The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

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The Continuum Mechanics Incremental Equations

The basic Problem
Establish the solution using an incremental formulation. Two main approaches exist for establishing equilibrium

- **Lagrangian Formulation**: Track the movement of all particles of the body, in their motion from an initial to a final configuration (pathline)

- **Eulerian Formulation**: The motion of the material through a stationary control volume is considered (streamlines). Mainly used in fluid mechanics (but also in large strain plasticity theories - e.g. generalized plasticity).
Spatial or Eulerian coordinates \((x)\): These coordinates are used to locate a point in space with respect to a fixed basis.

Material or Lagrangian coordinates \((X)\): These coordinates are used to label material points. If we sit on a material point, the label does not change with time.

**Example:** Assume that the motion is
\[
x = \phi(X, t) = X(1 + 2t + t^2)
\]
The inverse of the map gives us \(X\) in terms of \(x\), i.e.,
\[
X = \phi^{-1}(x, t) = \frac{x}{(1 + 2t + t^2)}
\]
Then, the displacement of the material point \(X\) is
\[
u(X, t) = \phi(X, t) - \phi(X, 0) = X(2t + t^2)
\]
The velocity of the material point is (**Langrangian Description**)
\[
v(X, t) = \frac{\partial u}{\partial t} = 2X(1 + t)
\]
Alternatively we can express the velocity in terms of \(x\) (**Eulerian Description**)
\[
\hat{v}(X, t) = v(\phi^{-1}(x, t), t) = \frac{2x(1 + t)}{(1 + 2t + t^2)}
\]
How can we evaluate stresses and forces at time $t$ since both the surface and the volume of the body are unknown? (In the linear case stiffness and equilibrium were evaluated based on the initial configuration)

We need to properly map both current strains and stresses to corresponding measures evaluated at previous configurations.
Lagrangian Formulation

In the future we introduce an appropriate notation:

\[ t \mathbf{x}_i =^0 \mathbf{x}_i + t \mathbf{u}_i, \quad i = 1, 2, 3 \]
\[ t + \Delta t \mathbf{x}_i =^0 \mathbf{x}_i + ^{t + \Delta t} \mathbf{u}_i \]

Increments in displacements from time \( t \) to \( t + \Delta t \) are related as:

\[ \mathbf{u}_i = ^{t + \Delta t} \mathbf{u}_i - t \mathbf{u}_i, \quad i = 1, 2, 3 \]

Reference configurations are indexed as e.g.

\[ ^{t + \Delta t} f_s^0 \mathbf{f}_i \]

where the left subscript indicates the reference configuration and the left superscript indicates at which configuration the quantity occurs.

**Note** if those quantities are the same the left subscript maybe omitted e.g:

\[ ^{t + \Delta t} T_{ij}^{t + \Delta t} \]

Differentiation is indexed as:

\[ ^{t + \Delta t} \mathbf{u}_{i,j}^0 = \frac{\partial ^{t + \Delta t} \mathbf{u}_i^0}{\partial ^0 \mathbf{x}_j} \]
The deformation gradient, strain and stress tensors

- As mentioned, we must try to establish a description of the volume we consider such that we can express the internal virtual work in terms of an integral over a volume we know!

- Further, we would like to be able to decompose the stresses and strains in an efficient manner, keeping track of how the volume stretches and rotates (rigidly).

We consider a body under deformation at times 0 and t:

\[ u = t_x - o_x \]
The deformation gradient

We now consider the change of an infinitesimal gradient vector

Definition

The deformation gradient maps $d^0x$ onto $d^t x$ through the following relation

$$d^t x = ^t_0 X d^0 x$$
The deformation gradient

We can write the deformation gradient as the Jacobian of the current configuration at time $t$, with respect to the initial configuration at time $0$.

