.3, Wednesday 10:00-12:00 or by email

Assistant
Adrian Egger, HIL E13.1, email: egger@ibk.baug.ethz.ch

Course Website

Lecture Notes and Homeworks will be posted at:
http://www.chatzi.ibk.ethz.ch/education/method-of-finite-elements-ii.html

Suggested Reading

- "Nonlinear Finite Elements for Continua and Structures", by T. Belytschko, W. K. Liu, and B. Moran, John Wiley and Sons, 2000
- "The Finite Element Method: Linear Static and Dynamic Finite Element Analysis", by T. J. R. Hughes, Dover Publications, 2000
- Lecture Notes by Carlos A. Felippa
  Nonlinear Finite Element Methods (ASEN 6107): `http://www.colorado.edu/engineering/CAS/courses.d/NFEM.d/Home.html`

Course Outline

- Review of the Finite Element method - Introduction to Non-Linear Analysis
- Non-Linear Finite Elements in solids and Structural Mechanics
  - Overview of Solution Methods
  - Continuum Mechanics & Finite Deformations
  - Lagrangian Formulation
  - Structural Elements
- Dynamic Finite Element Calculations
  - Integration Methods
  - Mode Superposition
- Eigenvalue Problems
- Special Topics
  - The Scaled Boundary Element & Extended Finite Element methods

Grading Policy

Performance Evaluation - Homeworks (100%)

Homework

- Homeworks are due in class within 3 weeks after assignment
- Computer Assignments may be done using any coding language (MATLAB, Fortran, C, MAPLE) - example code will be provided in MATLAB
- Commercial software such as ABAQUS and SAP will also be used for certain Assignments

Homework Sessions will be pre-announced and it is advised to bring a laptop along for those sessions

Help us Structure the Course!

Participate in our online survey:
`http://goo.gl/forms/ws0ASBLXiY`

Lecture #1: Structure

Let us review the Linear Finite Element Method

- Strong vs. Weak Formulation
- The Finite Element (FE) formulation
- The Iso-Parametric Mapping

Structural Finite Elements

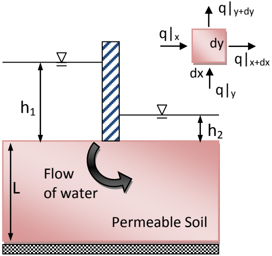
- The Bar Element
- The Beam Element

Example

- The Axially Loaded Bar

Classification of Engineering Systems

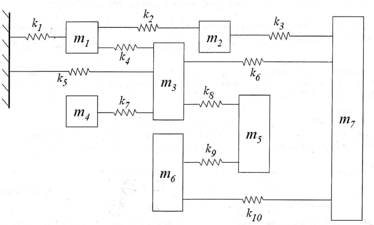
Continuous



$$k\left(\frac{\partial^2\phi}{\partial^2 x}+\frac{\partial^2\phi}{\partial^2 y}\right)=0$$

Laplace Equation

Discrete



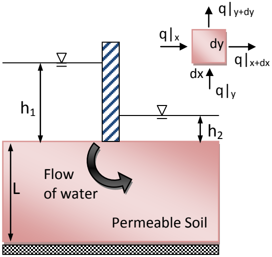
$$F = KX$$

Direct Stiffness Method

FEM: Numerical Technique for **approximating** the solution of **continuous systems**. We will use a displacement based formulation and a stiffness based solution (direct stiffness method).

Classification of Engineering Systems

Continuous



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Laplace Equation

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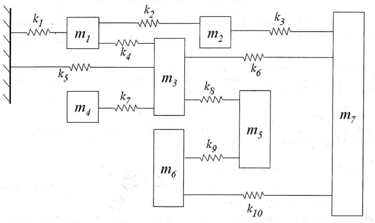


$$F = KX$$

Direct Stiffness Method

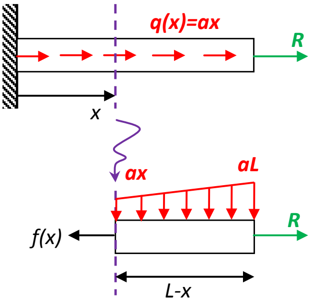
FEM: Numerical Technique for **approximating** the solution of **continuous systems**. We will use a displacement based formulation and a stiffness based solution (direct stiffness method).

How is the Physical Problem formulated?

The formulation of the equations governing the response of a system under specific loads and constraints at its boundaries is usually provided in the form of a differential equation. The differential equation also known as the **strong form** of the problem is typically extracted using the following sets of equations:

1. Equilibrium Equations
   ex. $f(x) = R + \frac{aL + ax}{2}(L - x)$
2. Constitutive Requirements Equations
   ex. $\sigma = E\epsilon$
3. Kinematics Relationships
   ex. $\epsilon = \frac{du}{dx}$

Axial bar Example



How is the Physical Problem formulated?

