Introduction to the Extended Finite Element Method

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Introduction

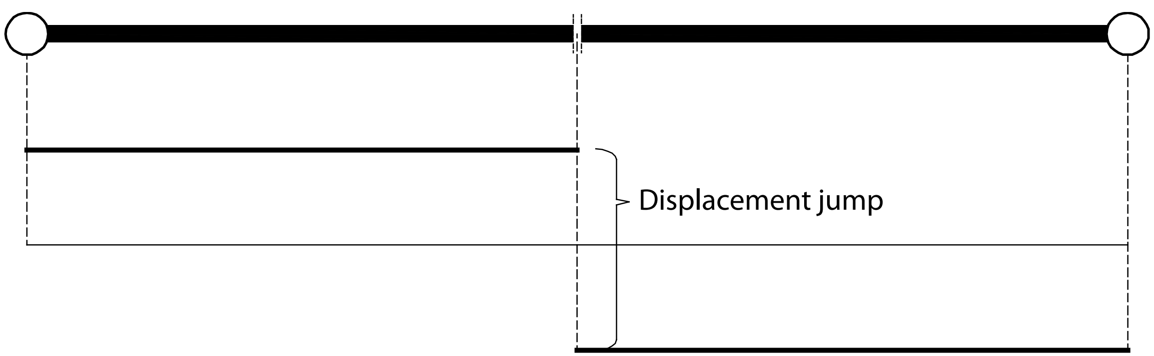
The Extended Finite Element Method (XFEM) is a numerical method, based on the Finite Element Method (FEM), that is especially designed for treating discontinuities.

Discontinuities are generally divided in strong and weak discontinuities.

Strong and Weak discontinuities

Strong discontinuities are discontinuities in the solution variable of a problem.

In structures, the solution variable is usually the displacements so strong discontinuities are displacement jumps, e.g. cracks and holes.

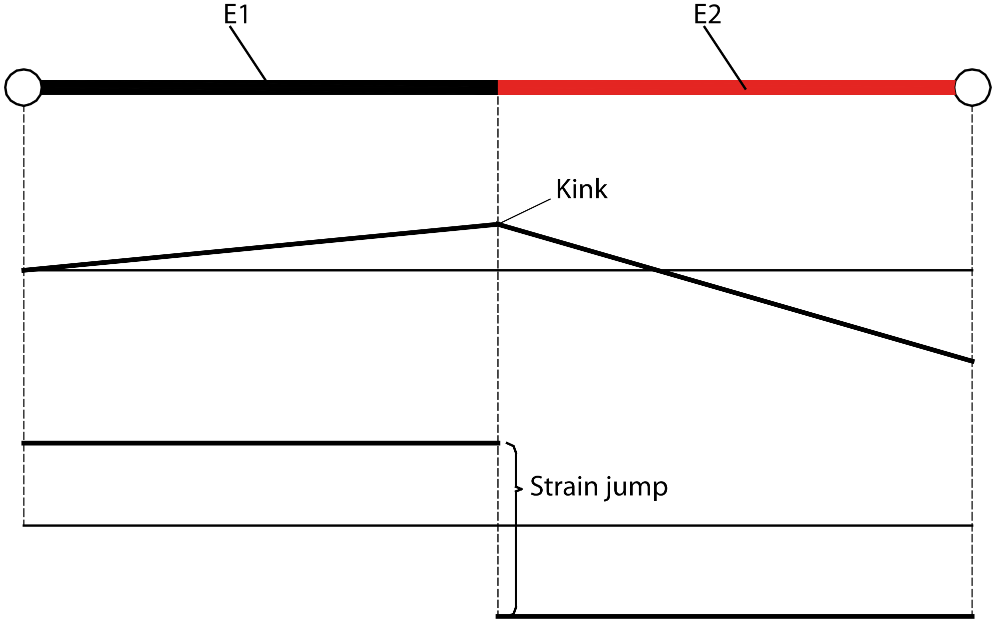


Cracked Bar

Strong and Weak discontinuities

Weak discontinuities are discontinuities in the derivatives of the solution variable.

In structures such discontinuities would invole kinks in the displacements (jump in the strains), as for example in bimaterial problems.



Bimaterial Bar

Fracture Mechanics

The biggest part of this presentation will be dealing with the modeling of strong discontinuities and more specificaly with cracks.

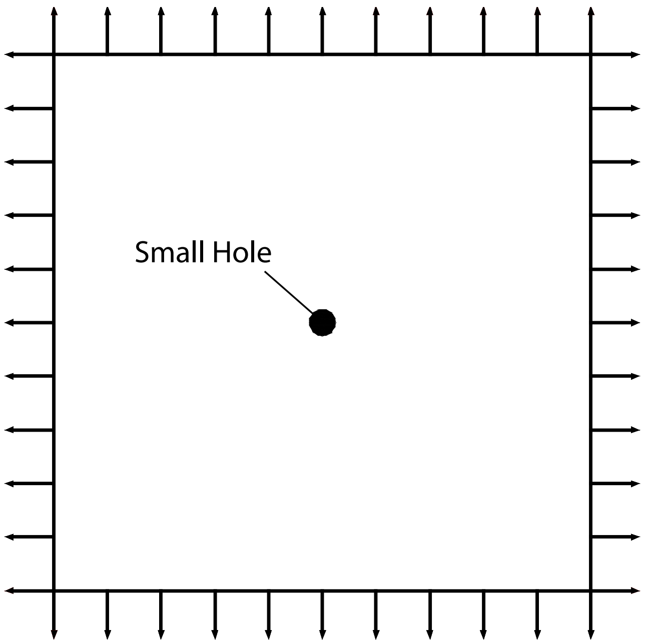
All formulations will be derived for the 2D cracked domain case and in the end the corresponding formulations for weak discontinuities will be given.

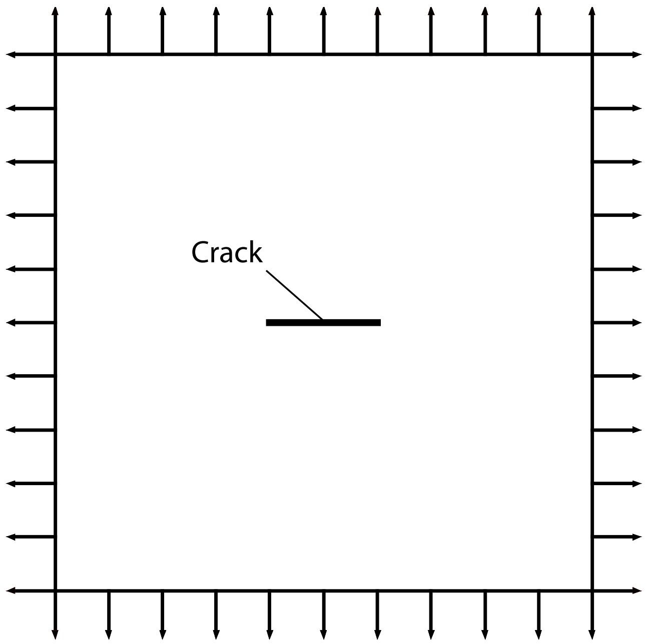
So some basic concepts of fracture mechanics will be briefly mentioned

Fracture Mechanics

- Problem Statement

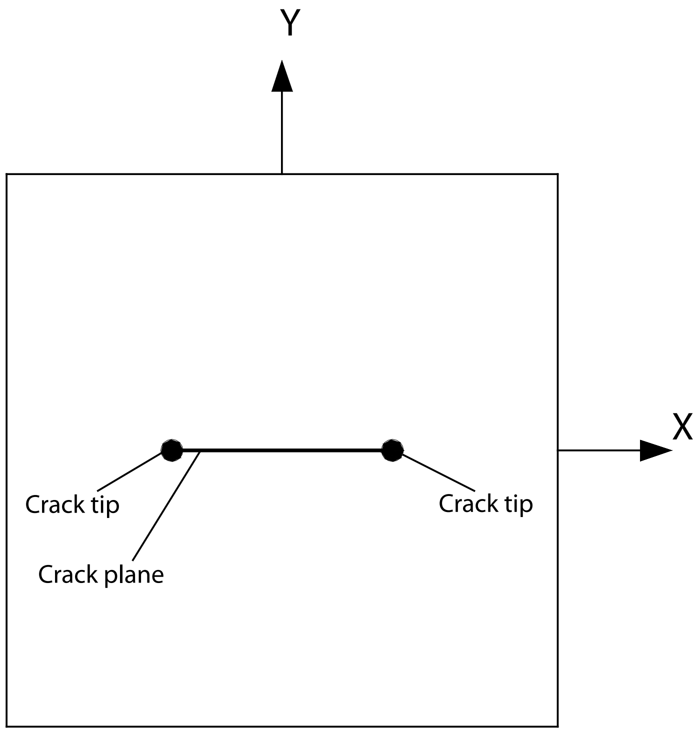
  Determine the stress, strain and displacement distribution in structures in the presence of flaws such as cracks and small holes.



