Applications of the Scaled Boundary Finite Element Method in Linear Elastic Fracture Mechanics

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Motivation I

- FEM has been extended to handle a multitude of problems:
  - Non-linearity
    - Geometric
    - Material
  - Dynamics
  - Contact problems
  - Crack problems
  - Optimization problems
  - Inverse problems

http://gem-innovation.com/services/mesh-independent-fem/
Motivation II

Only partially alleviated by:

- Vectorization and parallelization of code
- Multiscale schemes, sub-structuring, …
- Fast multipole boundary element method (FMBEM)
- Extended finite element method (xFEM)
- Etc.
Conceptual Comparison of FEM, BEM and SBFEM

**FEM:**
- High amount of DOF
- Crack surface discretized
- Discretization error in all directions

**BEM:**
- Discretization on boundary only
- Crack surface discretized
- Non-symmetric dense matrices

**SBFEM bounded domain:**
- Introduction of a scaling center
- Discretization on boundary only
- Crack surface not discretized
- Analytical solution in radial direction

**SBFEM unbounded domain:**
- Introduction of a scaling center
- Discretization on boundary only
- Analytical solution in radial direction
Legend:

- $P(\hat{x}, \hat{y}) = \text{point in domain}$
- $P(x, y) = \text{point on boundary}$
- $(\hat{u}, \hat{t}) = \text{boundary conditions of domain}$
- $(u^e, t^e) = \text{boundary conditions of elements (i.e. discretized boundary)}$
  - $\bullet$ = nodes of line element with coordinates $(x_n, y_n)$
  - $O = \text{scaling center and origin of normalized radial coordinate } \zeta$
  - $(\xi, \eta) = \text{scaled boundary coordinates with } 0 \leq \xi \leq 1 \text{ and } -1 \leq \eta \leq 1$
- $DOF_i = \text{degree of freedom of each node in global } (x,y)\text{-direction}$
- $(x, y) = \text{global coordinate system}$

\[
\begin{align*}
\hat{x}(\xi, \eta) &= \hat{x}_0 + \xi \ast x(\eta) = \hat{x}_0 + \xi \ast [N(\eta)] [x] \\
\bullet \; \hat{x}_0 &= \text{position of the scaling center} \\
\bullet \; [x] &= [x_1, x_2, ..., x_n]^T \\
\bullet \; [N(\eta)] &= \text{shape function}
\end{align*}
\]

\[
[u(\xi, \eta)] = [N^u(\eta)] [u(\xi)]
\]
SBFEM Fundamentals II:

- Applying the principle of virtual work* yields 2 equations:

  **Boundary:** \( \{P\} = \{E^0\}\{u\},_\xi + \{E^1\}^T\{u\} \)

  **Domain:** \([E^0] \xi^2 \{u(\xi)\},_\xi + ([E^0] + [E^1]^T - [E^1]) \xi \{u(\xi)\},_\xi - [E^2] \{u(\xi)\} = \{0\} \)

  where \([E^0] = \int_{\partial \Omega} [B^1(\eta)]^T [D][B^1(\eta)]|J|d\eta \)

  \([E^1] = \int_{\partial \Omega} [B^1(\eta)]^T [D][B^2(\eta)]|J|d\eta \)

  \([E^2] = \int_{\partial \Omega} [B^2(\eta)]^T [D][B^2(\eta)]|J|d\eta \)

- General solution for displacements is assumed of form:

  \( \{u(\xi)\} = c_1 \xi^{\lambda_1} \{\phi_1\} + c_2 \xi^{\lambda_2} \{\phi_2\} + \ldots + c_n \xi^{\lambda_n} \{\phi_n\} \)

  \( = [\phi] \xi^{[\lambda]} \{c\} \)

  * Derivation at end of presentation
SBFEM Fundamentals III:

- Solution assumed as a power series:
  - Notice the similarity to the mode superposition method!

\[ [u(\xi)] = \sum [\Phi_i][\xi^{-\lambda_i}][c_i] \]