Differential Formulation (Strong Form) in 2 Dimensions

Quite commonly, in engineering systems, the governing equations are of a second order (derivatives up to u'' or $\frac{\partial^2 u}{\partial^2 x}$) and they are formulated in terms of variable u, i.e. displacement:

Governing Differential Equation ex: general 2nd order PDE

$$A(x,y)\frac{\partial^2 u}{\partial^2 x} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial^2 y} = \phi(x,y,u,\frac{\partial u}{\partial y},\frac{\partial u}{\partial y})$$

Problem Classification

- $B^2 - AC < 0 \;\Rightarrow$ elliptic
- $B^2 - AC = 0 \;\Rightarrow$ parabolic
- $B^2 - AC > 0 \;\Rightarrow$ hyperbolic

Boundary Condition Classification

- Essential (Dirichlet): $u(x_0, y_0) = u_0$
  order $m-1$ at most for C^{m-1}
- Natural (Neumann): $\frac{\partial u}{\partial y}(x_0, y_0) = \dot{u}_0$
  order m to $2m-1$ for C^{m-1}

Differential Formulation (Strong Form) in 2 Dimensions

The previous classification corresponds to certain characteristics for each class of methods. More specifically,

- Elliptic equations are most commonly associated with a diffusive or dispersive process in which the state variable u is in an equilibrium condition.
- Parabolic equations most often arise in transient flow problems where the flow is down gradient of some state variable u. Often met in the heat flow context.
- Hyperbolic equations refer to a wide range of areas including elasticity, acoustics, atmospheric science and hydraulics.

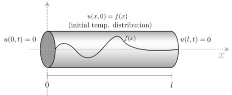
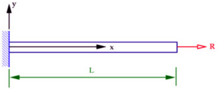
Reference Problem

Consider the following 1 Dimensional (1D) strong form (parabolic)

$$\frac{d}{dx}\left(c(x)\frac{d\mathbf{u}}{dx}\right) + \mathbf{f}(x) = 0$$

$$-\,c(0)\frac{d}{dx}\mathbf{u}(0) = C_1 \qquad \text{(Neumann BC)}$$

$$u(L) = 0 \qquad \text{(Dirichlet BC)}$$

Physical Problem (1D)		Diff. Equation	Quantities	Constitutive Law
One dimensional Heat flow		$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + Q = 0$	T=temperature A=area k=thermal conductivity Q=heat supply	Fourier $q = -k\, dT/dx$ $q =$ heat flux
Axially Loaded Bar		$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0$	u=displacement A=area E=Young's modulus B=axial loading	Hooke $\sigma = E du/dx$ $\sigma =$ stress

Approximating the Strong Form

The strong form requires strong continuity on the dependent field variables (usually displacements). Whatever functions define these variables have to be differentiable up to the order of the PDE that exist in the strong form of the system equations. Obtaining the exact solution for a strong form of the system equation is a quite difficult task for practical engineering problems.

The finite difference method can be used to solve the system equations of the strong form and obtain an approximate solution. However, this method usually works well for problems with simple and regular geometry and boundary conditions.

Alternatively we can use the **finite element method** on a **weak form** of the system. This weak form is usually obtained through energy principles which is why it is also known as variational form.

From Strong Form to Weak form

Three are the approaches commonly used to go from strong to weak form:

- Principle of Virtual Work
- Principle of Minimum Potential Energy
- Methods of weighted residuals (Galerkin, Collocation, Least Squares methods, etc)

Visit our Course page on Linear FEM for further details:
`http://www.chatzi.ibk.ethz.ch/education/method-of-finite-elements-i.html`

From Strong Form to Weak form - Approach #1

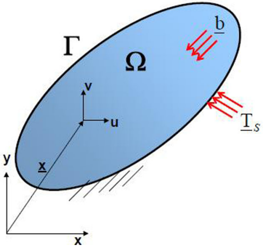
Principle of Virtual Work

For any set of compatible small virtual displacements imposed on the body in its state of equilibrium, the total internal virtual work is equal to the total external virtual work.

$$W_{int} = \int_{\Omega} \bar{\epsilon}^T \boldsymbol{\tau} d\Omega = W_{ext} = \int_{\Omega} \bar{\mathbf{u}}^T \mathbf{b} d\Omega + \int_{\Gamma} \bar{\mathbf{u}}^{ST} \mathbf{T_S} d\Gamma + \sum_i \bar{\mathbf{u}}^{iT} \mathbf{R_C}^i$$

where

- $\mathbf{T_S}$: surface traction (along boundary Γ)
- $\mathbf{b}$: body force per unit area
- $\mathbf{R_C}$: nodal loads
- $\bar{\mathbf{u}}$: virtual displacement
- $\bar{\epsilon}$: virtual strain
- $\boldsymbol{\tau}$: stresses



From Strong Form to Weak form - Approach #3

Galerkin's Method

Given an arbitrary weight function w, where

$$S = \{u | u \in \mathcal{C}^0, u(l) = 0\}, S^0 = \{w | w \in \mathcal{C}^0, w(l) = 0\}$$

$\mathcal{C}^0$ is the collection of all continuous functions.

Multiplying by w and integrating over Ω

$$\int_0^l w(x)[(c(x)u'(x))' + f(x)]dx = 0$$

$$[w(0)(c(0)u'(0) + C_1] = 0$$

Weak Form - 1D FEM

Using the divergence theorem (integration by parts) we reduce the order of the differential:

$$\int_0^l wg' dx = [wg]_0^l - \int_0^l gw' dx$$

The **weak form** is then reduced to the following problem. Also, in what follows we assume constant properties $c(x) = c = const$.