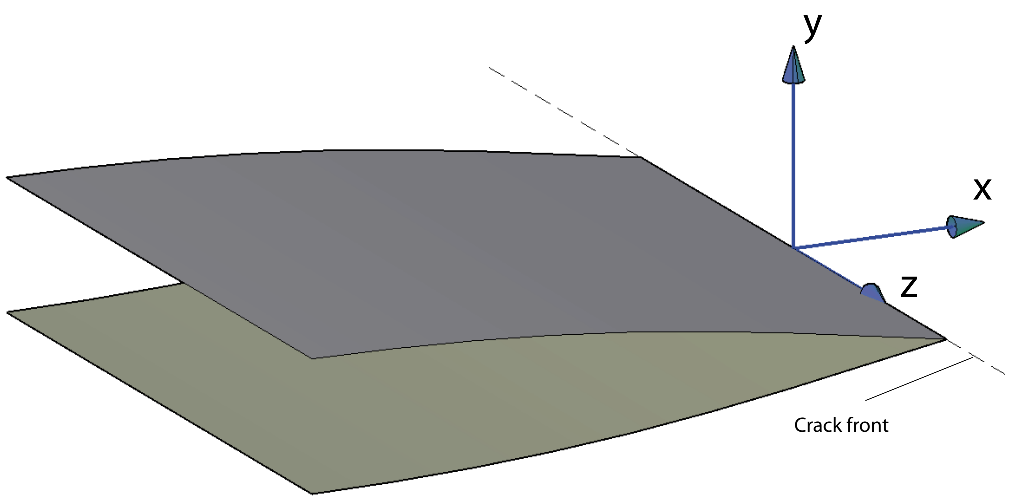


Fracture Mechanics

- Problem geometry (cracked domain case)



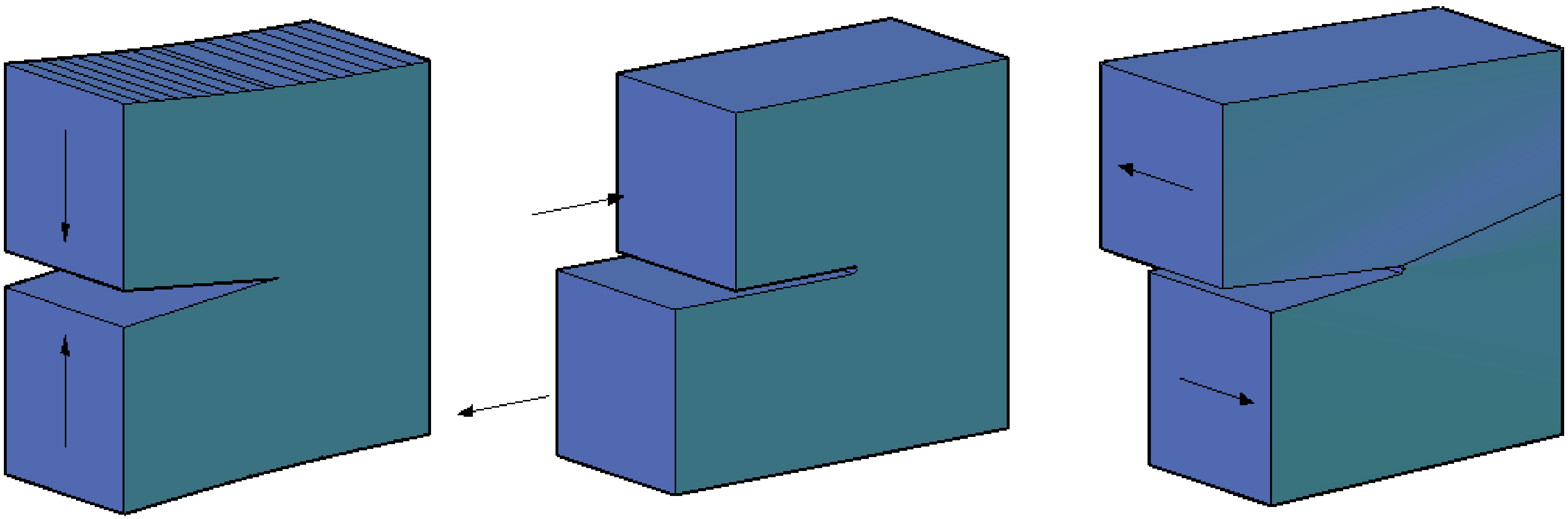
2D Crack



3D Crack

Fracture Mechanics

- Crack opening modes (i.e. how can the two crack surfaces deform)



Mode I *Mode II* *Mode III*

Fracture Mechanics

- Governing equations (2d cracked domain)

Equilibrium equations:

$$\nabla \boldsymbol{\sigma} = \mathbf{0} \quad \rightarrow \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \;, \; \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Kinematics equations:

$$\boldsymbol{\varepsilon} = \boldsymbol{\nabla}_s \boldsymbol{u} \;\rightarrow\; \varepsilon_{xx} = \frac{\partial u}{\partial x} \;, \varepsilon_{yy} = \frac{\partial v}{\partial y} \;, \; \gamma_{xx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Constitutive equations:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} \;\rightarrow \varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\, \sigma_{yy}\right),$$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\, \sigma_{yy}\right) \;, \; \gamma_{xy} = \frac{2\,(1+\nu)}{E}\, \tau_{xy}$$

Essential (Dirichlet) boundary conditions:

$\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_u

Natural (Neumann) boundary conditions:

$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \bar{\boldsymbol{t}}$ on Γ_t

$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{0}$ on Γ_{C^+} and Γ_{C^-} (traction condition for the crack)

where $\Gamma_u \cup \Gamma_t \cup \Gamma_{C^+} \cup \Gamma_{C^-}$ is the boundary of the body
$\boldsymbol{n}$ is a vector normal to the boundary

Fracture Mechanics

- Analytical solution for the crack problem

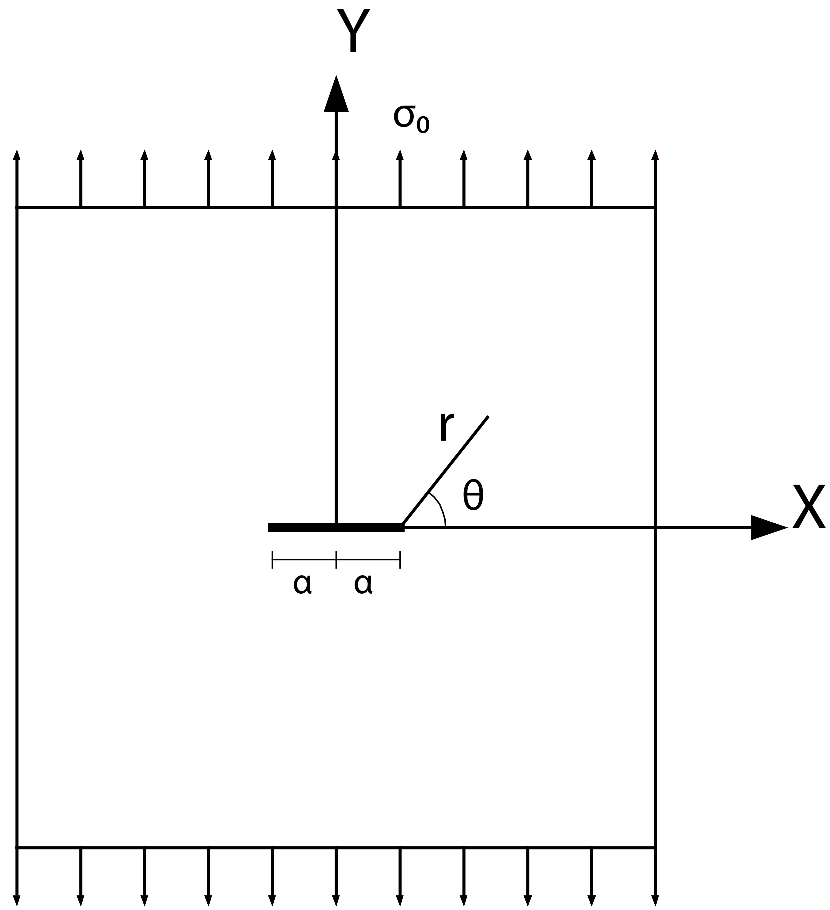
  Westergaard (1939) solved the problem using a complex Airy stress function. According to this solution the stress fields near the crack tip (for mode I loading) assume the form:

$$\sigma_{xx} = \sigma_0 \sqrt{\frac{\alpha}{2r}}\ cos\frac{\theta}{2}\left(1 - sin\frac{\theta}{2}\ sin\frac{3\theta}{2}\right) = \frac{K_I}{\sqrt{2\pi r}} cos\frac{\theta}{2}\left(1 - sin\frac{\theta}{2}\ sin\frac{3\theta}{2}\right)$$

$$\sigma_{yy} = \sigma_0 \sqrt{\frac{\alpha}{2r}}\ cos\frac{\theta}{2}\left(1 + sin\frac{\theta}{2}\ sin\frac{3\theta}{2}\right) = \frac{K_I}{\sqrt{2\pi r}} cos\frac{\theta}{2}\left(1 + sin\frac{\theta}{2}\ sin\frac{3\theta}{2}\right)$$

$$\sigma_{xy} = \sigma_0 \sqrt{\frac{\alpha}{2r}}\ sin\frac{\theta}{2}\ cos\frac{\theta}{2} cos\frac{3\theta}{2} = \frac{K_I}{\sqrt{2\pi r}} sin\frac{\theta}{2}\ cos\frac{\theta}{2} cos\frac{3\theta}{2}$$

– Westergaard solution parameters



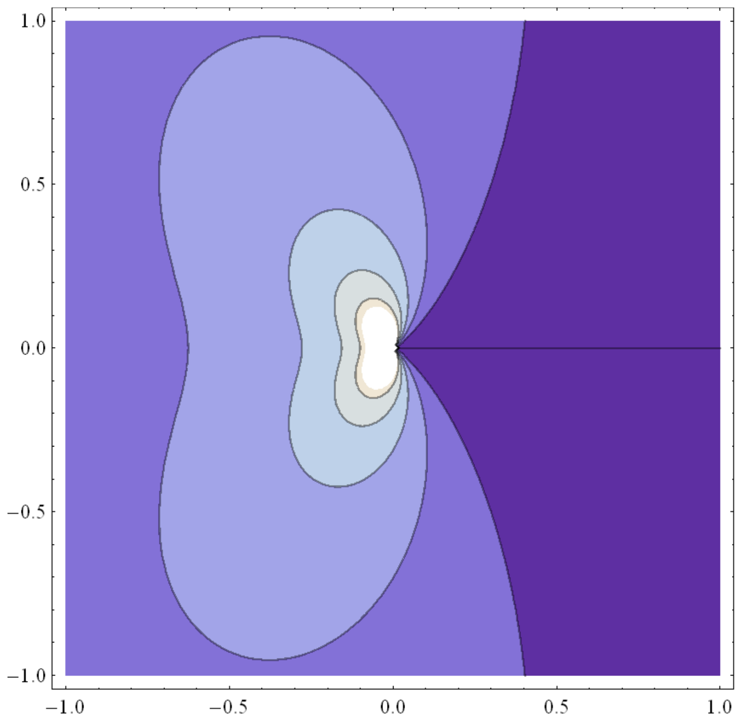
Mode I Stress Intensity Factor

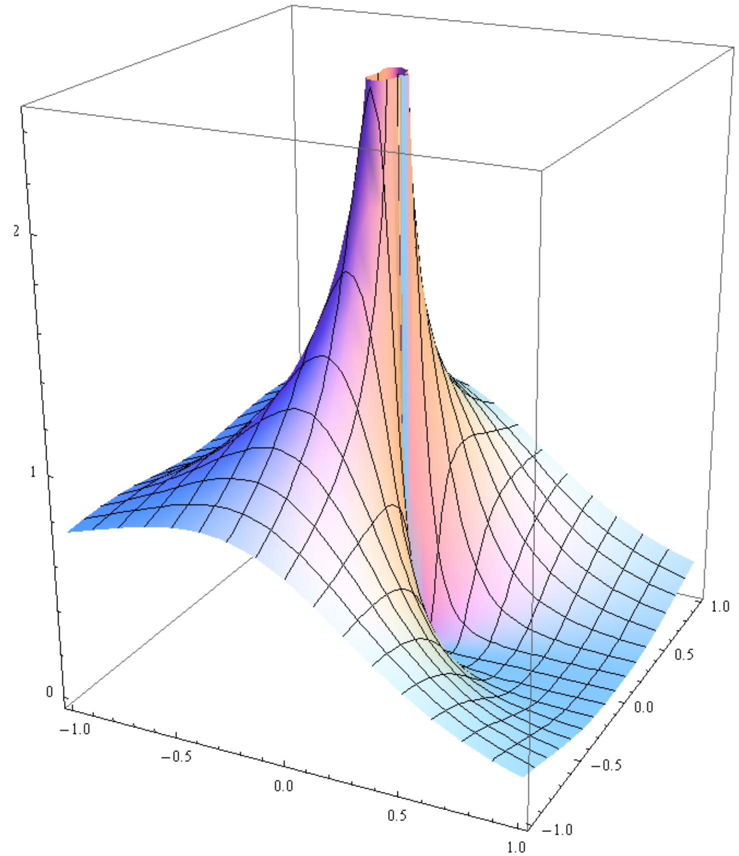
$$K_I = \lim_{\substack{r \to 0 \\ \theta \to 0}} \sigma_{yy}\sqrt{2\pi r} = \sigma_0\sqrt{\pi a}$$

Stress intensity factors as well as the Westergaard solutions are similar for modes II and III.

Fracture Mechanics

Contour and 3d plot for normal stresses σ_{xx}:



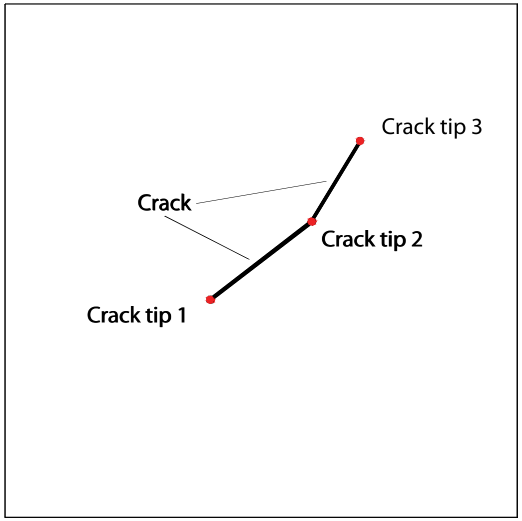


Stresses approach infinity near the crack tip

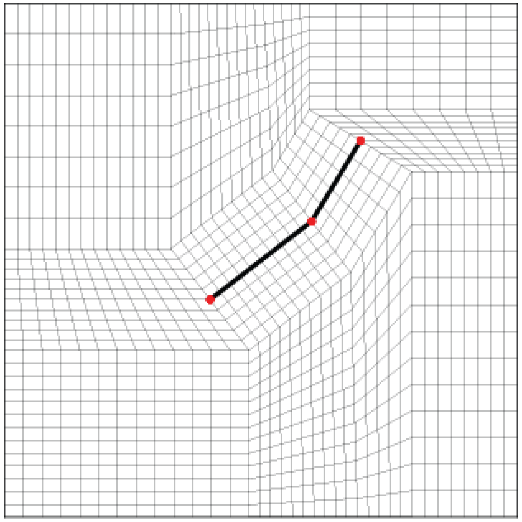
FEM Solution

In order to model the crack with FEM, the geometry has to be explicitly represented by the mesh, i.e. nodes have to be placed across the crack and on the crack tip.

