Applications of the Scaled Boundary Finite Element Method in Linear Elastic Fracture Mechanics

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Motivation I

- FEM has been extended to handle a multitude of problems:
  - Non-linearity
    - Geometric
    - Material
  - Dynamics
  - Contact problems
  - Crack problems
  - Optimization problems
  - Inverse problems

http://gem-innovation.com/services/mesh-independent-fem/
Motivation II

Only partially alleviated by:

- Vectorization and parallelization of code
- Multiscale schemes, sub-structuring, …
- Fast multipole boundary element method (FMBEM)
- Extended finite element method (xFEM)
- Etc.
Conceptual Comparison of FEM, BEM and SBFEM

**FEM:**
- High amount of DOF
- Crack surface discretized
- Discretization error in all directions

**BEM:**
- Discretization on boundary only
- Crack surface discretized
- Non-symmetric dense matrices

**SBFEM bounded domain:**
- Introduction of a scaling center
- Discretization on boundary only
- Crack surface not discretized
- Analytical solution in radial direction

**SBFEM unbounded domain:**
- Introduction of a scaling center
- Discretization on boundary only
- Analytical solution in radial direction
The strong form of the governing equations in 2D elastostatics is given as follows:

- **Equilibrium**: \( \nabla \cdot \sigma + b = 0 \) in \( \Omega \) (1a)
- **Constitutive**: \( \sigma = D\varepsilon \) in \( \Omega \) (1b)
- **Compatibility**: \( \varepsilon = \nabla^T u \) in \( \Omega \) (1c)
- **Boundary Conditions**: \( u = \hat{u} \) on \( \Gamma \) (1d)
  \( \sigma \cdot n = \hat{t} \) on \( \Gamma \) (1e)
SBFEM (PVV 2)

Legend:

\( P(\hat{x}, \hat{y}) = \) point in domain
\( P(x, y) = \) point on boundary
\((\bar{u}, \bar{t}) = \) boundary conditions of domain
\((\bar{u}^e, \bar{t}^e) = \) boundary conditions of elements (i.e. discretized boundary)

\( \bullet = \) nodes of line element with coordinates \((x_n, y_n)\)

\( O = \) scaling center and origin of normalized radial coordinate \(\xi\)

\( (\xi, \eta) = \) scaled boundary coordinates with \(0 \leq \xi \leq 1\) and \(-1 \leq \eta \leq 1\)

\( DOF_i = \) degree of freedom of each node in global \((x,y)\)-direction

\( (x, y) = \) global coordinate system

\[
\hat{\mathbf{u}}(\xi, \eta) = \hat{x}_0 + \xi \ast \mathbf{x}(\eta) = \hat{x}_0 + \xi \ast [N(\eta)]\mathbf{x}
\]

- \( \hat{x}_0 = \) position of the scaling center
- \([\mathbf{x}] = [x_1, x_2, ..., x_n]^T\)
- \([N(\eta)] = \) shape function

\[
[\mathbf{u}(\xi, \eta)] = [N^{\mathbf{u}}(\eta)][\mathbf{u}(\xi)]
\]
SBFEM (PVV 3)

- Geometry transformation

\[
x(\xi, \eta) = x_0 + \xi x(\eta) = x_0 + \xi [N(n)] \{x\} \quad (2a)
\]
\[
y(\xi, \eta) = y_0 + \xi y(\eta) = y_0 + \xi [N(n)] \{y\} \quad (2b)
\]

- Jacobian

\[
\begin{pmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{pmatrix} =
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{pmatrix} = J
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{pmatrix} \quad (3a)
\]

\[
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{pmatrix} = \frac{1}{|J|}
\begin{bmatrix}
\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \\
-\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}
\end{bmatrix}
\begin{pmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{pmatrix} = J^{-1}
\begin{pmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{pmatrix} \quad (3b)
\]

- Differential unit volume

\[dV = |J| \xi \, d\xi \, d\eta\]
SBFEM (PVV 4)

- The linear differential operator $L$ may thus be written as:

The derivatives may further be used to construct the linear operator $L$ in scaled boundary coordinates. In a first step, it is split. Then the partial derivatives are substituted:

$$[L] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial y} = [L^1] \frac{\partial}{\partial x} + [L^2] \frac{\partial}{\partial y} \tag{5a}$$

$$= \frac{1}{|J|} \left[ [L^1] \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) + [L^2] \left( \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} \right) \right] \tag{5b}$$

$$=[b^1(\eta)] \frac{\partial}{\partial \xi} + \frac{1}{\xi} [b^2(\eta)] \frac{\partial}{\partial \eta} \tag{5c}$$

with $[b^1(\eta)] = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & 0 \\ 0 & -\frac{\partial x}{\partial \eta} \end{bmatrix}$ and $[b^2(\eta)] = \frac{1}{|J|} \begin{bmatrix} -\frac{\partial y}{\partial \xi} & 0 \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \end{bmatrix}$
SBFEM (PVV 5)