- \([\Phi_i]\) = eigenvector
- \(\lambda_i\) = corresponding eigenvalue
- \([c_i]\) = integration constant

http://web.sbe.hw.ac.uk/acme2011/Handout_Scaled_boundary_methods_CA.pdf
Leads to a Hamiltonian eigenvalue problem of \([Z]\):
- Introducing a new variable leads to a first order differential equation

\[
\{X(\zeta)\} = \begin{cases}
\{u(\zeta)\} \\
\{q(\zeta)\}
\end{cases} \quad \Rightarrow \quad \zeta\{X(\zeta)\},_{\zeta} = -[Z]\{X(\zeta)\}
\]

with:
\[
[Z] = \begin{bmatrix}
[E^0]^{-1}[E^1]^T & -[E^0]^{-1} \\
-[E^2] + [E^1][E^0]^{-1}[E^1]^T & -[E^1][E^0]^{-1}
\end{bmatrix}
\]

- Substituting the general solution into the obtained equation:

\[
[Z][\Phi] = [\Phi][\lambda] = \begin{bmatrix}
[\Phi_{u1}] & [\Phi_{u2}]
\end{bmatrix}
\begin{bmatrix}
[\lambda_n] \\
[\lambda_p]
\end{bmatrix}
\]

- Having calculated the decomposition

Displacements:
\[
\{u(\zeta)\} = ([\Phi_{u1}]_{\zeta}^{-[\lambda_n]}\{c_1\} + [\Phi_{u2}]_{\zeta}^{-[\lambda_p]}\{c_2\}) \quad \Rightarrow \quad \{R\} = \{q(\zeta = 1)\} = [K]\{u(\zeta = 1)\}
\]

Forces:
\[
\{q(\zeta)\} = ([\Phi_{q1}]_{\zeta}^{-[\lambda_n]}\{c_1\} + [\Phi_{q2}]_{\zeta}^{-[\lambda_p]}\{c_2\})
\]

\[
[K_{bounded}] = +[\Phi_{q1}][\Phi_{u1}]^{-1}
\]
\[
[K_{unbounded}] = -[\Phi_{q2}][\Phi_{u2}]^{-1}
\]
Basic Analysis Procedure

+ 
Reduction of problem dimension by 1
Semi-analytical solution in radial direction
Bounded and unbounded domains

- 
Assembly of multiple coefficient matrices
Schur/Eigen decomposition
Fully populated stiffness matrix
Comparison to ABAQUS reference solution

ABAQUS:

#DOF = 50'000+
Real time = 11s

Time savings:

$10^2 \times @ \sim 0.1\%$
Cost of using higher order elements

0.8 seconds

Computational Time vs #DOF

#DOF

360 DOF

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Stress Intensity factors (SIFs)

- Analog to stress concentration factors in i.e. tunneling
- In fracture mechanics predicts stress distribution near crack tip and is useful for providing a failure criterion:

\[
\sigma_{ij}(r, \theta) = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \text{higher order terms}
\]

\( K = \text{stress intensity factor} \)

\( f_{ij} = \text{function of load and geometry} \)

\[
K_1 = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{yy}(r, 0) \\
K_{II} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{yz}(r, 0) \\
K_{III} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{yz}(r, 0)
\]

### What is being compared?

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ABAQUS: Contour Integral

- Integral based on tractions and displacements

\[
K_I = \frac{E}{2(1-\nu^2)\Gamma} \int \left[ t_n U_{nI}^* + t_s U_{sI}^* - u_n T_{nI}^* - u_s T_{sI}^* \right] d\Gamma
\]

\[
K_{II} = \frac{E}{2(1-\nu^2)\Gamma} \int \left[ t_n U_{nII}^* + t_s U_{sII}^* - u_n T_{nII}^* - u_s T_{sII}^* \right] d\Gamma
\]

- Requires information about crack propagation direction
- Cannot predict how a crack will propagate