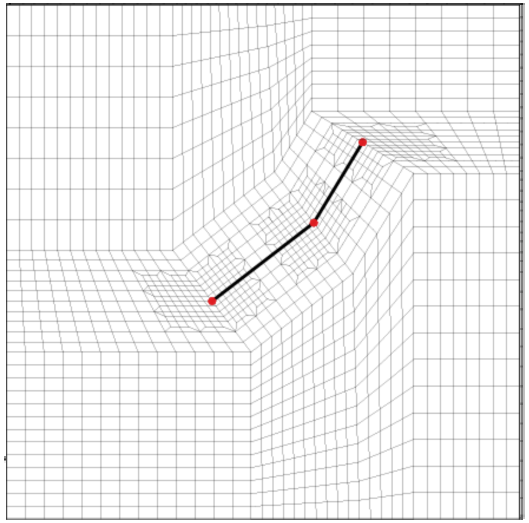
Example:



Cracked domain / *Crack propagation*



FE Mesh / *Remeshing*



Crack tip / *Mesh refinement*

FEM Solution

Remarks:

- Mesh refinement is usually necessary near the crack tips in order to represent the asymptotic fields asociated with the crack tips.
- As the crack propagates remeshing is needed which is computationally expensive especially in complex geometries and 3D domains.
- In some cases when remeshing, results need to be projected from one mesh to the other which further increases the computational cost.

Partition of Unity

The mathematical background of XFEM is the partition of unity concept.

Partition of unity is a set of n functions f_i that satisfy the relationship:

$\sum_{i=1}^{n} f_i(x) = 1$ or more generally

$$\sum_{i=1}^{n} f_i(x) g(x) = g(x)$$

Finite element interpolation functions also satisfy the partition of unity condition:

$\sum_{i=1}^{n} N_i(X) = 1$ or $\sum_{i=1}^{n} N_i(X) f(X) = f(X)$

Partition of Unity

By taking advandage of this property it is possible to enrich the finite element aproximation space:

$$u(x) = \sum_{i=1}^{n} N_i(X) u_i$$

in order to represent certain known caracteristics of the solution of the problem at hand:

$$u(x) = \sum_{i=1}^{n} N_i(X) u_i + \sum_{i=1}^{n} N_i(X) \sum_{j=1}^{k} p_j a_{ij} \quad \text{where}$$

p_j are the enrichment functions,

u_i are the FE degrees of freedom,

a_{ij} are extra degrees of freedom

Partition of Unity

In the above equation two factors have to be determined:

1. The type of enrichment functions used (next section).
2. The parts of the approximation that are going to be enriched.

In the case of cracks, the nature of the discontinuity is local since stress, strain and displacement fields are discontinuous or singular only near the crack tips or along the crack, so enrichment should be local too, i.e. only nodes near the crack are enriched.

This matter is going to be addressed in more detail in a following section.

Near Tip Enrichment

The displacement expressions of the Westergaard solution are:

$$u(x,y) = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2sin^2\left(\frac{\theta}{2}\right)\right]$$

$$+\frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}}sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 + 2cos^2\left(\frac{\theta}{2}\right)\right]$$

$$v(x,y) = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2cos^2\left(\frac{\theta}{2}\right)\right]$$

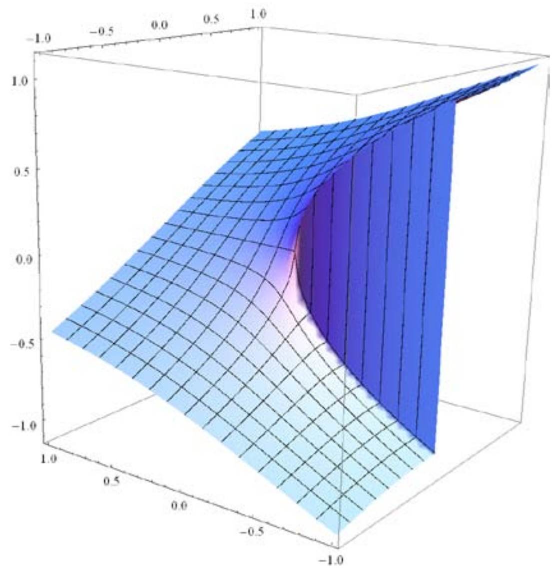
$$-\frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}}cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 - 2sin^2\left(\frac{\theta}{2}\right)\right]$$

Near Tip Enrichment

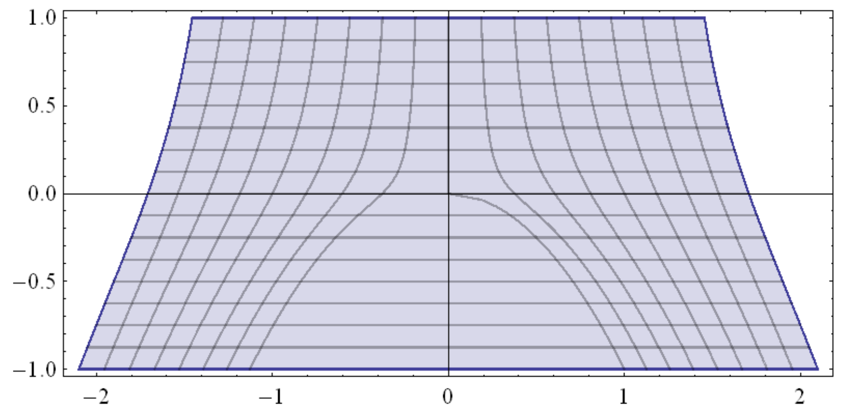
It can be shown that the above functions can be spanned by the basis:

$$\left\{\sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin(\theta)\sin\left(\frac{\theta}{2}\right), \sqrt{r}\sin(\theta)\cos\left(\frac{\theta}{2}\right)\right\}$$

So this will be the enrichment functions used for the crack tip.



$\sqrt{r}\, sin(\theta/2)$

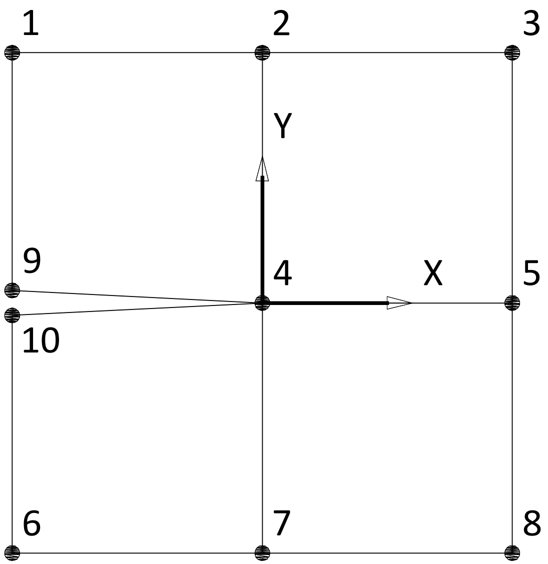


Deformed crack for $u_x = \sqrt{r}\, sin(\theta/2)$

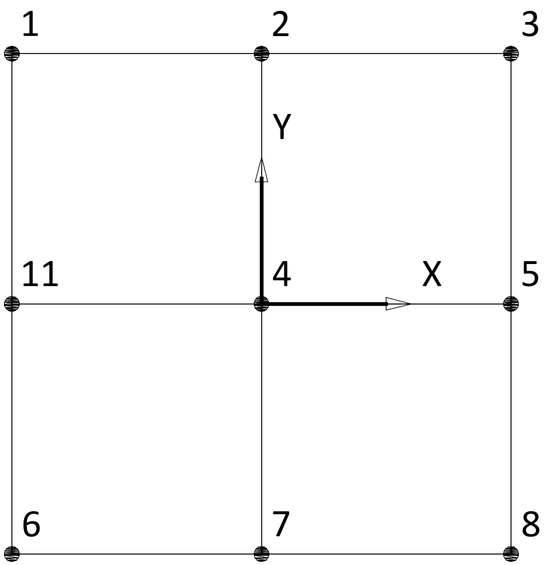
Jump Enrichment

A simple example is considered first in order to demostrate the concept, the results will the be generalized to more complex cases:

The objective is to represent mesh 1 using mesh 2 plus some enrichment terms



Mesh 1



Mesh 2

Jump Enrichment

The finite element approximation for mesh 1 is:

$$u = \sum_{i=1}^{10} u_i \, N_i$$

By defining :

$$a = \frac{u_9 + u_{10}}{2}, \quad b = \frac{u_9 - u_{10}}{2}$$

We can express u_9 and u_{10} in terms of a and b :

$$u_9 = a + b, \quad u_{10} = a - b$$

Jump Enrichment

By replacing u_9 and u_{10} in the finite element approximation we have:

$$u = \sum_{i=1}^{8} u_i\, N_i + a(N_9 + N_{10}) + b(N_9 + N_{10})H(x)$$

where $H(x)$ is a discontinuous or 'jump' function:

$$H(x) = \begin{cases} 1 & \text{for } y > 0 \\ -1 & \text{for } y < 0 \end{cases}$$

By considering mesh 2 we replace $N_9 + N_{10}$ by N_{11} and a by u_{11}:

$$u = \sum_{i=1}^{8} u_i\, N_i + u_{11}N_{11} + b\, N_{11}H(x)$$

Jump Enrichment

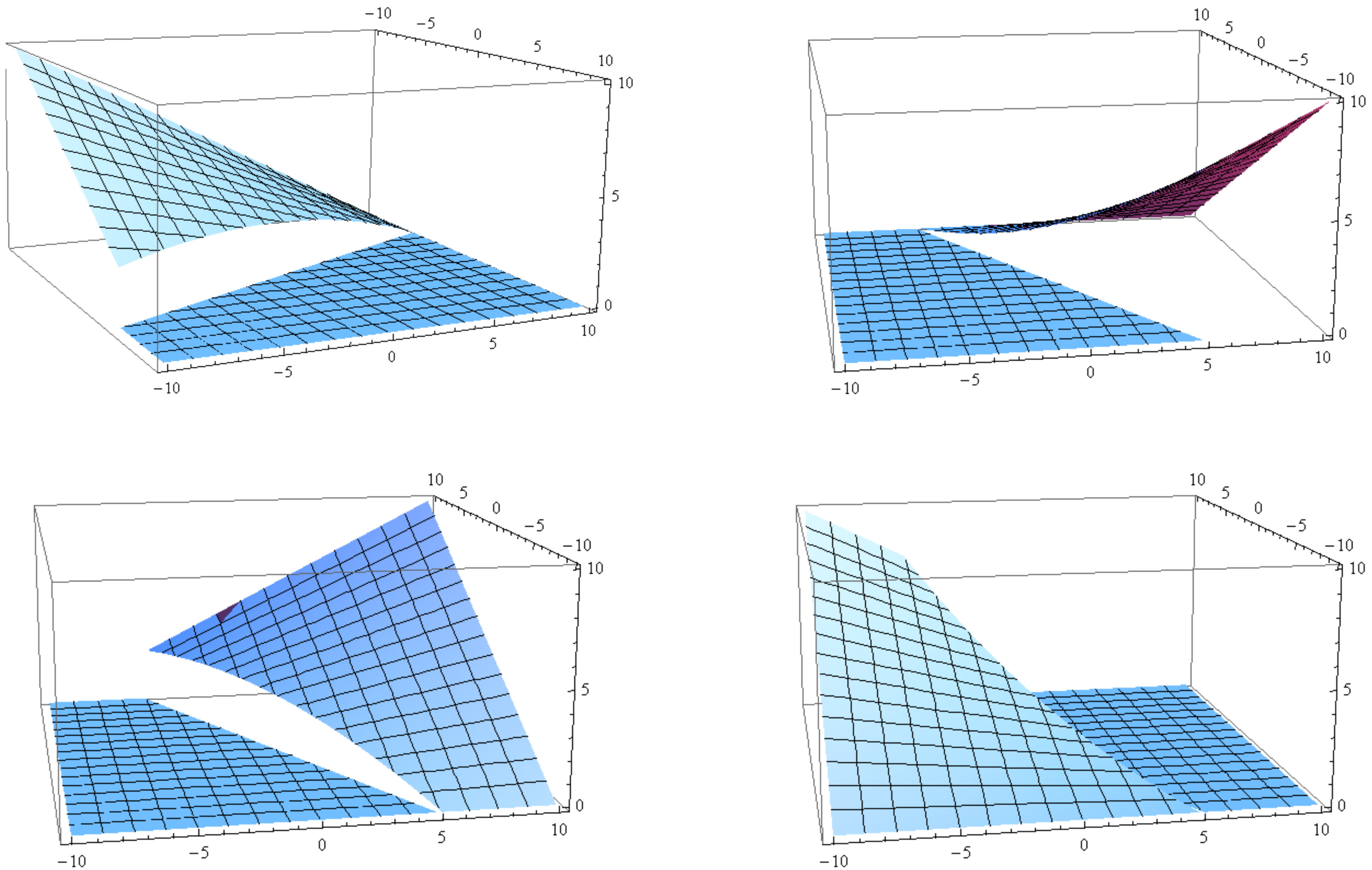
The above equation is equivalent to the finite element approximation for mesh 2 plus the discontinuous enrichment term.

By generalizing the above concept we can define jump enrichment functions for the general case which assume the form:

$$u_{enr} = \sum_{j \in J} b_j N_i \, H(X)$$

where $H(X) = 1$ on one side of the crack and $H(X) = -1$ on the other

Jump Enrichment



Jump enrichment terms for arbitrary crack orientation

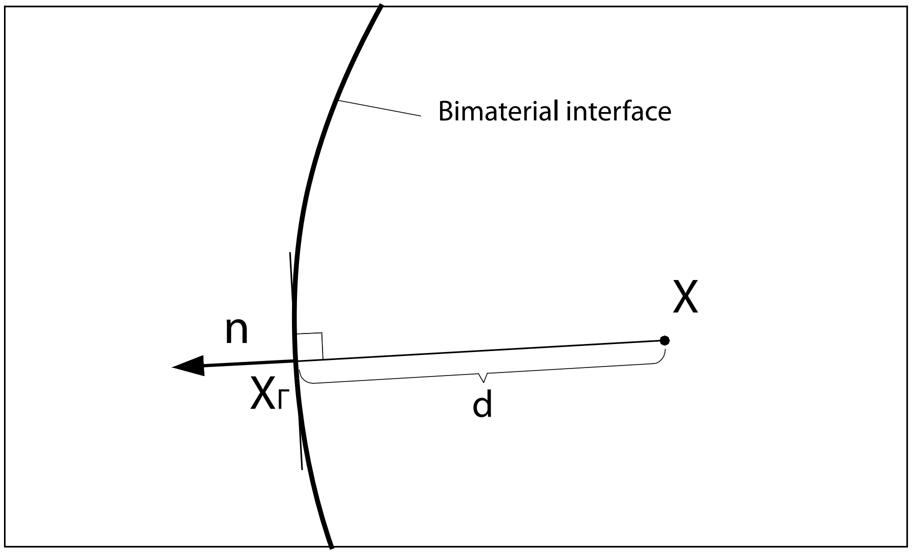
Signed Distance Function

For weak discontinuities, the enrichment function used is the absolute value of the signed distance function:

$$x(X) = |\xi(X)| \quad \text{where} \quad \xi(X) = d\, sign(\boldsymbol{n}\,(\boldsymbol{X} - \boldsymbol{X}_{\boldsymbol{\Gamma}}))$$

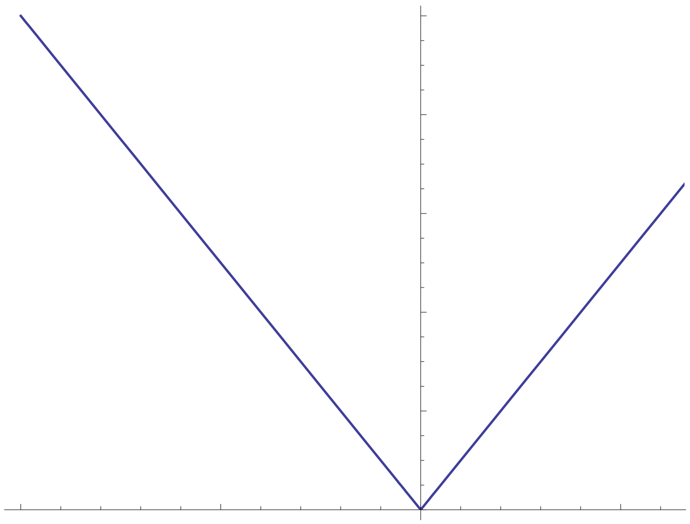
$d = \|\boldsymbol{X} - \boldsymbol{X}_{\boldsymbol{\Gamma}}\|$ the distance from point $\boldsymbol{X}$ to the interface

$$sign(x) = \begin{cases} -1 \text{ for } x < 0 \\ \;\;\,1 \text{ for } x > 0 \end{cases}$$