- Assuming an analytical solution in radial direction:

\[ \{u(\xi, \eta)\} = [N^u(\eta)]\{u(\xi)\} \]

\[ [N^u(\eta)] = [N^1(\eta)I_n, N^2(\eta)I_n, \ldots, N^2(\eta)I_n] \]

- And therefore the strains and stresses become:

\[ \{\varepsilon(\xi, \eta)\} = [B^1(\eta)]\{u(\xi)\},_\xi + \frac{1}{\xi}[B^2(\eta)]\{u(\xi)\} \]  

(9)

where

\[ [B^1(\eta)] = [b^1(\eta)][N^u(\eta)] \]  

(10a)

\[ [B^2(\eta)] = [b^2(\eta)][N^u(\eta)],_\eta \]  

(10b)

And as a consequence the stresses follow as:

\[ \{\sigma(\xi, \eta)\} = [D]\left([B^1(\eta)]\{u(\xi)\},_\xi + \frac{1}{\xi}[B^2(\eta)]\{u(\xi)\}\right) \]
SBFEM (PVV 6)

- Setting up the virtual work formulation:

Implementing the above in as a virtual work statement we consider \( \{ \delta \mathbf{u}(\xi, \eta) \} \) and \( \{ \delta \mathbf{e}(\xi, \eta) \} \) as the virtual displacements and strains given as:

\[
\{ \delta \mathbf{u}(\xi, \eta) \} = [\mathbf{N}(\eta)] \{ \delta \mathbf{u}(\xi) \} \quad (12a)
\]
\[
\{ \delta \mathbf{e}(\xi, \eta) \} = [\mathbf{L}] \{ \delta \mathbf{u}(\xi, \eta) \} \quad (12b)
\]

As such, the virtual work statement in scaled boundary coordinates becomes:

\[
\int_{V} \{ \delta \mathbf{e}(\xi, \eta) \}^T \{ \sigma(\xi, \eta) \} dV - \int_{\partial \Omega} \{ \delta \mathbf{u}(\eta) \}^T \{ \mathbf{t}(\eta) \} = 0
\]

(13)
SBFEM (PVV 7)

Considering the term representing the internal virtual work first it can be rearranged

$$\int_V \left\{ \delta \varepsilon(\xi, \eta) \right\}^T \{ \sigma(\xi, \eta) \} dV$$

$$= \int_V \left[ [B^1(\eta)] \{ \delta u(\xi) \} ,_\xi + \frac{1}{\xi} [B^2(\eta)] \{ \delta u(\xi) \} \right]^T$$

$$\times [D] \left( [B^1(\eta)] \{ u(\xi) \} ,_\xi + \frac{1}{\xi} [B^2(\eta)] \{ u(\xi) \} \right) dV$$

$$= \int_{\partial \Omega} \int_{\xi=0}^{\xi=1} \left\{ \delta u(\xi) \right\}^T [B^1(\eta)]^T [D][B^1(\eta)] [\xi \{ u(\xi) \} ,_\xi | J] d\xi d\eta$$

$$+ \int_{\partial \Omega} \int_{\xi=0}^{\xi=1} \left\{ \delta u(\xi) \right\}^T [B^1(\eta)]^T [D][B^2(\eta)] [\{ u(\xi) \} ,_\xi | J] d\xi d\eta$$

$$+ \int_{\partial \Omega} \int_{\xi=0}^{\xi=1} \left\{ \delta u(\xi) \right\}^T [B^2(\eta)]^T [D][B^1(\eta)] [\{ u(\xi) \} ,_\xi | J] d\xi d\eta$$

$$+ \int_{\partial \Omega} \int_{\xi=0}^{\xi=1} \left\{ \delta u(\xi) \right\}^T [B^2(\eta)]^T [D][B^2(\eta)] \frac{1}{\xi} [\{ u(\xi) \} ,_\xi | J] d\xi d\eta$$

(14)
SBFEM (PVV 8)