ABAQUS DEMO
Extended finite element method (xFEM) I

- Goal: Separate geometry from mesh
  - XFEM achieves this by locally enriching the FE approximation with local partitions of unity enrichment functions
Extended finite element method (xFEM) II

- xFEM aims to overcome the shortcomings of FEM
- Does so by introducing two kinds of enrichment
  - Jump enrichment
  - Tip enrichment

\[ u = \sum_{i \in I} N_i u_i + \sum_{j \in j} b_j N_i H(X) \]
\[ + \sum_{k \in K} N_i \sum_{l=1}^{4} c_{kl} F_l(X) \]

- Achieves:
  - Higher accuracy for stresses at crack tip
  - Less remeshing required
  - Level set method used to efficiently track cracks
xFEM: Jump enrichment

\[ \sum_{j \in J} b_j N_i H(X) \]

Heaviside step function
courtesy of Kostas Agathos
xFEM: Tip enrichment

- Analytical solution for the crack problem solved by Westergaard (1939) using a complex Airy stress function
  
  \[ u(x, y) = \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \cos \left( \frac{\theta}{2} \right) \left[ \kappa - 1 + 2\sin^2 \left( \frac{\theta}{2} \right) \right] \]
  
  \[ v(x, y) = \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \sin \left( \frac{\theta}{2} \right) \left[ \kappa + 1 - 2\cos^2 \left( \frac{\theta}{2} \right) \right] \]
  
  \[ + \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \sin \left( \frac{\theta}{2} \right) \left[ \kappa + 1 + 2\cos^2 \left( \frac{\theta}{2} \right) \right] \]
  
  \[ - \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \cos \left( \frac{\theta}{2} \right) \left[ \kappa - 1 - 2\sin^2 \left( \frac{\theta}{2} \right) \right] \]

- These can be spanned by the following basis, which are used as enrichment functions for the crack tip
  
  \[ \{ \sqrt{r} \sin \left( \frac{\theta}{2} \right), \sqrt{r} \cos \left( \frac{\theta}{2} \right), \sqrt{r} \sin(\theta) \sin \left( \frac{\theta}{2} \right), \sqrt{r} \sin(\theta) \cos \left( \frac{\theta}{2} \right) \} \]

Deformed crack for \( u_x = \sqrt{r} \sin(\theta/2) \) courtesy of Kostas Agathos
xFEM DEMO
SBFEM: Analytical limit in radial direction

- Strain:
  \[ \{ \mathbf{e}(\xi, \eta) \} = [\mathbf{B}^1(\eta)] \{ \mathbf{u}(\xi) \},_\xi + \frac{1}{\xi} [\mathbf{B}^2(\eta)] \{ \mathbf{u}(\xi) \} \]

- Stress:
  \[ \{ \sigma(\xi, \eta) \} = [\mathbf{D}] \left( [\mathbf{B}^1(\eta)] \{ \mathbf{u}(\xi) \},_\xi + \frac{1}{\xi} [\mathbf{B}^2(\eta)] \{ \mathbf{u}(\xi) \} \right) \]

- Take limit as \( \xi \to 0 \); Singularity for eigenvalues \(-1 < \lambda < 0\)
  \[ \{ \sigma^{(s)}(\xi, \eta) \} = \left[ \Gamma_i(\eta) \right] \xi^{-[\lambda s]} {}^{[\xi]} \{ \mathbf{c}^{(s)} \} \]
  where: \( \Gamma_i = \begin{cases} \Gamma_{xx} \\ \Gamma_{yy} \\ \Gamma_{xy} \end{cases} \)
  \[ \mathbf{D} \{ -\lambda_i \mathbf{B}_1(\eta) + \mathbf{B}_2(\eta) \} \Phi_i \]

- By matching expressions with the exact solution:
  \[ \left\{ \begin{array}{c} K_L \\ K_{II} \end{array} \right\} = \sqrt{2\pi L_0} \left( \sum_{i=I,II} c_i \Gamma_{yy}(\eta = \eta_A)_i \right) \]
  \[ \sum_{i=I,II} c_i \Gamma_{xy}(\eta = \eta_A)_i \]
Effects of Stress Smoothing