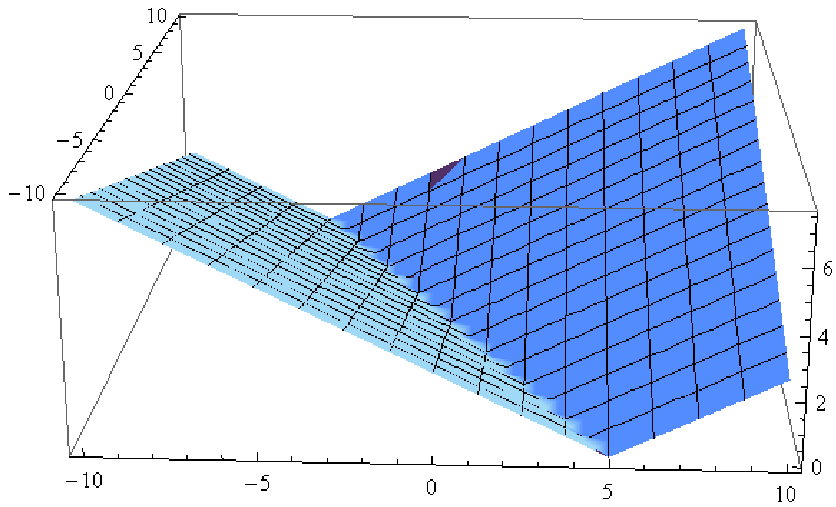


Signed Distance Function

Absolute value of the signed distance function:



1D absolute value of the signed distance function



2D absolute value of the signed distance function for arbitrary discontinuity

XFEM Displacement Approximation

Finally the displacement approximation for a cracked domain assumes the form:

$$u = \sum_{i\in I} N_i u_i + \sum_{j\in J} b_j N_i\, H(X) + \sum_{k\in K} N_i \sum_{l=1}^{4} c_{kl} F_l(X)$$

where $H(X), F_l(X)$ the jump and tip enrichment functions

b_j, c_{kl} additional degrees of freedom

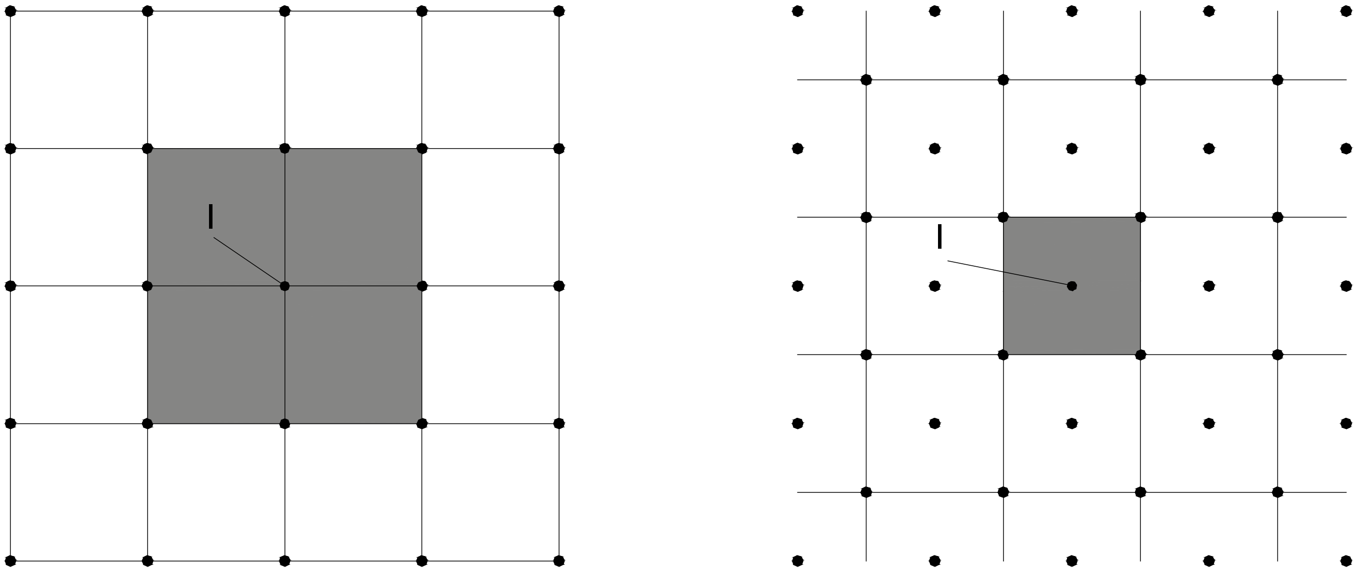
I, J, K the sets of all nodes, jump enriched nodes and tip enriched nodes respectively

The sets of nodes J and K have to be defined

Selection of enriched nodes

In order to select the nodes to be enriched the following definition is necessary:

The support of a node is the set of elements that contain that node.



Nodal support of external and internal node

Selection of enriched nodes

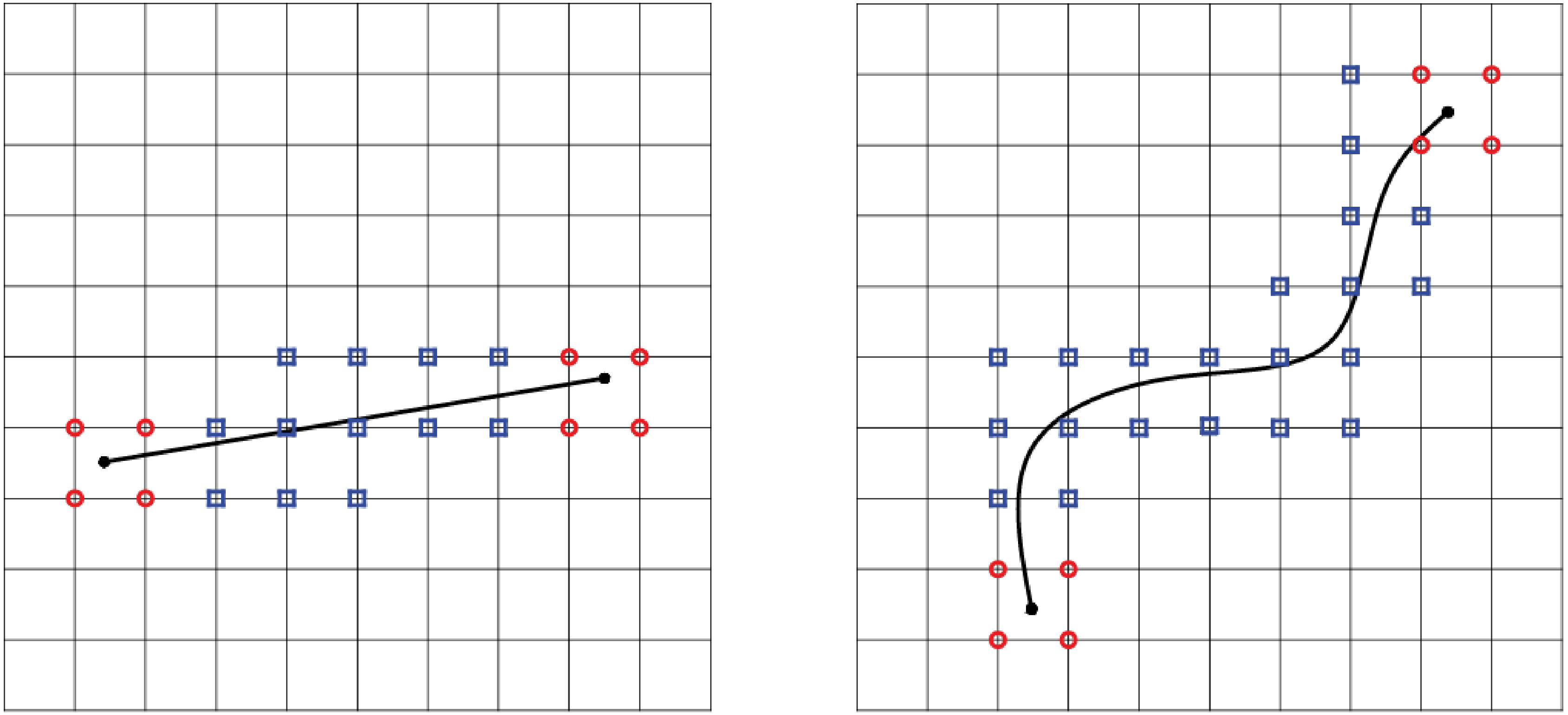
The selection of enriched nodes is done based on the following rules:

- Nodes whose nodal support is intersected by the crack belong to set J and therefore are enriched with jump fuctions

- Nodes whose nodal support contains a crack tip belong to set K and are enriched with the tip enrichment functions

Selection of enriched nodes

Node enrichment examples:



tip enrichment

jump enrichment

Level Set Method

In order to facilitate the evaluation of the enrichment functions and their derivatives, which is necessary for the calculation of the stiffness matrices, the level set method (LSM) is employed in most XFEM implementations.