By applying Green’s theorem, i.e. integration by parts, the integrals containing \( \{ \delta \mathbf{u}(\xi) \} \xi \) are converted, which leads to the following formulation:

\[
\int_V \{ \delta \mathbf{\varepsilon}(\xi, \eta) \}^T \{ \mathbf{\sigma}(\xi, \eta) \} dV = \int_{\partial \Omega} \left. \left\{ \delta \mathbf{u}(\xi) \right\}^T \left[ \mathbf{B}^1(\eta) \right]^T \left[ \mathbf{D} \right] \left[ \mathbf{B}^1(\eta) \right] \xi \{ \mathbf{u}(\xi) \} , \xi \right|_\xi = 1 d\eta \] 

\[
- \int_{\partial \Omega} \left. \left\{ \delta \mathbf{u}(\xi) \right\}^T \left[ \mathbf{B}^1(\eta) \right]^T \left[ \mathbf{D} \right] \left[ \mathbf{B}^1(\eta) \right] \right| d\xi d\eta 
\times \left\{ \{ \mathbf{u}(\xi) \} , \xi + \{ \mathbf{u}(\xi) \} , \xi \xi \right\} | \mathbf{J}| d\xi d\eta 
\]

\[
+ \int_{\partial \Omega} \left. \left\{ \delta \mathbf{u}(\xi) \right\}^T \left[ \mathbf{B}^1(\eta) \right]^T \left[ \mathbf{D} \right] \left[ \mathbf{B}^2(\eta) \right] \{ \mathbf{u}(\xi) \} \right|_\xi = 1 d\eta 
\]

\[
- \int_{\partial \Omega} \int_{\xi = 0}^{\xi = 1} \left. \left\{ \delta \mathbf{u}(\xi) \right\}^T \left[ \mathbf{B}^1(\eta) \right]^T \left[ \mathbf{D} \right] \left[ \mathbf{B}^2(\eta) \right] \{ \mathbf{u}(\xi) \} , \xi \right| \mathbf{J} d\xi d\eta 
\]

\[
+ \int_{\partial \Omega} \int_{\xi = 0}^{\xi = 1} \left. \left\{ \delta \mathbf{u}(\xi) \right\}^T \left[ \mathbf{B}^2(\eta) \right]^T \left[ \mathbf{D} \right] \left[ \mathbf{B}^1(\eta) \right] \{ \mathbf{u}(\xi) \} , \xi \right| \mathbf{J} d\xi d\eta 
\]

\[
+ \int_{\partial \Omega} \int_{\xi = 0}^{\xi = 1} \left. \left\{ \delta \mathbf{u}(\xi) \right\}^T \left[ \mathbf{B}^2(\eta) \right]^T \left[ \mathbf{D} \right] \left[ \mathbf{B}^2(\eta) \right] \frac{1}{\xi} \{ \mathbf{u}(\xi) \} \right| \mathbf{J} d\xi d\eta 
\]
SBFEM (PVV 9)

- Introducing some substitutions

\[{E^0} = \int_{\partial \Omega} [B^1(\eta)]^T [D][B^1(\eta)]|J|d\eta \quad (16a)\]

\[{E^1} = \int_{\partial \Omega} [B^1(\eta)]^T [D][B^2(\eta)]|J|d\eta \quad (16b)\]

\[{E^2} = \int_{\partial \Omega} [B^2(\eta)]^T [D][B^2(\eta)]|J|d\eta \quad (16c)\]

- Leads to some significant simplifications

\[\int_V \{\delta \varepsilon(\xi, \eta)\}^T \{\sigma(\xi, \eta)\} dV = \{\delta u\}^T \{[E^0]\{u\}, \xi + [E^1]^T \{u\}\} \]

\[- \int_{\xi=0}^{\xi=1} \delta u(\xi)^T \{[E^0] \xi \{u(\xi)\}, \xi \xi \]

\[+ \{[E^0] + [E^1]^T - [E^1]\{u(\xi)\}, \xi - [E^2] \frac{1}{\xi} \{u(\xi)\}\} d\xi \quad (17)\]
SBFEM (PVV 10)

Having performed the necessary derivations for the internal virtual work, let us shift our focus to the external virtual work term:

$$\int_{\partial \Omega} \{ \delta u(\eta) \}^T \{ t(\eta) \} d\eta = \{ \delta u \}^T \int_{\partial \Omega} \{ N(\eta) \}^T \{ t(\eta) \} d\eta$$

(18)

Assuming that no other forces are acting on the domain other than the traction on the domain boundary, these may be identified as the equivalent nodal forces due to boundary tractions also termed \{P\}. Equating the internal virtual work to the external virtual work statements we can formulate the complete virtual work equation:

$$\{ \delta u \}^T \{ [E^0] \{ u \},\xi + [E^1]^T \{ u \} - \{ P \} \}$$

$$- \int_{\xi=0}^{\xi=1} \{ \delta u(\xi) \}^T \{ [E^0] \{ u(\xi) \},\xi \}$$

$$+ \{ [E^0] + [E^1]^T - [E^1] \} \{ u(\xi) \},\xi - [E^2] \frac{1}{\xi} \{ u(\xi) \} \} d\xi = \{ 0 \}$$
SBFEM (PVV 11)