Find $u(x) \in \mathcal{S}$ such that:

$$\int_0^l w' c u' dx = \int_0^l wf dx + w(0) C_1$$

$$S = \{u | u \in \mathcal{C}^0, u(l) = 0\}$$

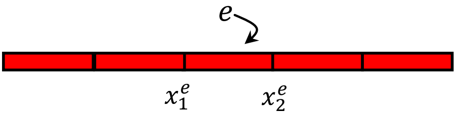
$$S^0 = \{w | w \in \mathcal{C}^0, w(l) = 0\}$$

Notes:

1. Natural (Neumann) boundary conditions, are imposed on the secondary variables like forces and tractions.
   For example, $\frac{\partial u}{\partial y}(x_0, y_0) = \dot{u}_0$.
2. Essential (Dirichlet) or geometric boundary conditions, are imposed on the primary variables like displacements.
   For example, $u(x_0, y_0) = u_0$.
3. A solution to the strong form will also satisfy the weak form, but not vice versa. Since the weak form uses a lower order of derivatives it can be satisfied by a larger set of functions.
4. For the derivation of the weak form we can choose any weighting function w, since it is arbitrary, so we usually choose one that satisfies homogeneous boundary conditions wherever the actual solution satisfies essential boundary conditions. Note that this does not hold for natural boundary conditions!

How to derive a solution to the weak form?

Step #1: Follow the **FE** approach:
Divide the body into finite elements, e, connected to each other through nodes.



Then break the overall integral into a summation over the finite elements:

$$\sum_e \left[\int_{x_1^e}^{x_2^e} w' c u' dx - \int_{x_1^e}^{x_2^e} w f dx - w(0) C_1 \right] = 0$$

Step #2: Approximate the continuous displacement using a discrete equivalent:

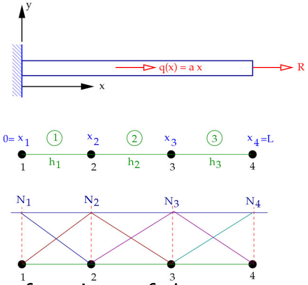
Galerkin's method assumes that the approximate (or **trial**) solution, u, can be expressed as a linear combination of the nodal point displacements u_i, where i refers to the corresponding **node number**.

$$u(x) \approx u^h(x) = \sum_i N_i(x) u_i = \mathbf{N}(x)\mathbf{u}$$

where bold notation signifies a vector and $N_i(x)$ are the **shape functions**. In fact, the shape function can be any mathematical formula that helps us interpolate what happens at points that lie within the nodes of the mesh. In the **1-D case** that we are using as a reference, $N_i(x)$ are defined as 1st degree polynomials indicating a **linear interpolation**.

1D FE formulation: Galerkin's Method

Shape function Properties:



- Bounded and Continuous
- One for each node
- $N_i^e(x_j^e) = \delta_{ij}$, where
  $$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The shape functions can be written as piecewise functions of the x coordinate:

$$N_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \leq x < x_i \\ \dfrac{x i + 1 - x}{x_{i+1} - x_i}, & x_i \leq x < x_{i+1} \\ 0, & \textit{otherwise} \end{cases}$$

This is not a convenient notation. Instead of using the global coordinate x, things become simplified when using coordinate ξ referring to the local system of the element (see page 25).

1D FE formulation: Galerkin's Method

Step #3: Approximate $w(x)$ using a discrete equivalent:

The weighting function, w is usually (although not necessarily) chosen to be of the same form as u

$$w(x) \approx w^h(x) = \sum_i N_i(x) w_i = \mathbf{N}(x)\mathbf{w}$$

i.e. for 2 nodes:
$\mathbf{N} = [N_1 \quad N_2]$ $\quad \mathbf{u} = [u_1 \quad u_2]^T$ $\quad \mathbf{w} = [w_1 \quad w_2]^T$

Alternatively we could have a Petrov-Galerkin formulation, where $w(x)$ is obtained through the following relationships:

$$w(x) = \sum_i (N_i + \delta \frac{h^e}{\sigma} \frac{dN_i}{dx}) w_i$$

$$\delta = coth(\frac{Pe^e}{2}) - \frac{2}{Pe^e} \qquad coth = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

1D FE formulation: Galerkin's Method

Step #4: Substituting into the weak formulation and rearranging terms we obtain the following in **matrix notation**:

$$\int_0^l w' c u' dx - \int_0^l w f dx - w(0) C_1 = 0 \Rightarrow$$

$$\int_0^l (\mathbf{w}^T \mathbf{N}^T)' c (\mathbf{N}\mathbf{u})' dx - \int_0^l \mathbf{w}^T \mathbf{N}^T f dx - \mathbf{w}^T \mathbf{N}(0)^T C_1 = 0$$

Since **w**, **u** are vectors, each one containing a set of discrete values corresponding at the nodes i, it follows that the above set of equations can be rewritten in the following form, i.e. as a summation over the w_i, u_i components (**Einstein notation**):

$$\int_0^l \left(\sum_i u_i \frac{dN_i(x)}{dx} \right) c \left(\sum_j w_j \frac{dN_j(x)}{dx} \right) dx$$

$$- \int_0^l f \sum_j w_j N_j(x) dx - \sum_j w_j N_j(x) C_1 \Big|_{x=0} = 0$$

1D FE formulation: Galerkin's Method

This is rewritten as,

$$\sum_j w_j \left[\int_0^l \left(\sum_i c u_i \frac{dN_i(x)}{dx} \frac{dN_j(x)}{dx} \right) - f N_j(x) dx + \left(N_j(x) C_1 \right)\Big|_{x=0} \right] = 0$$

The above equation has to hold $\forall w_j$ since the weighting function $w(x)$ is an arbitrary one. Therefore the following **system of equations** has to hold:

$$\int_0^l \left(\sum_i c u_i \frac{dN_i(x)}{dx} \frac{dN_j(x)}{dx} \right) - f N_j(x) dx + \left(N_j(x) C_1 \right)\Big|_{x=0} = 0 \quad j = 1, ..., n$$