$$
{0}^t X = \begin{bmatrix}
\frac{\partial^t x_1}{\partial^t x_1} & \frac{\partial^t x_1}{\partial^t x_2} & \frac{\partial^t x_1}{\partial^t x_3} \\
\frac{\partial^t x_2}{\partial^t x_1} & \frac{\partial^t x_2}{\partial^t x_2} & \frac{\partial^t x_2}{\partial^t x_3} \\
\frac{\partial^t x_3}{\partial^t x_1} & \frac{\partial^t x_3}{\partial^t x_2} & \frac{\partial^t x_3}{\partial^t x_3}
\end{bmatrix}
$$

The deformation gradient describes the stretches and rotations that the material fibers have undergone from time zero to time $t$.

$$
{0}^t X = (0 \nabla^{t x^T})^T, \quad \text{where} \quad 0 \nabla = \begin{bmatrix}
\frac{\partial}{\partial^0 x_1} \\
\frac{\partial}{\partial^0 x_2} \\
\frac{\partial}{\partial^0 x_3}
\end{bmatrix}
$$

and

$$
{t}^x T = \begin{bmatrix}
{t}x_1 \\
{t}x_2 \\
{t}x_3
\end{bmatrix}
$$

It can be shown that

$$
{0}^t X = (0^t X)^{-1}
$$
The deformation gradient

Decomposition of the deformation gradient

The deformation gradient is also used to measure the stretch of a material fiber and the change in angle between fibers due to the deformation. For this we use the

The deformation gradient can be decomposed into a unique product of two matrices

\[ t_0X = t_0R^t_0U \]

\[ t_0U: \text{Symmetric stretch matrix} \]

\[ t_0R: \text{Orthogonal rotation matrix} \]

This is referred to as a Polar Decomposition
Decomposition of the deformation gradient

We continue by rewriting the deformation gradient

\[ X = RU = RUR^T R = VR \]

where \( U \): the right stretch matrix and \( V \): the left stretch matrix

Further it can be shown that:

\[ U = R_L \Lambda R_L^T \]

where \( \Lambda \): the principal stretches and \( R_L \): the Direction of principal stretches

\[ V = R_E \Lambda R_E^T \]

where \( R_E \): the Base vectors of principal stretches in the stationary coordinate system
The deformation gradient

Consider a bar under stretch and rotation

We continue by rewriting the deformation gradient

\[ X = RU \quad \text{Decomposition} \]

It is simpler to consider the deformation in two steps

**Stretching**

\[
U = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

**Rotation**

\[
R = \begin{bmatrix}
cos\theta & -sin\theta & 0 \\
sin\theta & cos\theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
The deformation gradient, strain and stress tensors

**General Case**

Assuming both rotation \( U \) and stretch \( R \): \( X = RU \)

\[
U = \begin{bmatrix}
\frac{l}{L} & 0 & 0 \\
0 & \frac{h}{H} & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \qquad \quad R = \begin{bmatrix}
cos\theta & -sin\theta & 0 \\
sin\theta & cos\theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

which yields

\[
X = \begin{bmatrix}
\frac{l}{L}cos\theta & -\frac{h}{H}sin\theta & 0 \\
\frac{l}{L}sin\theta & \frac{h}{H}cos\theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
The deformation gradient

Using the decomposition of the deformation gradient we define the right and left Cauchy-Green deformation tensors:

The **right Cauchy-Green** deformation tensor:

\[ C = X^T X = (RU)^T RU = U^T R^T RU = U^2 \]

since \( R \) is orthogonal, hence \( RR^T = R^T R = I \)

The **left Cauchy-Green** deformation tensor:

\[ B = XX^T = VR^T RV = V^2 \]
The strain tensor

From deformations to strains:
The strain may be understood in terms of the stretch $\lambda = \frac{l}{L}$.
When dealing with large displacements the one dimensional strain measures are defined as:

- Using time 0 as reference
  
  **Green - Lagrange** strain: $E = \frac{1}{2} \left( \frac{l^2}{L^2} - 1 \right) = \frac{1}{2} (\lambda^2 - 1)$
  
  **Tensor Equivalent:** $t^+ \Delta t \epsilon_0 = \frac{1}{2} (C - I)$

- Using time $t$ as reference
  
  **Almansi** strain: $A = \frac{1}{2} \left( 1 - \frac{L^2}{l^2} \right) = \frac{1}{2} (1 - \lambda^{-2})$
  
  **Tensor Equivalent:** $t^+ \Delta t \epsilon_t = \frac{1}{2} (I - B^{-1})$
The strain tensor

Proof: From deformations to strains
The strain may be understood as the stretch per unit length. We can assess the strain through the inner product between two infinitesimal vectors before and after deformation.