In order for this equation to hold for all $\xi$, which implies continuously satisfied in radial direction and only compliant in the finite element sense in the tangential direction, both of the following conditions must be met:

$$\{P\} = \{E^0\}\{u\},\xi + \{E^1\}^T\{u\}$$  \hspace{1cm} (20)

$$\{E^0\}\xi^2\{u(\xi)\},\xi\xi + \{E^0\} + \{E^1\}^T - \{E^1\}\{u(\xi)\},\xi$$

$$- \{E^2\}\{u(\xi)\} = \{0\} \hspace{1cm} (21)$$

The above equation is termed the scaled boundary finite element equation in displacement.
SBFEM (Solution 1)

The general solution to the scaled boundary finite element equation can be written in power series form as:

\[ \{ u(\xi) \} = c_1 \xi^{-\lambda_1} \{ \phi_1 \} + c_2 \xi^{-\lambda_2} \{ \phi_2 \} + \ldots + c_n \xi^{-\lambda_n} \{ \phi_n \} \]

\[ = [\phi] \xi^{[-\lambda]} \{ c \} \quad (18) \]

where \( \lambda_i \) and \( \{ \phi_i \} \) are the corresponding eigenvalues and vectors respectively. The boundary conditions determine the integration constants \( c_i \).
SBFEM (Solution 2)

- Notice the similarity to the mode superposition method

\[ [u(\xi)] = \sum [\Phi_i] [\xi - \lambda_i] [c_i] \]

- \([\Phi_i]\) = eigenvector
- \(\lambda_i\) = corresponding eigenvalue
- \([c_i]\) = integration constant

http://web.sbe.hw.ac.uk/acme2011/Handout_Scaled_boundary_methods_CA.pdf
SBFEM (Solution 3)

- A quadratic eigenproblem results from substituting the general solution into the scaled boundary finite element equation in displacements:

  \[
  \left[ \lambda \right]^2 [E^0] - \lambda [[E^1]^T - [E^1]] - [E^2] \{\phi\} = \{0\}
  \]
  \[
  \{q\} = [[E^1]^T - \lambda [E^0]] \{\phi\}
  \]

- The boundary forces are now expressed as in an equivalent modal formulation. Conceptually, they represent the nodal force modes \{q\} required to balance the corresponding displacement modes \{\phi\}.
SBFEM (Solution 4)

- Linearizing the quadratic eigenproblem is obtained by rearranging the force mode equation and substituting into the scaled boundary finite element equation in displacements.

\[
\begin{align*}
\begin{bmatrix}\lambda\end{bmatrix}\{\phi\} &= \left[E^0\right]^{-1}\left[E^1\right]^T\{\phi\} - \{q\} \\
\lambda\left[E^0\right]\left[E^0\right]^{-1}\left[E^1\right]^T\{\phi\} - \{q\} - \lambda\left[E^1\right]^T\{\phi\} \\
+ \left[E^1\right]\left[E^0\right]^{-1}\left[E^1\right]^T\{\phi\} - \{q\} - [E^2]\{\phi\} &= \{0\}
\end{align*}
\]

which is equivalent to:

\[
\begin{bmatrix}\lambda\end{bmatrix}\{q\} = \left[E^1\right]\left[E^0\right]^{-1}\left[E^1\right]^T\{\phi\} - \{q\} - [E^2]\{\phi\}
\]
SBFEM (Solution 5)

The linearised, combined form of the quadratic eigenproblem can thus be expressed more compactly in matrix notation as:

\[
[Z] \begin{pmatrix} \phi \\ q \end{pmatrix} = [\lambda] \begin{pmatrix} \phi \\ q \end{pmatrix}
\]

where

\[
[Z] = \begin{bmatrix} [E^0]^{-1}[E^1]^T & -[E^0]^{-1} \\ [E^1][E^0]^{-1}[E^1]^T - [E^2] & -[E^1][E^0]^{-1} \end{bmatrix}
\]

(22)
SBFEM (Solution 6)

Since eigenvalues of opposite sign contribute to the response of the system in fundamentally different ways, which will be detailed in the following paragraph, they should be sorted accordingly.