After reorganizing and moving the summation outside the integral, this becomes:

$$\sum_i \left[\int_0^l c \frac{dN_i(x)}{dx} \frac{dN_j(x)}{dx} \right] u_i = \int_0^l f N_j(x) dx + \left(N_j(x) C_1 \right)\Big|_{x=0} = 0 \quad j = 1, ..., n$$

1D FE formulation: Galerkin's Method

We finally obtain the following discrete system in matrix notation:

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

where writing the integral from 0 to l as a summation over the subelements we obtain:

$$\mathbf{K} = \mathcal{A}_e \mathbf{K}^e \longrightarrow \mathbf{K}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}_{,x}^T c \mathbf{N}_{,x} dx = \int_{x_1^e}^{x_2^e} \mathbf{B}^T c \mathbf{B} dx$$

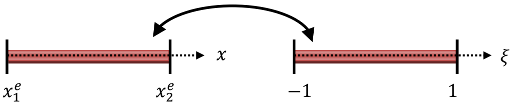
$$\mathbf{f} = \mathcal{A}_e \mathbf{f}^e \longrightarrow \mathbf{f}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}^T \mathbf{f} dx + \mathbf{N}^T h|_{x=0}$$

where $\mathcal{A}$ is not a sum but an assembly (see page and, x denotes differentiation with respect to x.

In addition, $\mathbf{B} = \mathbf{N}_{,x} = \dfrac{dN(x)}{dx}$ is known as the strain-displacement matrix.

Iso-Parametric Mapping

This is a way to move from the use of global coordinates (i.e.in (x, y)) into normalized coordinates (usually (ξ, η)) so that the finally derived stiffness expressions are uniform for elements of the same type.



Shape Functions in Natural Coordinates

$$x(\xi) = \sum_{i=1,2} N_i(\xi) x_i^e = N_1(\xi) x_1^e + N_2(\xi) x_2^e$$

$$N_1(\xi) = \frac{1}{2}(1 - \xi), \qquad N_2(\xi) = \frac{1}{2}(1 + \xi)$$

1D FE formulation: Iso-Parametric Formulation

Map the integrals to the natural domain $\longrightarrow$ **element stiffness** matrix.
Using the chain rule of differentiation for $N(\xi(x))$ we obtain:

$$\mathbf{K}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}_{,x}^T c \mathbf{N}_{,x} dx = \int_{-1}^{1} (\mathbf{N}_{,\xi}\xi_{,x})^T c (\mathbf{N}_{,\xi}\xi_{,x}) x_{,\xi} d\xi$$

$$\text{where } \mathbf{N}_{,\xi} = \frac{d}{d\xi} \left[\begin{array}{cc} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{array} \right] = \left[\begin{array}{cc} \frac{-1}{2} & \frac{1}{2} \end{array} \right]$$

$$\text{and } x_{,\xi} = \frac{dx}{d\xi} = \frac{x_2^e - x_1^e}{2} = \frac{h}{2} = J \text{ (Jacobian) and } h \text{ is the element length}$$

$$\xi_{,x} = \frac{d\xi}{dx} = J^{-1} = 2/h$$

From all the above,

$$\mathbf{K}^e = \frac{c}{x_2^e - x_1^e} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]$$

Similary, we obtain the **element load vector**:

$$\mathbf{f}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}^T \mathbf{f} dx + \mathbf{N}^T h|_{x=0} = \int_{-1}^{1} \mathbf{N}^T(\xi) \mathbf{f} x_{,\xi} d\xi + \mathbf{N}^\mathbf{T}(\mathbf{x}) h|_{x=0}$$

Note: the iso-parametric mapping is only done for the integral.

Discussion & Limitations

- What are some of the main Assumptions & Simplifications we have performed so far in the Linear Theory?
- What do you imagine are the added problems beyond this realm?

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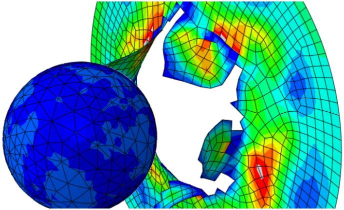
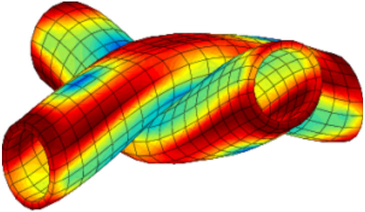



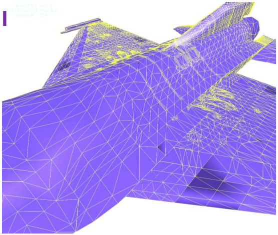

The beam element belongs in the so called **Structural Finite Elements**

Why?