SIF Error [%] raw vs exact solution

SIF Error [%] recovered vs exact solution

Calculated SBFEM SIF Error [%]

 Exact solution

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8

1 1.5 2

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8

1 1.5 2

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8

1 1.5 2

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8

1 1.5 2
Error Estimator for Stress Intensity Factors

Plain Stress SIF Comparison for 3-Node Elements vs #DOF

Calculated Error in SIF [%]

-5 -4 -3 -2 -1 0 1

#DOF [1]

-50 100 200 300

Recovered SIF

Raw SIF

SIF error estimator

- exact solution
- non-smoothed
- smoothed
- estimated error
- deviation of error estimator
SBFEM DEMO
Numerical experiments: Stress intensity factors

1. \( \sigma = 0.1 \text{ GPa} \)
   \( \epsilon \to 0 \)
   \( E = 200 \text{ GPa} \)
   \( \nu = 0.3 \)

2. \( \tau = 0.1 \text{ GPa} \)
   \( y \)
   \( x \)
   \( E = 200 \text{ GPa} \)
   \( \nu = 0.3 \)

3. \( \sigma = 0.1 \text{ GPa} \)
   \( \gamma = 75 \text{ mm} \)
   \( \pi/4 \)
   \( 2b = 150 \text{ mm} \)

150 mm

75 mm

75 mm

75 mm
Numerical Experiment 1

SIF K1

DOF

10^6

1.0%
0.5%
0.1%

Note: The graph shows the variation of SIF K1 with DOF for different mesh sizes and element types. The legend indicates the specific configurations tested, such as ABAQUS, XFEM, and SBFEM, with varying node counts.
Numerical Experiment 1

SIF K1

seconds

10^{-1} 10^{1} 10^{2}

1.0% 0.5% 0.1%

1.0% 0.5% 0.1%

4.4 4.36 4.32 4.3 4.29 4.26 4.24 4.22 4.2

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10^2
Numerical Experiment 2

Graph showing SIF K1 vs. time in seconds for various methods and mesh sizes.
Numerical Experiment 2
Numerical Experiment 3

Graphs showing the relationship between SIF K1 and SIF K2 against seconds. The graphs include data points for different mesh densities: 1.55 * mesh, 2.55 * mesh, and 3.55 * mesh. The data is compared against ABAQUS, XFEM, and SBFEM results.
Observations

- Incredibly fast convergence of SBFEM using few DOF
- Exceptional speed considering non-optimized code
- Stress recovery methods dramatically improve accuracy at virtually no additional computational cost

- Always choose at least two elements per side
- Three elements per side are better
- Do not use two node elements if possible
Conclusion

- SBFEM combines many of the desirable characteristics of FEM and BEM into one method with additional benefits of its own:
  - Analytical solution in radial direction:
    - Higher accuracy per DOF
    - permits elegant and efficient calculation of stress intensity factors
  - Stress recovery enhances results greatly
    - Must only be performed on the boundary
    - Large workload can be performed in advance
  - No change necessary to solution process to extract crack related phenomena (i.e. SIFs and T-stress of various orders of singularity)
  - Dense and fully populated matrices:
    - Higher order elements don’t (noticeably) impact performance
Questions
SBFEM derivation I

The strong form of the governing equations in 2D elastostatics is given as follows:

- equilibrium: \( \nabla \cdot \sigma + b = 0 \) in \( \Omega \) (1a)
- constitutive: \( \sigma = D\varepsilon \) in \( \Omega \) (1b)
- compatibility: \( \varepsilon = \nabla^T u \) in \( \Omega \) (1c)
- boundary conditions: \( u = \hat{u} \) on \( \Gamma \) (1d)
  \( \sigma \cdot n = \hat{t} \) on \( \Gamma \) (1e)
SBFEM derivation II