\[
[Z] \begin{bmatrix} \phi \\ q \end{bmatrix} = \begin{bmatrix} \lambda \end{bmatrix} \begin{bmatrix} \phi \\ q \end{bmatrix} = \begin{bmatrix} \lambda_{neg} \\ \lambda_{pos} \end{bmatrix} \begin{bmatrix} [\phi_1] [\phi_2] \\ [O_1] [O_2] \end{bmatrix} \tag{24}
\]
SBFEM (Solution 7)

Having determined the eigenvalues and eigenvectors, these can be substituted into the general solution (Eqn. [18]) with the aim of determining the domain stiffness matrix.

\[ \{u(\xi)\} = [\phi_1] \xi^{[-\lambda_{neg}]} \{c_1\} + [\phi_2] \xi^{[-\lambda_{pos}]} \{c_2\} \]

\[ \{P_{bounded}\} = [q_1] \{c_1\} \]

\[ \{c_1\} = [\Phi_1]^{-1} \{u(\xi = 1)\} \]

\[ \{P_{bounded}\} = [q_1][\Phi_1]^{-1} \{u(\xi = 1)\} \]

\[ K_{bounded} = +[q_1][\Phi_1]^{-1} \]
# SBFEM implementation

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  processing | input processing                 |        |
|          | for each subdomain              |        |
|          | for each element                |        |
|          | for each integration point      | $10_a-b$ |
|          | evaluate coefficient matrices   | $14_a-c$ |
|          | assemble coefficient matrices   |        |

| solution | form Hamiltonian               | 23     |
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|          | $U = K^{-1}F$                  |        |

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Stress Intensity factors (SIFs)

- Analog to stress concentration factors in i.e. tunneling
- In fracture mechanics predicts stress distribution near crack tip and is useful for providing a failure criterion:

\[
\sigma_{ij}(r, \theta) = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \text{higher order terms}
\]

\(K\) = stress intensity factor

\(f_{ij}\) = function of load and geometry
SIFs by SBFEM (1)

- **Strain:**
  \[
  \{\varepsilon(\xi, \eta)\} = [B^1(\eta)]\{u(\xi)\},\xi + \frac{1}{\xi}[B^2(\eta)]\{u(\xi)\}
  \]

- **Stress:**
  \[
  \{\sigma(\xi, \eta)\} = [D]\left([B^1(\eta)]\{u(\xi)\},\xi + \frac{1}{\xi}[B^2(\eta)]\{u(\xi)\}\right)
  \]

- **Take limit as** \(\xi \to 0\); **Singularity for eigenvalues** \(-1 < \lambda_s < 0\)
SIFs by SBFEM (2)

- Take limit as $\xi \to 0$; Singularity for eigenvalues $-1 < \lambda_s < 0$

$$\{\sigma^{(s)}(\xi, \eta)\} = \left[ \Gamma_i(\eta) \right] \xi^{\lambda_s} \left\{ c^{(s)} \right\}$$

where: $\Gamma_i = \begin{cases} \Gamma_{xx} \\ \Gamma_{yy} \\ \Gamma_{xy} \end{cases}$

$$= D[\lambda_i \phi_1(\eta) + \phi_2(\eta)] \phi_i$$

- By matching expressions with the exact solution:

$$\begin{bmatrix} \phi_{1,1} & \cdots & \phi_{1,n} \\ \vdots & \ddots & \vdots \\ \phi_{n,1} & \cdots & \phi_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^1_{0.5} \\ \phi_{1,1} \\ \vdots \\ \phi_{i,1} \\ \vdots \\ \phi_{i+5,1} \\ \vdots \\ \phi_{n,1} \\ \phi_{i,1} \\ \vdots \\ \phi_{i,j} \\ \vdots \\ \phi_{i+5,j} \\ \vdots \\ \phi_{n,j} \\ \phi_{i,1} \\ \vdots \\ \phi_{i,n} \\ \vdots \\ \phi_{i+5,n} \\ \vdots \\ \phi_{n,n} \end{bmatrix}$$

$$\begin{bmatrix} K_I \\ K_{II} \end{bmatrix} = \sqrt{2\pi L_0} \begin{cases} \sum_{i=I,II} c_i \Gamma_{yy}(\eta = \eta_A)_i \\ \sum_{i=I,II} c_i \Gamma_{xy}(\eta = \eta_A)_i \end{cases}$$
Stress recovery for SIFs (1)

\[ \sigma^{rec}(\eta) = \{P\}\{a\} \]

\[
\{P\} = \begin{bmatrix} \eta^0_i \\ \eta^1_i \\ \vdots \\ \eta^n_i \end{bmatrix}, \quad \{a\} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \sigma^{raw} = \begin{bmatrix} \sigma^1_{\text{raw}} \\ \sigma^2_{\text{raw}} \\ \vdots \\ \sigma^n_{\text{raw}} \end{bmatrix}
\]