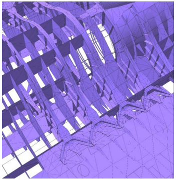
F-16 Aeroelastic Structural Model

FEM model:
150000 Nodes

Exterior model 95% are shell elements



Internal structure zoom. Some Brick and tetrahedral elements

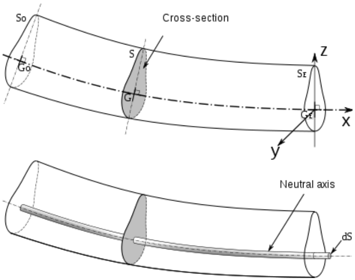


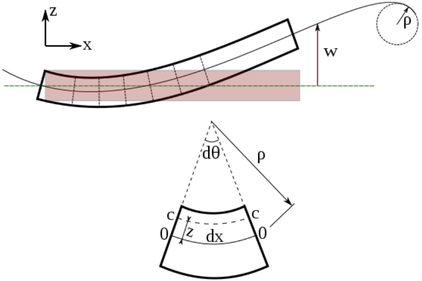
http://www.colorado.edu/engineering/CAS/Felippa.d/FelippaHome.d/Home.html

Two main beam theories:

- Euler-Bernoulli theory (Engineering beam theory) -slender beams
- Timoshenko theory thick beams

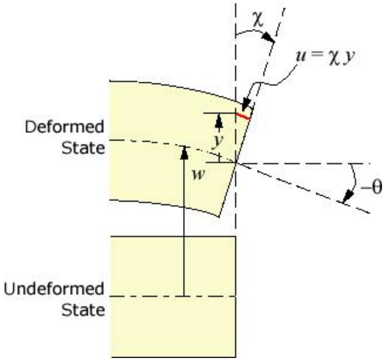
Euler - Bernoulli Beam





Euler Bernoulli Beam Assumptions - Kirchhoff Assumptions

- Normals remain straight (they do not bend)
- Normals remain unstretched (they keep the same length)
- Normals remain normal (they always make a right angle to the neutral plane)

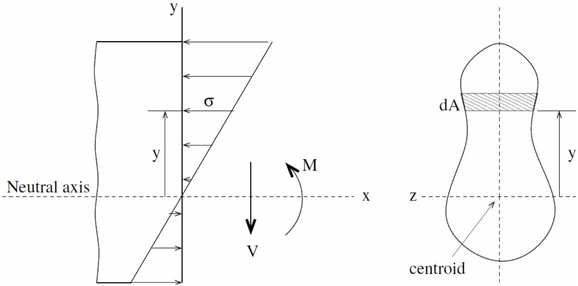


The Euler-Bernoulli Beam theory (small deformations)

$$\sigma = -\frac{M}{I} y$$

$$\epsilon = \frac{\sigma}{E}$$

$$\frac{d^2 v}{dx^2} = \frac{M}{EI}$$



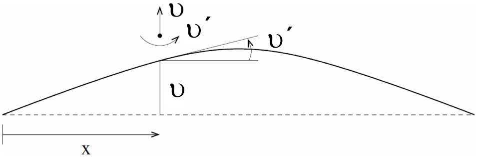
σ - normal stress

M - bending moment

ε - normal strain

v - displacement of the centroid

EI - bending stiffness



Beam Elements - Strong Form

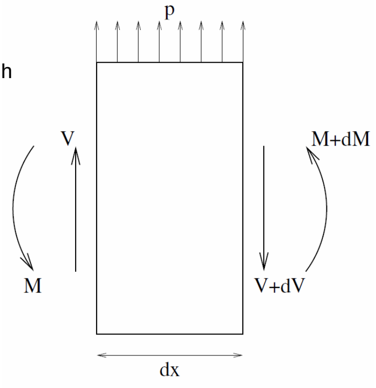
Equilibrium

$$\frac{dV}{dx} = p$$ – distributed load per unit length

$$\frac{dM}{dx} = V$$ – shear force

Combining the equations

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - p = 0$$



(S)

(1) $$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - p(x) = 0 \quad \text{in} \quad \Omega$$

(2) $$v = \bar{v} \quad \text{displacement} \quad \text{on } \Gamma_u$$

(3) $$\frac{dv}{dx} = -\bar{\theta} \quad \text{angle} \quad \text{on } \Gamma_\theta$$

(4) $$EI\frac{d^2v}{dx^2} = \bar{M} \quad \text{moment} \quad \text{on } \Gamma_M$$

(5) $$-EI\frac{d^3v}{dx^3} = \bar{S} \quad \text{shear force} \quad \text{on } \Gamma_S$$

$$\Gamma_u \cap \Gamma_S = \emptyset \qquad \Gamma_u \cup \Gamma_S = \Gamma$$

$$\Gamma_\theta \cap \Gamma_M = \emptyset \qquad \Gamma_\theta \cup \Gamma_M = \Gamma$$

Free end with applied load

$$EI\frac{d^2v}{dx^2} = \bar{M} \text{ on } \Gamma_M$$

$$-EI\frac{d^3v}{dx^3} = \bar{S} \text{ on } \Gamma_S$$

Simple support

$$EI\frac{d^2v}{dx^2} = 0 \text{ on } \Gamma_M$$

$$v = 0 \text{ on } \Gamma_u$$

Clamped support

$$\frac{dv}{dx} = 0 \text{ on } \Gamma_\theta$$

$$v = 0 \text{ on } \Gamma_u$$

Basic Formulas-Shape Functions

The order of the shape functions is chosen based on the problem's strong form. Moreover, the shape functions are usually expressed in the iso-parametric coordinate system

- **Bar Element**

For the standard bar element, the strong form is $AE\dfrac{d^2\mathbf{u}}{dx^2} + f(x) = 0$. The homogeneous form is $AE\frac{d^2\mathbf{u}}{dx^2} = 0$ which is a 2nd order ODE with known solution: $u(x) = C_1 x + C_2$, i.e., a 1st degree polynomial.