The level set method is also a powerful tool for tracking moving interfaces, which makes it’s use very common in problems such as crack propagation.

Level Set Method

In the LSM curves are not represented explicitly, the level set functions are used instead.

The level set functions are essentialy surfaces and their value for a given point in the plane is the point’s height, i.e. the point’s level (hence the name level set functions).

The set of points for which the falue of the level set function is zero is the curve itself. So the curve is the intersection of the surface with the xy plane.

Level Set Method

Examples of level set functions:

- Signed distance function

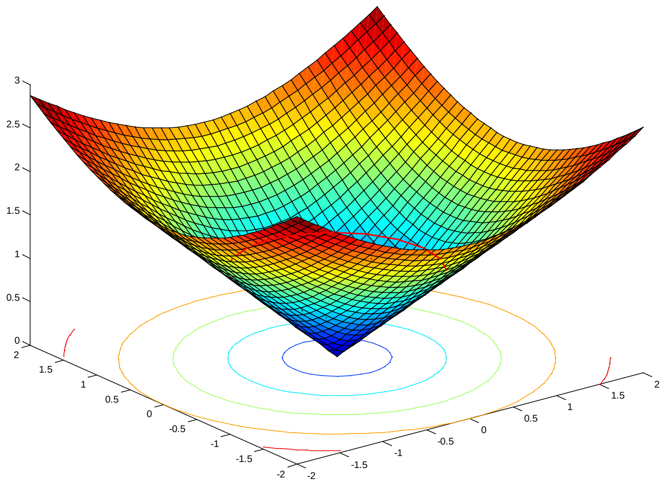
$$\phi(X) = d\ sign(\boldsymbol{n}\ (\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{\Gamma}}))$$

- Circular level set function

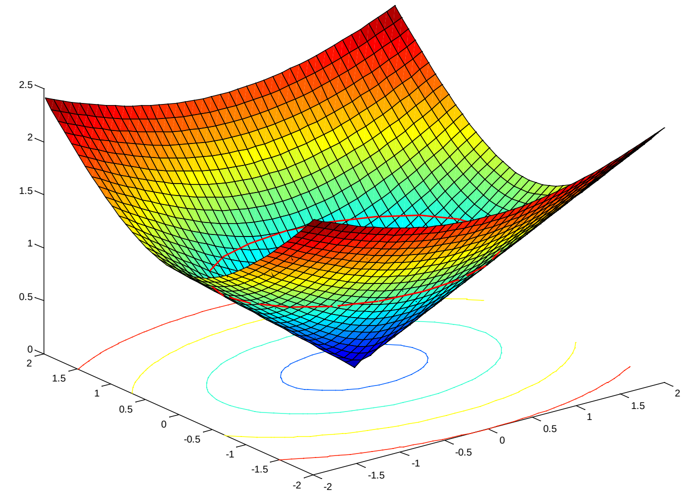
$$\phi(X) = \{\|\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{c}}\| - \boldsymbol{r}_{\boldsymbol{c}}\}$$

- Other types such as polygonal elliptical etc.

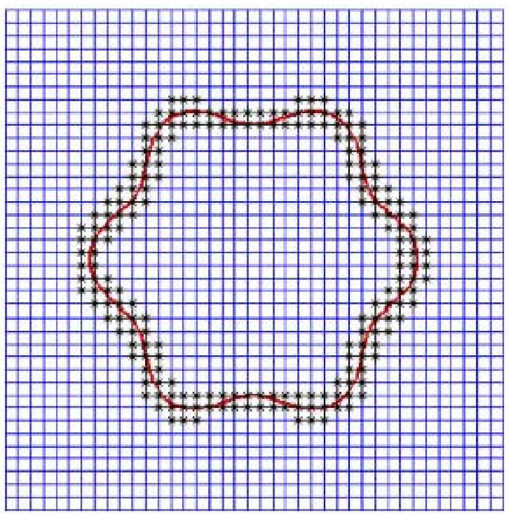
Level Set Method



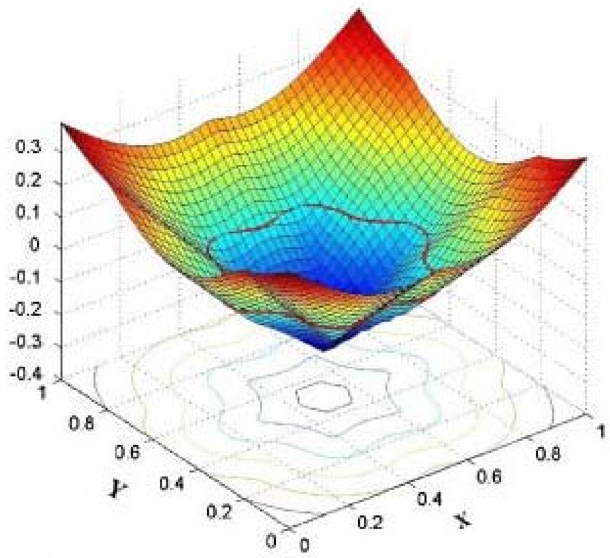
Circular level set function



Elliptical level set function



Non-regular discontinuity

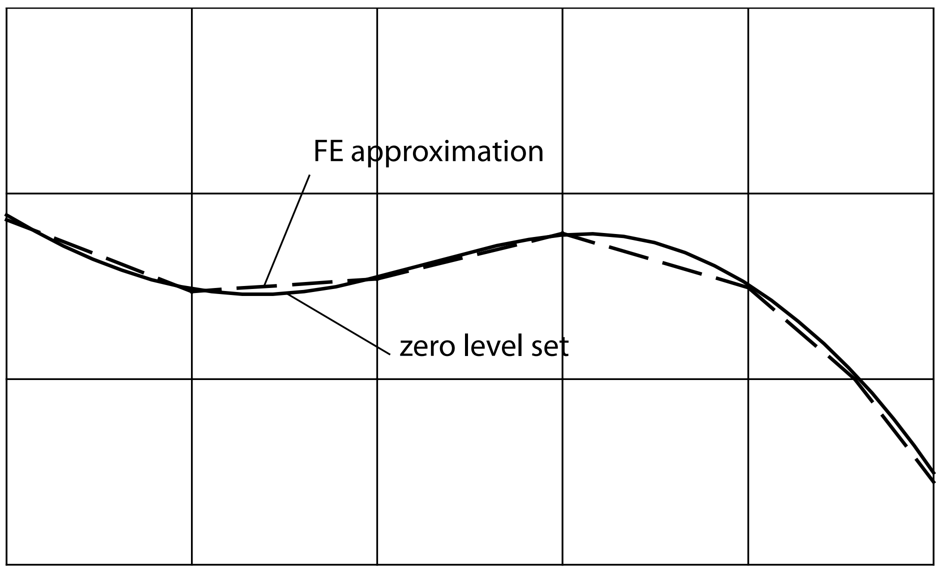


Non-regular level set function

FE Approximation of Level Sets

Level set functions (and their derivatives) are usually approximated by the FE shape functions, so that the level set values are calculated only for nodal points:

$$\phi = \sum N_i(\boldsymbol{x})\phi_i \;\; \rightarrow \;\; \frac{\partial \phi}{\partial x} = \sum N_{i,x}(\boldsymbol{x})\phi_i\,, \quad \frac{\partial \phi}{\partial y} = \sum N_{i,y}(\boldsymbol{x})\phi_i$$