\[
[\tilde{P}] = \begin{bmatrix}
\eta^0_1 & \eta^2_1 & \cdots & \eta^i_1 & \cdots & \eta^n_1 \\
\eta^0_2 & \eta^2_2 & \cdots & \eta^i_2 & \cdots & \eta^n_2 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\eta^0_i & \eta^2_i & \cdots & \eta^i_i & \cdots & \eta^n_i \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\eta^0_n & \eta^2_n & \cdots & \eta^i_n & \cdots & \eta^n_n
\end{bmatrix}
\]

\[
[\tilde{P}](\{\sigma^{raw}\} - [\tilde{P}]\{a\}) = 0
\]

\[
\{a\} = ([\tilde{P}]^T[\tilde{P}])^{-1}[\tilde{P}]^T\{\sigma^{raw}\}
\]
Stress recovery for SIFs (2)

- Accelerate procedure

\[ \sigma^s(\xi, \eta) = \sum_{i=1}^{2} c_i \xi^{-\lambda_i-1} \Gamma^s_p \]

\[
\begin{bmatrix}
    K_{I}^{rec} \\
    K_{II}^{rec}
\end{bmatrix} = \sqrt{2\pi L_0} \left\{ \sum_{i=1}^{2} c_i \Gamma^s_{p,yy}(\eta = \eta_A) \right\} 
\begin{bmatrix}
    \sum_{i=1}^{2} c_i \Gamma^s_{p,yy}(\eta = \eta_A) \\
    \sum_{i=1}^{2} c_i \Gamma^s_{p,xy}(\eta = \eta_A)
\end{bmatrix}
\]

Based on the difference in raw and recovered stresses, a novel error estimator accompanying the porposed recovery scheme for SIFs can be deduced.

\[
\{e^*_{\sigma}(\xi, \eta)\} = \{\sigma^*(\xi, \eta)\} - \{\sigma_h(\xi, \eta)\} \quad (41)
\]
Optimal placement of scaling center
What is being compared?

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<td>Matlab tic - toc</td>
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ABAQUS: Contour Integral

- Integral based on tractions and displacements

\[
K_I = \frac{E}{2(1-v^2)K_I^*} \int_{\Gamma} [t_n U_n^* + t_s U_s^* - u_n T_n^* - u_s T_s^*] d\Gamma
\]

\[
K_{II} = \frac{E}{2(1-v^2)K_{II}^*} \times \int_{\Gamma} [t_n U_n^* + t_s U_s^* - u_n T_n^* - u_s T_s^*] d\Gamma
\]

- Requires information about crack propagation direction
- Cannot predict how a crack will propagate


ABAQUS DEMO
Extended finite element method (XFEM) I

- Goal: Separate geometry from mesh
  - XFEM achieves this by locally enriching the FE approximation with local partitions of unity enrichment functions
Extended finite element method (XFEM) II

- xFEM aims to overcome the shortcomings of FEM
- Does so by introducing two kinds of enrichment
  - Jump enrichment
  - Tip enrichment

\[ u = \sum_{i \in I} N_i u_i + \sum_{j \in J} b_j N_i H(X) \]
\[ + \sum_{k \in K} N_i \sum_{l=1}^{4} c_{kl} F_l(X) \]

- Achieves:
  - Higher accuracy for stresses at crack tip
  - Less remeshing required
  - Level set method used to efficiently track cracks
XFEM: Jump enrichment

\[ \sum_{j \in J} b_j N_i H(X) \]

Heaviside step function

courtesy of Kostas Agathos
XFEM: Tip enrichment

- Analytical solution for the crack problem solved by Westergaard (1939) using a complex Airy stress function

\[
\begin{align*}
    u(x, y) &= \frac{K_l}{\mu} \sqrt{\frac{r}{2\pi}} \cos \left(\frac{\theta}{2}\right) \left[\kappa - 1 + 2\sin^2 \left(\frac{\theta}{2}\right)\right] \\
    v(x, y) &= \frac{K_l}{\mu} \sqrt{\frac{r}{2\pi}} \sin \left(\frac{\theta}{2}\right) \left[\kappa + 1 - 2\cos^2 \left(\frac{\theta}{2}\right)\right] \\
    + \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \sin \left(\frac{\theta}{2}\right) \left[\kappa + 1 + 2\cos^2 \left(\frac{\theta}{2}\right)\right] \\
    - \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \cos \left(\frac{\theta}{2}\right) \left[\kappa - 1 - 2\sin^2 \left(\frac{\theta}{2}\right)\right]
\end{align*}
\]