The shape functions $\mathbf{N}(\xi)$ are therefore selected as 1st degree polynomials:

$$\mathbf{N}(\xi) = \begin{bmatrix} \dfrac{1}{2}(1-\xi) \\ \dfrac{1}{2}(1+\xi) \end{bmatrix} \begin{matrix} \rightarrow u_1 \\ \rightarrow u_2 \end{matrix}$$

Basic Formulas-Shape Functions

- **Beam Element**

For the standard beam element, the strong form is $\frac{d^2}{dx^2}\left(EI\frac{d^2 \upsilon}{dx^2}\right) + f(x) = 0$.

The homogeneous form is $\frac{d^2}{dx^2}\left(EI\frac{d^2 \upsilon}{dx^2}\right) = 0$ which is a 4th order ODE with known solution: $u(x) = C_1 x3 + C_2 x^2 + C_3 x + C_4$, i.e., a 3rd degree polynomial.

The shape functions $\mathbf{H}(\xi)$ are therefore selected as 3rd degree polynomials:

$$\mathbf{H}(\xi) = \begin{bmatrix} \frac{1}{4}(1-\xi)^2(2+\xi) \\ \frac{1}{4}(1-\xi)^2(1+\xi) \\ \frac{1}{4}(1+\xi)^2(2+\xi) \\ \frac{1}{4}(1+\xi)^2(\xi-1) \end{bmatrix} \begin{matrix} \rightarrow u_1 \\ \rightarrow \theta_1 \\ \rightarrow u_2 \\ \rightarrow \theta_2 \end{matrix}$$

Basic Formulas-Stiffness Matrices

Going from Strong to Weak form yields the expressions of the equivalent stiffness matrix for each element:

- **Bar Element**

$$K_{ij} = \int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx$$

Using matrix notation, the definition of the strain-displacement matrix $\mathbf{B} = \dfrac{dN(x)}{dx}$, and the definition of the Jacobian (since we are looking for derivatives in term of x and not ξ ultimately): $J = \frac{dx}{d\xi} = \frac{x_2^e - x_1^e}{2} = \frac{h}{2}$ we obtain:

$$\mathbf{K}^e = \int_0^L \mathbf{B}^T EA \mathbf{B} dx = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where h is the length of the element.

Basic Formulas-Stiffness Matrices

Going from Strong to Weak form yields the expressions of the equivalent stiffness matrix for each element:

- **Beam Element**

$$K_{ij} = \int_0^L \frac{d^2 H_j(x)}{dx^2} EI \frac{d^2 H_i(x)}{dx^2} dx$$

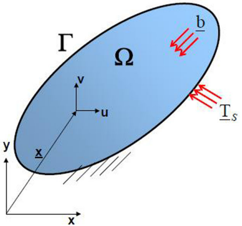
Using matrix notation, the definition of the strain-displacement matrix $\mathbf{B} = \dfrac{d^2 H(x)}{dx^2}$, and the definition of the Jacobian (since we are looking for derivatives in term of x and not ξ ultimately): $J = \frac{dx}{d\xi} = \frac{l}{2}$ we obtain:

$$\mathbf{K}^e = \int_0^L \mathbf{B}^T EI \mathbf{B} dx = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

where l is the length of the element.

Governing Equations

Equilibrium Eq:	$\nabla_s \boldsymbol{\sigma} + \mathbf{b} = 0$	$\in \Omega$
Kinematic Eq:	$\boldsymbol{\epsilon} = \nabla_s u$	$\in \Omega$
Constitutive Eq:	$\boldsymbol{\sigma} = D \cdot \boldsymbol{\epsilon}$	$\in \Omega$
Traction B.C.:	$\boldsymbol{\tau} \cdot n = T_s$	$\in \Gamma_t$
Displacement B.C:	$\mathbf{u} = \mathbf{u}_\Gamma$	$\in \Gamma_u$



Hooke's Law - Constitutive Equation

Plane Stress

$$\boldsymbol{\tau}_{zz} = \boldsymbol{\tau}_{xz} = \boldsymbol{\tau}_{yz} = 0, \boldsymbol{\epsilon}_{zz} \neq 0$$

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

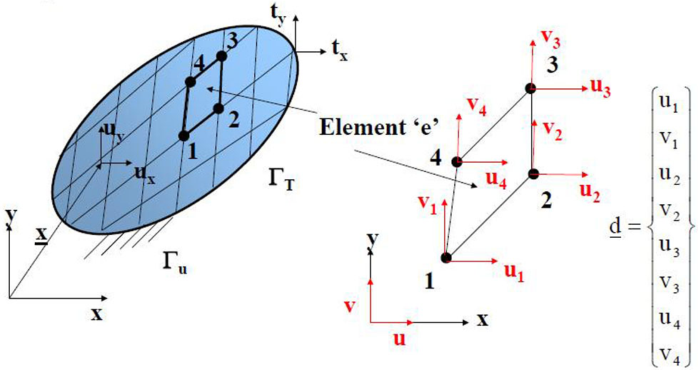
Plane Strain

$$\boldsymbol{\epsilon}_{zz} = \gamma_{xz} = \gamma_{yz} = 0, \boldsymbol{\sigma}_{zz} \neq 0$$

$$D = \frac{E}{(1-\nu)(1+\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

2D FE formulation: Discretization

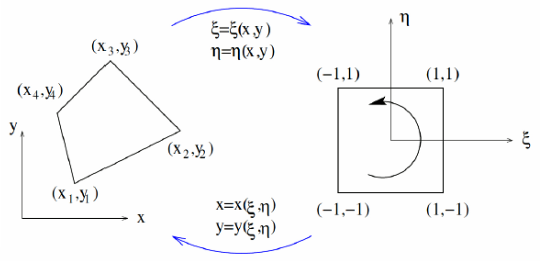
Divide the body into finite elements connected to each other through nodes