- **Geometry transformation**

  \[
  x(\xi, \eta) = x_0 + \xi x(\eta) = x_0 + \xi [\textbf{N}(n)] \{x\} \tag{2a}
  \]

  \[
  y(\xi, \eta) = y_0 + \xi y(\eta) = y_0 + \xi [\textbf{N}(n)] \{y\} \tag{2b}
  \]

- **Jacobian**

  \[
  \begin{bmatrix}
  \frac{\partial}{\partial \xi} \\
  \frac{\partial}{\partial \eta}
  \end{bmatrix} = \begin{bmatrix}
  \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
  \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
  \end{bmatrix} \begin{bmatrix}
  \frac{\partial}{\partial x} \\
  \frac{\partial}{\partial y}
  \end{bmatrix} = \textbf{J} \cdot \begin{bmatrix}
  \frac{\partial}{\partial x} \\
  \frac{\partial}{\partial y}
  \end{bmatrix} \tag{3a}
  \]

  \[
  \begin{bmatrix}
  \frac{\partial x}{\partial \xi} \\
  \frac{\partial y}{\partial \eta}
  \end{bmatrix} = \frac{1}{|\textbf{J}|} \begin{bmatrix}
  \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \\
  -\frac{\partial x}{\partial \eta} + \frac{\partial x}{\partial \xi}
  \end{bmatrix} \begin{bmatrix}
  \frac{\partial}{\partial \xi} \\
  \frac{\partial}{\partial \eta}
  \end{bmatrix} = \textbf{J}^{-1} \cdot \begin{bmatrix}
  \frac{\partial}{\partial \xi} \\
  \frac{\partial}{\partial \eta}
  \end{bmatrix} \tag{3b}
  \]

- **Differential unit volumen**

  \[dV = |\textbf{J}| \xi d\xi d\eta\]
The linear differential operator $L$ may thus be written as:

The derivatives may further be used to construct the linear operator $L$ in scaled boundary coordinates. In a first step, it is split. Then the partial derivatives are substituted:

$$
[L] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial y} = [L^1] \frac{\partial}{\partial x} + [L^2] \frac{\partial}{\partial y} \quad (5a)
$$

$$
= \frac{1}{|J|} \left[ [L^1] \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) + [L^2] \left( \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} \right) \right] \quad (5b)
$$

$$
=[b^1(\eta)] \frac{\partial}{\partial \xi} + \frac{1}{\xi}[b^2(\eta)] \frac{\partial}{\partial \eta} \quad (5c)
$$

with $[b^1(\eta)] = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & 0 \\ 0 & -\frac{\partial x}{\partial \eta} \end{bmatrix}$, $[b^2(\eta)] = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \xi} & 0 \\ 0 & \frac{\partial x}{\partial \xi} \end{bmatrix}$. 

SBFEM derivation IV

- Assuming an analytical solution in radial direction:
  \[
  \{u(\xi, \eta)\} = [N^u(\eta)]\{u(\xi)\}
  \]
  \[
  [N^u(\eta)] = [N^1(\eta)I_n, N^2(\eta)I_n, \ldots, N^2(\eta)I_n]
  \]

- And therefore the strains and stresses become:
  \[
  \{\varepsilon(\xi, \eta)\} = [B^1(\eta)]\{u(\xi)\},_\xi + \frac{1}{\xi} [B^2(\eta)]\{u(\xi)\} \quad (9)
  \]

where

\[
[B^1(\eta)] = [b^1(\eta)][N^u(\eta)] \quad (10a)
\]
\[
[B^2(\eta)] = [b^2(\eta)][N^u(\eta)],_\eta \quad (10b)
\]

And as a consequence the stresses follow as:

\[
\{\sigma(\xi, \eta)\} = [D] \left( [B^1(\eta)]\{u(\xi)\},_\xi + \frac{1}{\xi} [B^2(\eta)]\{u(\xi)\} \right)
\]
SBFEM derivation V

- Setting up the virtual work formulation:

Implementing the above in as a virtual work statement we consider \( \{ \delta \mathbf{u}(\xi, \eta) \} \) and \( \{ \delta \mathbf{e}(\xi, \eta) \} \) as the virtual displacements and strains given as:

\[
\{ \delta \mathbf{u}(\xi, \eta) \} = [\mathbf{N}(\eta)] \{ \delta \mathbf{u}(\xi) \} \quad \text{(12a)}
\]
\[
\{ \delta \mathbf{e}(\xi, \eta) \} = [\mathbf{L}] \{ \delta \mathbf{u}(\xi, \eta) \} \quad \text{(12b)}
\]

As such, the virtual work statement in scaled boundary coordinates becomes:

\[
\int_{V} \{ \delta \mathbf{e}(\xi, \eta) \}^T \{ \mathbf{\sigma}(\xi, \eta) \} dV - \int_{\partial \Omega} \{ \delta \mathbf{u}(\eta) \}^T \{ \mathbf{t}(\eta) \} = 0
\]

(13)
SBFEM derivation VI

Considering the term representing the internal virtual work first it can be rearranged

\[ \int_V \{ \delta \varepsilon(\xi, \eta) \}^T \{ \sigma(\xi, \eta) \} dV \]

\[ = \int_V \left[ [B^1(\eta)] \{ \delta u(\xi) \},_{\xi} + \frac{1}{\xi} [B^2(\eta)] \{ \delta u(\xi) \} \right]^T \]

\[ \times [D] \left( [B^1(\eta)] \{ u(\xi) \},_{\xi} + \frac{1}{\xi} [B^2(\eta)] \{ u(\xi) \} \right) dV \]

\[ = \int_{\partial \Omega} \int_{\xi = 0}^{\xi = 1} \{ \delta u(\xi) \} \{ [B^1(\eta)] \}^T [D] [B^1(\eta)] \{ u(\xi) \},_{\xi} |J| d\xi d\eta \]

\[ + \int_{\partial \Omega} \int_{\xi = 0}^{\xi = 1} \{ \delta u(\xi) \} \{ [B^1(\eta)] \}^T [D] [B^2(\eta)] \{ u(\xi) \},_{\xi} |J| d\xi d\eta \]

\[ + \int_{\partial \Omega} \int_{\xi = 0}^{\xi = 1} \{ \delta u(\xi) \} \{ [B^2(\eta)] \}^T [D] [B^1(\eta)] \{ u(\xi) \},_{\xi} |J| d\xi d\eta \]

\[ + \int_{\partial \Omega} \int_{\xi = 0}^{\xi = 1} \{ \delta u(\xi) \} \{ [B^2(\eta)] \}^T [D] [B^2(\eta)] \frac{1}{\xi} \{ u(\xi) \},_{\xi} |J| d\xi d\eta \]

\[ (14) \]
SBFEM derivation VII

By applying Green’s theorem, i.e. integration by parts, the integrals containing \( \{\delta u(\xi)\}^T \xi \) are converted, which leads to the following formulation:

\[
\int_V \{\delta \epsilon(\xi, \eta)\}^T \{\sigma(\xi, \eta)\} dV
= \int_{\partial \Omega} \{\delta u(\xi)\}^T [B^1(\eta)]^T [D][B^1(\eta)] \xi \{u(\xi)\}, \xi | d\eta \Big|_{\xi=1}
- \int_{\partial \Omega} \{\delta u(\xi)\}^T [B^1(\eta)]^T [D][B^1(\eta)]
\times \{\{u(\xi)\}_\xi + \{u(\xi)\}_{\xi\xi}\} | J | d\xi d\eta
+ \int_{\partial \Omega} \{\delta u(\xi)\}^T [B^1(\eta)]^T [D][B^2(\eta)] \{u(\xi)\} | J | d\eta \Big|_{\xi=1}
- \int_{\partial \Omega} \int_{\xi=0}^{\xi=1} \{\delta u(\xi)\}^T [B^1(\eta)]^T [D][B^2(\eta)] \{u(\xi)\}, \xi | J | d\xi d\eta
+ \int_{\partial \Omega} \int_{\xi=0}^{\xi=1} \{\delta u(\xi)\}^T [B^2(\eta)]^T [D][B^1(\eta)] \{u(\xi)\}, \xi | J | d\xi d\eta
+ \int_{\partial \Omega} \int_{\xi=0}^{\xi=1} \{\delta u(\xi)\}^T [B^2(\eta)]^T [D][B^2(\eta)] \frac{1}{\xi} \{u(\xi)\} | J | d\xi d\eta
\]
SBFEM derivation VIII