- These can be spanned by the following basis, which are used as enrichment functions for the crack tip

\[
\{\sqrt{r} \sin \left(\frac{\theta}{2}\right), \sqrt{r} \cos \left(\frac{\theta}{2}\right), \sqrt{r} \sin(\theta) \sin \left(\frac{\theta}{2}\right), \sqrt{r} \sin(\theta) \cos \left(\frac{\theta}{2}\right)\}
\]
XFEM DEMO
SBFEM DEMO
Numerical experiments:

1. $K_{II} = 1$
   $E = 200 \, [N/mm^2]$
   $\nu = 0.3$
   $a = L/2$

2. $E = 200 \, [N/mm^2]$
   $\nu = 0.3$
   $a = L/4$

3. $E = 200 \, [N/mm^2]$
   $\nu = 0.3$
   $a = L/4$
   $\alpha = \pi/4$
Numerical example 1 (SBFEM)

$L_2$ displacement error, HSchur decomposition

$L_2$ stress error, HSchur decomposition

- $K_{II} = 1$
- $E = 200 \text{ N/mm}^2$
- $\nu = 0.3$
- $\alpha = b/2$
Numerical example 1 (SBFEM)

Error in $K_2$, recovered stress

Error estimator for $K_2$
Numerical example 2

<table>
<thead>
<tr>
<th>Eqn. #</th>
<th>SBFEM</th>
<th>(X)FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Schur</td>
<td>HSchur</td>
</tr>
<tr>
<td>23</td>
<td>$bn^2$</td>
<td>$bn^2$</td>
</tr>
<tr>
<td>22</td>
<td>$25 \times 2n^3$</td>
<td>$40n^3 + 205n^2$</td>
</tr>
<tr>
<td>24</td>
<td>$3n^3$</td>
<td>$3n^3$</td>
</tr>
<tr>
<td>diagonalize</td>
<td>$2n^3$</td>
<td>$2n^3$</td>
</tr>
<tr>
<td>29</td>
<td>$2/3n^3$</td>
<td>$2/3n^3$</td>
</tr>
<tr>
<td>$U = K^{-1}F$</td>
<td>$1/3n^3$</td>
<td>$1/3n^3$</td>
</tr>
<tr>
<td>27</td>
<td>$1/3n^3$</td>
<td>$1/3n^3$</td>
</tr>
<tr>
<td>Total</td>
<td>$\frac{619}{3}n^3 + bn^2$</td>
<td>$\frac{139}{3}n^3 + (205 + b)n^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Task</th>
<th>98 DOF</th>
<th>198 DOF</th>
<th>322 DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>invert $[E^0]$</td>
<td>0.53</td>
<td>0.63</td>
<td>0.43</td>
</tr>
<tr>
<td>HSchur</td>
<td>52.26</td>
<td>64.03</td>
<td>70.18</td>
</tr>
<tr>
<td>form K</td>
<td>0.38</td>
<td>0.30</td>
<td>0.25</td>
</tr>
<tr>
<td>invert K</td>
<td>6.87</td>
<td>3.88</td>
<td>2.64</td>
</tr>
<tr>
<td>form C</td>
<td>3.69</td>
<td>0.43</td>
<td>0.37</td>
</tr>
<tr>
<td>stress recovery</td>
<td>36.03</td>
<td>30.63</td>
<td>26.11</td>
</tr>
<tr>
<td>SIF</td>
<td>0.24</td>
<td>0.10</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Computational requirements for solution
Numerical example 2
Numerical example 2
Numerical example 3

![Graphs showing error in $K_2$ vs DOF and error in $K_2$ vs flops](image-url)
Numerical example 3

Comparison of condition number evolution

- Cond(K) vs. # DOF
- Log-log scale
- Various methods: 2-N HSchur, 3-N HSchur, 4-N HSchur, 5-N HSchur, 6-N HSchur, XFEM

Diagram showing a rectangular plate with boundary conditions and loadings.
Numerical example 3

Table 7 Values of computed $K_1$ as a function of the crack inclination angle ranging from 10° to 80°. For SBFEM, 3-noded elements were chosen with 1 to 4 elements discretizing each side. For Abaqus and XFEM, $K_1$ was calculated using fine meshes. The normed values correspond to the ratio of SBFEM $K_1$ solution to the Abaqus reference solution.