Shape Functions in Natural Coordinates

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta), \qquad N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta), \qquad N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

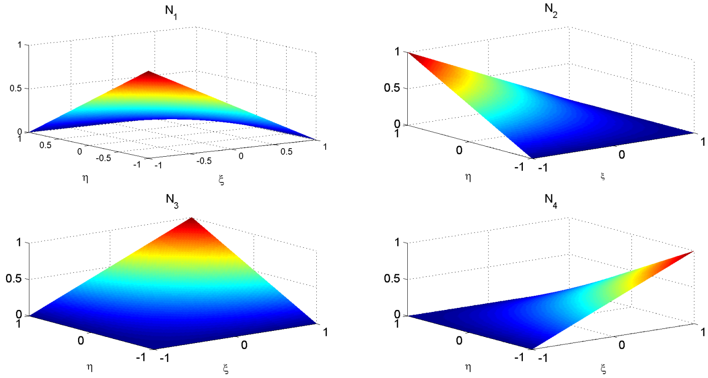


Iso-parametric Mapping

$$x = \sum_{i=1}^{4} N_i(\xi,\eta) x_i^e$$

$$y = \sum_{i=1}^{4} N_i(\xi,\eta) y_i^e$$

Bilinear Shape Functions

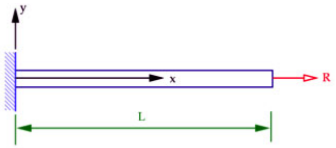


$$\mathbf{K^{tot}} \cdot d = f^{tot} \quad \text{where} \quad \mathbf{K^e} = \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega, \quad f^e = \int_{\Omega^e} N^T B d\Omega + \int_{\Gamma_T^e} N^T t_s d\Gamma$$

*The total stiffness $\mathbf{K^{tot}}$ is obtained via assembly of the element matrices $\mathbf{K^e}$.

The same holds for the force vectors.

A. Constant End Load



Given: Length L, Section Area A, Young's modulus E
Find: stresses and deformations.

Assumptions:
The cross-section of the bar does not change after loading.
The material is linear elastic, isotropic, and homogeneous.
The load is centric.
End-effects are not of interest to us.

Axially Loaded Bar Example

A. Constant End Load

Strength of Materials Approach

From the **equilibrium equation**, the axial force at a random point x along the bar is:

$$\mathbf{f}(x) = \mathbf{R}(= const) \Rightarrow \boldsymbol{\sigma}(x) = \frac{\mathbf{R}}{A}$$

From the **constitutive equation (Hooke's Law)**:

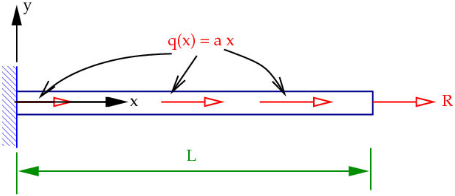
$$\boldsymbol{\epsilon}(x) = \frac{\boldsymbol{\sigma}(x)}{E} = \frac{\mathbf{R}}{AE}$$

Hence, the deformation $\delta(x)$ is obtained from kinematics as:

$$\boldsymbol{\epsilon} = \frac{\delta(x)}{x} \Rightarrow \delta(x) = \frac{\mathbf{R}x}{AE}$$

Note: The stress & strain is independent of x for this case of loading.

B. Linearly Distributed Axial + Constant End Load



From the **equilibrium equation**, the axial force at random point x along the bar is:

$$\mathbf{f}(x) = \mathbf{R} + \frac{aL + ax}{2}(L - x) = \mathbf{R} + \frac{a(L^2 - x^2)}{2}(\text{ depends on x})$$

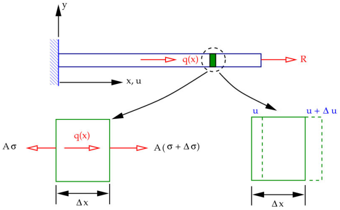
In order to now find stresses & deformations (which depend on x) we have to repeat the process for every point in the bar. This is computationally inefficient.

Axially Loaded Bar Example

From the **equilibrium equation**, for an infinitesimal element:

$$A\boldsymbol{\sigma} = \mathbf{q}(x)\Delta x + A(\boldsymbol{\sigma} + \Delta\sigma) \Rightarrow A \underbrace{lim}_{\Delta x \to 0} \frac{\Delta \boldsymbol{\sigma}}{\Delta x} + \mathbf{q}(x) = 0 \Rightarrow A\frac{d\boldsymbol{\sigma}}{dx} + \mathbf{q}(x) = 0$$

$$\text{Also, } \epsilon = \frac{d\mathbf{u}}{dx}, \boldsymbol{\sigma} = E\epsilon, \mathbf{q}(x) = ax \Rightarrow AE\frac{d^2\mathbf{u}}{dx^2} + ax = 0$$



Strong Form

$$AE\frac{d^2\mathbf{u}}{dx^2} + ax = 0$$

$\mathbf{u}(0) = 0$ essential BC

$$\mathbf{f}(L) = \mathbf{R} \Rightarrow AE \left.\frac{d\mathbf{u}}{dx}\right|_{x=L} = \mathbf{R} \text{ natural BC}$$

Analytical Solution

$$\mathbf{u}(x) = \mathbf{u}_{hom} + \mathbf{u}_p \Rightarrow \mathbf{u}(x) = C_1 x + C_2 - \frac{ax^3}{6AE}$$