- Use time 0 as reference

\[
{\dot{x}_1} \cdot {\dot{x}_2} - {x_1} \cdot {x_2} = (X{\dot{x}_1}) \cdot (X{\dot{x}_2}) - {x_1} \cdot {x_2} \\
= {x_1} \cdot (C - I) \cdot {x_2}
\]

where, Green - Lagrange strain: \( \epsilon_0^t = \frac{1}{2} (C - I) \)

- Use time \( t \) as reference

\[
{\dot{x}_1} \cdot {\dot{x}_2} - {x_1} \cdot {x_2} = {\dot{x}_1} \cdot {\dot{x}_2} - (X^{-1}{\dot{x}_1}) \cdot (X^{-1}{\dot{x}_2}) \\
= {\dot{x}_1} \cdot (I - B^{-1}) \cdot {\dot{x}_2}
\]

where, Almansi strain: \( \epsilon_0^t = \frac{1}{2} (I - B^{-1}) \)
The deformation gradient, strain and stress tensors

From deformations to strains: 1D Example

Assume the following deformation gradient matrix:

\[
X = \begin{bmatrix}
  \frac{l}{L} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

i.e. pure stretch

\[
d^t x_1 \cdot d^t x_2 - d^0 x_1 \cdot d^0 x_2 = \left( \frac{l}{L} d^0 x_1 \right) \cdot \left( \frac{l}{L} d^0 x_2 \right) - d^0 x_1 \cdot d^0 x_2
\]

\[
= d^0 x_1 \cdot \left( \frac{l^2}{L^2} - 1 \right) \cdot d^0 x_2
\]

or equivalently

\[
d^t x_1 \cdot d^t x_2 - d^0 x_1 \cdot d^0 x_2 = d^t x_1 \cdot d^t x_2 - \left( \frac{L}{l} d^t x_1 \right) \cdot \left( \frac{L}{l} d^t x_2 \right)
\]

\[
= d^t x_1 \cdot \left( 1 - \frac{L^2}{l^2} \right) \cdot d^t x_2
\]
The deformation gradient, strain and stress tensors

From deformations to strains: 3D

In terms of tensor components this yields:

**Green - Lagrange** strains:

\[
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)
\]

**Almansi** strains:

\[
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial t x_j} + \frac{\partial u_j}{\partial t x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial t x_i} \frac{\partial u_k}{\partial t x_j} \right)
\]
The stress tensors

Finally we need to establish the stresses

We start by introducing the **Cauchy stresses**: 

\[
\mathbf{\sigma} = \sigma_{ij} = \begin{bmatrix} T^{(e_1)} \\ T^{(e_2)} \\ T^{(e_3)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \\
\equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}
\]

The **Cauchy stress** tensor relates forces at the current configuration to areas at the current configuration.
The stress tensors

Since the Cauchy tensor is not known a-priori, forces and areas are mapped through the deformation gradient to the reference configuration.

Definition

The Second Piola-Kirchoff stress tensor

\[
\begin{align*}
\mathbf{S} &= \frac{\rho}{t} \mathbf{X} \tau \mathbf{X}^T \\
\end{align*}
\]

where \( \rho \) is the mass density of the body at time \( t \) and \( \frac{\rho}{t} = \text{det}(\mathbf{X}) \)

- These are so-called work conjugate to the Green - Lagrange strains
- The mapping retains the symmetry of the Cauchy tensor
- Rigid body motions (translations/rotations) do not induce strains/stresses
- The components of the Piola-Kirchoff stress tensor have little physical meaning and in practice, Cauchy stresses must be calculated