<table>
<thead>
<tr>
<th>Type</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
</tr>
</thead>
<tbody>
<tr>
<td>36 DOF</td>
<td>1.09394</td>
<td>1.03033</td>
<td>0.87814</td>
<td>0.69875</td>
<td>0.50031</td>
<td>0.31072</td>
<td>0.15275</td>
<td>0.05671</td>
</tr>
<tr>
<td>50 DOF</td>
<td>1.09307</td>
<td>1.02125</td>
<td>0.87838</td>
<td>0.69869</td>
<td>0.50038</td>
<td>0.31028</td>
<td>0.15225</td>
<td>0.04827</td>
</tr>
<tr>
<td>74 DOF</td>
<td>1.09390</td>
<td>1.01546</td>
<td>0.87856</td>
<td>0.69873</td>
<td>0.50042</td>
<td>0.31006</td>
<td>0.15200</td>
<td>0.04691</td>
</tr>
<tr>
<td>98 DOF</td>
<td>1.09486</td>
<td>1.01544</td>
<td>0.87871</td>
<td>0.69865</td>
<td>0.50046</td>
<td>0.31001</td>
<td>0.15189</td>
<td>0.04682</td>
</tr>
<tr>
<td>Normed</td>
<td>1.00475</td>
<td>1.00444</td>
<td>1.00581</td>
<td>1.00735</td>
<td>1.00538</td>
<td>1.00515</td>
<td>0.99140</td>
<td>0.99272</td>
</tr>
<tr>
<td>Abaqus ref.</td>
<td>1.08968</td>
<td>1.01094</td>
<td>0.87364</td>
<td>0.69355</td>
<td>0.49778</td>
<td>0.30843</td>
<td>0.15321</td>
<td>0.04716</td>
</tr>
<tr>
<td>XFEM</td>
<td>1.09278</td>
<td>1.01336</td>
<td>0.87647</td>
<td>0.69119</td>
<td>0.50215</td>
<td>0.30976</td>
<td>0.15172</td>
<td>0.04707</td>
</tr>
</tbody>
</table>

Table 8 Values of computed $K_2$ as a function of the crack inclination angle ranging from 10° to 80° using the same methodology as in Table 7.

<table>
<thead>
<tr>
<th>Type</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
</tr>
</thead>
<tbody>
<tr>
<td>36 DOF</td>
<td>0.13061</td>
<td>0.29445</td>
<td>0.41963</td>
<td>0.49134</td>
<td>0.50066</td>
<td>0.44532</td>
<td>0.33312</td>
<td>0.17540</td>
</tr>
<tr>
<td>50 DOF</td>
<td>0.13018</td>
<td>0.29185</td>
<td>0.41719</td>
<td>0.49132</td>
<td>0.50072</td>
<td>0.44541</td>
<td>0.33329</td>
<td>0.17823</td>
</tr>
<tr>
<td>74 DOF</td>
<td>0.12990</td>
<td>0.29034</td>
<td>0.41713</td>
<td>0.49123</td>
<td>0.50073</td>
<td>0.44569</td>
<td>0.33344</td>
<td>0.17862</td>
</tr>
<tr>
<td>98 DOF</td>
<td>0.12935</td>
<td>0.28989</td>
<td>0.41703</td>
<td>0.49126</td>
<td>0.50073</td>
<td>0.44572</td>
<td>0.33348</td>
<td>0.17862</td>
</tr>
<tr>
<td>Normed</td>
<td>0.99768</td>
<td>1.00279</td>
<td>1.00322</td>
<td>1.00338</td>
<td>1.00190</td>
<td>1.00051</td>
<td>0.99779</td>
<td>0.99700</td>
</tr>
<tr>
<td>Abaqus ref.</td>
<td>0.12965</td>
<td>0.28908</td>
<td>0.41569</td>
<td>0.48961</td>
<td>0.49978</td>
<td>0.44550</td>
<td>0.33422</td>
<td>0.17915</td>
</tr>
<tr>
<td>XFEM</td>
<td>0.12946</td>
<td>0.28948</td>
<td>0.41683</td>
<td>0.49037</td>
<td>0.49991</td>
<td>0.44586</td>
<td>0.33454</td>
<td>0.17923</td>
</tr>
</tbody>
</table>
Conclusion

- SBFEM combines many of the desireable characteristics of FEM and BEM into one method with additional benefits of its own:
  - Analytical solution in radial direction:
    - Higher accuracy per DOF
    - permits elegant and efficient calculation of stress intensity factors
  - Stress recovery enhances results greatly
    - Must only be performed on the boundary
    - Large workload can be performed in advance
  - No change necessary to solution process to extract crack related phenomena (i.e. SIFs of various orders of singularity)
  - Dense and fully populated matrices:
    - Higher order elements don’t (noticeably) impact performance
Questions