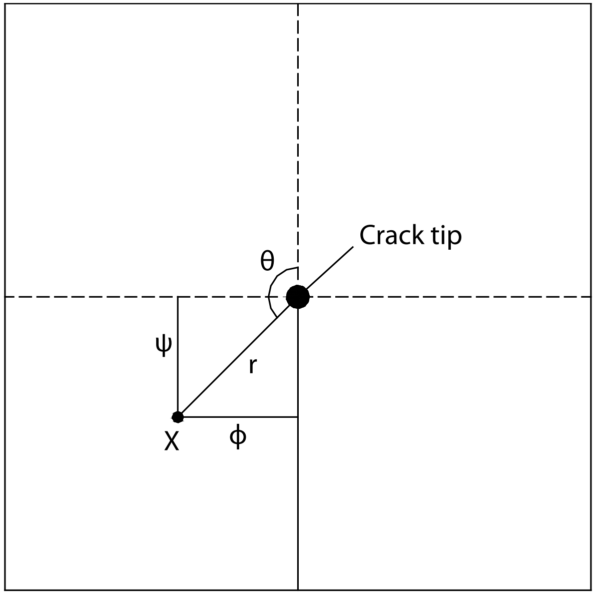


FE approximation of level sets

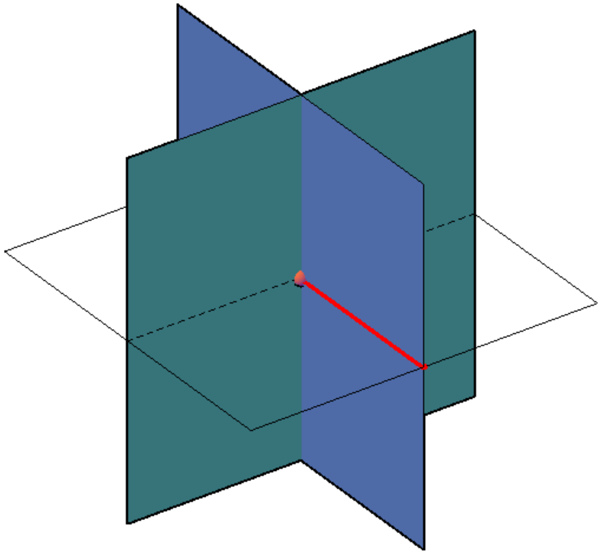
Crack Representation with Level Sets

The crack front is represented as a section of two normal planes (lines) and the level set functions are defined as the signed distances of any given point from those two planes.

$$r = \sqrt{\phi^2 + \psi^2}\,,\;\; \theta = \arctan\left(\frac{\phi}{\psi}\right),\;\; H(X) = H(\phi)$$



2D crack



3D crack

Weak Form and Discretization

The weak form of the problem is:

$$\int_V \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{C} : \delta\boldsymbol{\varepsilon}(\boldsymbol{v})\, dV = \int_V \boldsymbol{f_b} \cdot \delta\boldsymbol{v}\, dV + \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \delta\boldsymbol{v}\, d\Gamma$$

which results in

$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}$ or in element level $\boldsymbol{K_e}\boldsymbol{u_e} = \boldsymbol{f_e}$

where $\boldsymbol{K_e} = \int_V \boldsymbol{B}^T\, \boldsymbol{D}\, \boldsymbol{B}\, dV$

$$\boldsymbol{f_e} = \int_V \boldsymbol{f_b} \cdot \boldsymbol{N}^T\, dV + \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{N}^T\, d\Gamma$$

Weak Form and Discretization

Matrices $\boldsymbol{B}$ of the enriched elements contain apart from the standard terms the enrichment terms, which for the case of 2D elasticity would assume the form:

– Jump enriched nodes

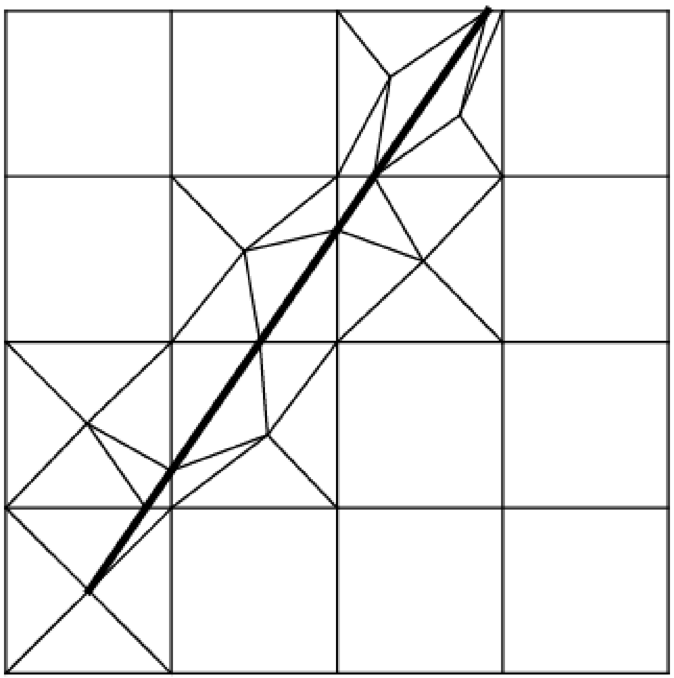
$$\boldsymbol{B}_i = \begin{bmatrix} (N_i H)_{,x} & 0 \\ 0 & (N_i H)_{,y} \\ (N_i H)_{,y} & (N_i H)_{,x} \end{bmatrix}$$

– Tip enriched nodes

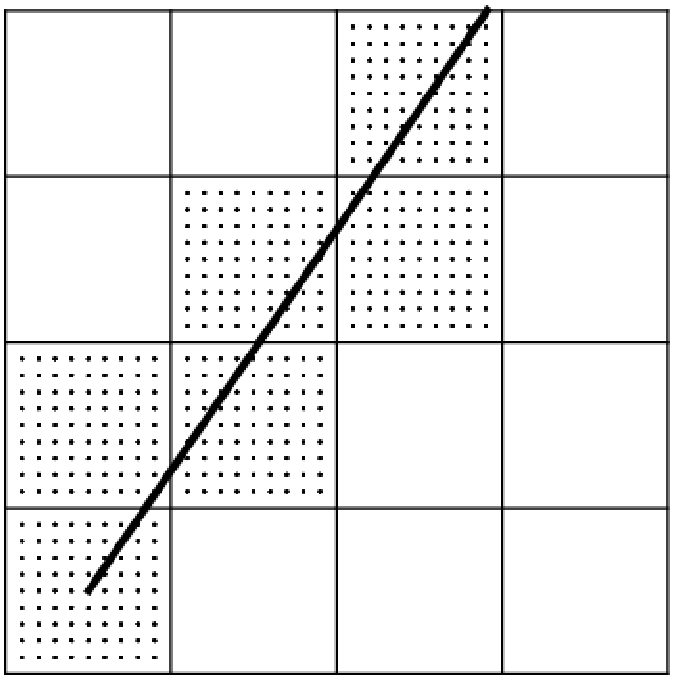
$$\boldsymbol{B}_i^a = \begin{bmatrix} (N_i F_a)_{,x} & 0 \\ 0 & (N_i F_a)_{,y} \\ (N_i F_a)_{,y} & (N_i F_a)_{,x} \end{bmatrix} \quad a = 1, \dots, 4$$

Numerical Integration

Gauss quadrature is not apropriate for the numerical integration of the discontinuous enrichment functions, so usually one of the following approaches is employed:



Division into subtriangles



Division into subquads

Post processing

Apart from stresses strains and displacements, one quantity of interest when post processing XFEM results is the stress intensity factors.

Their calculation is based on the evaluation of an integral (interaction integral) over an area around the crack tip. The procedure is similar to the one for the FE case.

Stress intensity factors are necessary for the calculation of the stress fields around the crack tip as well as for determining the direction of crack propagation.

Extentions

The method can be extended in a very straightforward manner to more general and complex problems such as:

- Crack propagation
- Branched and intersecting cracks
- Plastic enrichment
- Nonlinear finite elements
- Dynamic problems

So a wide variety of applications can be treated.

References

References/recomended reading:

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