- Introducing some substitutions

\[ E^0 = \int_{\partial\Omega} [B^1(\eta)]^T [D][B^1(\eta)] |J| d\eta \]  \hspace{1cm} (16a)

\[ E^1 = \int_{\partial\Omega} [B^1(\eta)]^T [D][B^2(\eta)] |J| d\eta \]  \hspace{1cm} (16b)

\[ E^2 = \int_{\partial\Omega} [B^2(\eta)]^T [D][B^2(\eta)] |J| d\eta \]  \hspace{1cm} (16c)

- Leads to some significant simplifications

\[ \int_V \{ \delta \varepsilon(\xi, \eta) \}^T \{ \sigma(\xi, \eta) \} dV = \{ \delta u \}^T \left\{ [E^0] \{ u \}, \xi + [E^1] \{ u \} \right\} \]

\[ - \int_{\xi=1} \left\{ \delta u(\xi) \right\}^T \left\{ [E^0] \xi \{ u(\xi) \}, \xi \right\} \]

\[ + \left\{ [E^0] + [E^1]^T - [E^1] \right\} \left\{ u(\xi) \right\}, \xi - [E^2] \frac{1}{\xi} \left\{ u(\xi) \right\} \right\} d\xi \]  \hspace{1cm} (17)
SBFEM derivation IX

Having performed the necessary derivations for the internal virtual work, let us shift our focus to the external virtual work term:

\[ \int_{\partial \Omega} \{\delta \mathbf{u}(\eta)\}^T \{\mathbf{t}(\eta)\} d\eta = \{\delta \mathbf{u}\}^T \int_{\partial \Omega} \{\mathbf{N}(\eta)\}^T \{\mathbf{t}(\eta)\} d\eta \]

(18)

Assuming that no other forces are acting on the domain other than the traction on the domain boundary, these may be identified as the equivalent nodal forces due to boundary tractions also termed \{\mathbf{P}\}. Equating the internal virtual work to the external virtual work statements we can formulate the complete virtual work equation:

\[
\{\delta \mathbf{u}\}^T \left\{[\mathbf{E}^0][\mathbf{u}],_\xi + [\mathbf{E}^1]^T \{\mathbf{u}\} - \{\mathbf{P}\}\right\} \\
- \int_{\xi=1}^\xi \{\delta \mathbf{u}(\xi)\}^T \left\{[\mathbf{E}^0]\xi \{\mathbf{u}(\xi)\},_\xi\right\} \\
+ \left\{[\mathbf{E}^0] + [\mathbf{E}^1]^T - [\mathbf{E}^1]\right\} \{\mathbf{u}(\xi)\},_\xi - [\mathbf{E}^2] \frac{1}{\mu} \{\mathbf{u}(\xi)\} \right\} d\xi = \{0\}
\]
SBFEM derivation X

In order for this equation to hold for all $\xi$, which implies continuously satisfied in radial direction and only compliant in the finite element sense in the tangential direction, both of the following conditions must be met:

$$\{P\} = [E^0]\{u\},_\xi + [E^1]^T\{u\} \quad (20)$$

$$[E^0]\xi^2\{u(\xi)\},_{\xi\xi} + [[E^0] + [E^1]^T - [E^1]]\xi\{u(\xi)\},_\xi - [E^2]\{u(\xi)\} = \{0\} \quad (21)$$

The above equation is termed the scaled boundary finite element equation in displacement.