C_1, C_2 are determined from the BC

Axially Loaded Bar Example

An analytical solution cannot always be found

Approximate Solution - The Galerkin Approach (#3): Multiply by the weight function **w** and integrate over the domain

$$\int_0^L AE\frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx + \int_0^L ax\mathbf{w}dx = 0$$

Apply integration by parts

$$\int_0^L AE\frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx = \left[AE\frac{d\mathbf{u}}{dx}\mathbf{w}\right]_0^l - \int_0^L AE\frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx \Rightarrow$$
$$\int_0^L AE\frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx = \left[AE\frac{d\mathbf{u}}{dx}(L)\mathbf{w}(L) - AE\frac{d\mathbf{u}}{dx}(0)\mathbf{w}(0)\right] - \int_0^L AE\frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx$$

But from BC **w**e have $u(0) = 0$, $AE\frac{d\mathbf{u}}{dx}(L)\mathbf{w}(L) = \mathbf{R}\mathbf{w}(L)$, therefore the approximate weak form can be written as

$$\int_0^L AE\frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx = \mathbf{R}\mathbf{w}(L) + \int_0^L ax\mathbf{w}dx$$

Axially Loaded Bar Example

In Galerkin's method we assume that the approximate solution, **u** can be expressed as

$$\mathbf{u}(x) = \sum_{j=1}^{n} u_j N_j(x)$$

w is chosen to be of the same form as the approximate solution (but with arbitrary coefficients w_i),

$$\mathbf{w}(x) = \sum_{i=1}^{n} w_i N_i(x)$$

Plug **u**(x),**w**(x) into the approximate weak form:

$$\int_0^L AE \sum_{j=1}^{n} u_j \frac{dN_j(x)}{dx} \sum_{i=1}^{n} w_i \frac{dN_i(x)}{dx} dx = \mathbf{R} \sum_{i=1}^{n} w_i N_i(L) + \int_0^L ax \sum_{i=1}^{n} w_i N_i(x) dx$$

w_i is arbitrary, so the above has to hold $\forall\ w_i$:

$$\sum_{j=1}^{n} \left[\int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx \right] u_j = \mathbf{R} N_i(L) + \int_0^L ax N_i(x) dx \quad i = 1 \dots n$$

which is a system of n equations that can be solved for the unknown coefficients u_j.

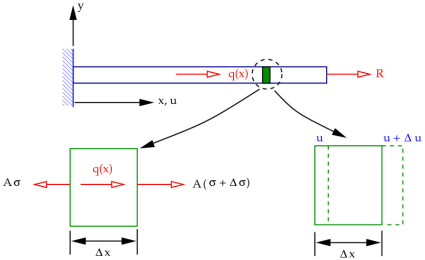
Axially Loaded Bar Example

The matrix form of the previous system can be expressed as

$$\mathbf{K}_{ij} u_j = f_i \text{ where } K_{ij} = \int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx$$

$$\text{and } f_i = \mathbf{R} N_i(L) + \int_0^L ax N_i(x) dx$$

Finite Element Solution - using 2 discrete elements, of length h (3 nodes)
From the iso-parametric formulation we know the element stiffness matrix
$\mathbf{K}^e = \frac{AE}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Assembling the element stiffness matrices we get:



$$\mathbf{K}^{tot} = \begin{bmatrix} K_{11}^e & K_{12}^1 & 0 \\ K_{12}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ 0 & K_{12}^2 & K_{22}^2 \end{bmatrix} \Rightarrow$$

$$\mathbf{K}^{tot} = \frac{AE}{h} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Axially Loaded Bar Example

We also have that the element load vector is

$$f_i = \mathbf{R} N_i(L) + \int_0^L a x N_i(x) dx$$

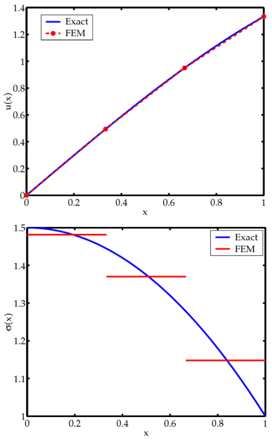
Expressing the integral in iso-parametric coordinates $N_i(\xi)$ we have:

$$\frac{d\xi}{dx} = \frac{2}{h}, x = N_1(\xi)x_1^e + N_2(\xi)x_2^e, \Rightarrow$$

$$f_i = \mathbf{R}|_{i=4} + \int_0^L a(N_1(\xi)x_1^e + N_2(\xi)x_2^e)N_i(\xi)\frac{2}{h}d\xi$$

Axially Loaded Bar Example

After the vectors are formulated we proceed with solving the main equation $\mathbf{Ku} = \mathbf{f} \Rightarrow \mathbf{u} = \mathbf{K}^{-1}\mathbf{f}$.
The results are plotted below using 3 elements:



Notice how the approximation is able to track the displacement $u(x)$, despite the fact that in reality the solution is a cubic function of x (remember the analytical solution).

Since the **shape functions used, $N_i(x)$, are linear** the displacement is approximated as: $u(x) = \sum_i N_i(x)u_i$, where u_i corresponds to nodal displacements.

The **strain is then obtained as** $\epsilon = \frac{du}{dx} \Rightarrow \epsilon = \frac{dN}{dx}u_i$ where in slide 25 we have defined $B = \frac{dN}{dx}$ to be the so-called strain-displacement matrix.