The Eigensystem Realization Algorithm (ERA)

\[ x_{i+1} = Ax_i + Bu_i \]
\[ y_i = Cx_i + Du_i \]
Workflow overview

Data assembly

Decomposition

Matrix Realization

Eigenvalue problem solving

Extract system properties

Assemble the selected data sets into a Hankel Matrix and a Shifted Hankel Matrix
Workflow overview

- Data assembly
- Decomposition
- Matrix Realization
- Eigenvalue problem solving
- Extract system properties

Decompose the Hankel Matrix using Singular Value Decomposition
Workflow overview

Data assembly

Decomposition

Matrix Realization

Extract the new controllability and observability matrix; Calculate the system realization matrix

Eigenvalue problem solving

Extract system properties

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Workflow overview

Data assembly → Decomposition → Matrix Realization

Solve the eigenvalue problem for the system realization matrix

Extract system properties
Workflow overview

1. Data assembly
2. Decomposition
3. Matrix realization
4. Eigenvalue problem solving
5. Extract system properties

Calculate natural frequencies and damping factors using the obtained eigenvalues.
**Note:** The ERA is implemented for the case of free response data. Therefore Impact (Hammer, drop-weight) tests would be generally suitable.
Preprocessing

Selection

![Input data graph](image)

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Preprocessing

Selection
(keep the part that corresponds to free response)
Preprocessing

Selection

Data taken to calculations

Acceleration [m/s²]

Samples @ 128Hz

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The ERA works by exploiting the relationship of the series of outputs from different points (channels) of the structure to fundamental system properties (Markov Parameters).

**Hankel Matrix:**

$$
\begin{bmatrix}
  y_1 & y_2 & \ldots & y_n \\
  y_2 & y_3 & \ldots & y_{n+1} \\
  y_3 & y_4 & \ldots & y_{n+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  y_n & y_{n+1} & \ldots & y_{n+k}
\end{bmatrix} = H_1
$$

**Shifted Hankel Matrix:**

$$
\begin{bmatrix}
  y_2 & y_3 & \ldots & y_{n+1} \\
  y_3 & y_4 & \ldots & y_{n+2} \\
  y_4 & y_5 & \ldots & y_{n+3} \\
  \vdots & \vdots & \ddots & \vdots \\
  y_{n+1} & y_{n+2} & \ldots & y_{n+k+1}
\end{bmatrix} = H_2
$$
Decomposition

Assume the state – space representation of a dynamic system

\[ x_{i+1} = Ax_i + Bu_i \]
\[ y_i = Cx_i + Du_i \]
Decomposition

Assume the state – space representation of a dynamic system

\[ x_{i+1} = Ax_i + Bu_i \]
\[ y_i = Cx_i + Du_i \]

\[
\begin{align*}
u_0 &= 1 \\
u_k &= 0 & \text{if } k > 0 \\
x_0 &= 0 \\
D &= 0
\end{align*}
\]

Assume an impulse force, at \( t = 0 \), and 0 Initial Conditions
Decomposition

Assume the state – space representation of a dynamic system

\[
\begin{align*}
    x_{i+1} &= Ax_i + Bu_i \\
    y_i &= Cx_i + Du_i
\end{align*}
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Assume an impulse force, at \( t = 0 \), and 0 Initial Conditions

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\begin{align*}
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Decomposition

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u_0 &= 1 \\
u_k &= 0 \quad \text{if } k > 0 \\
x_0 &= 0 \\
D &= 0
\end{align*}
\]

By iterating system in time

\[
\begin{align*}
x_0 &= 0 \\
x_1 &= Ax_0 + B = B \\
x_2 &= Ax_1 = AB \\
x_3 &= Ax_2 = A^2B \\
\ldots
\end{align*}
\]

\[
\begin{align*}
y_0 &= 0 \\
y_1 &= CB \\
y_2 &= CAB
\end{align*}
\]
Decomposition

\[ x_{i+1} = Ax_i + Bu_i \]
\[ y_i = Cx_i + Du_i \]

\[ u_0 = 1 \]
\[ u_k = 0 \quad \text{if} \quad k > 0 \]
\[ x_0 = 0 \]
\[ D = 0 \]

These constant parameters are termed & are system characteristics:

Markov Parameters

Iterate the system in time starting from I.C.

\[ y_0 = 0 \]
\[ y_1 = CB \]
\[ y_2 = CAB \]
Decomposition

By constructing the Hankel matrix of the Markov Parameters $y_i$:

\[
\begin{bmatrix}
y_1 & y_2 & \ldots & y_n \\
y_2 & y_3 & \ldots & y_{n+1} \\
y_3 & y_4 & \ldots & y_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
y_n & y_{n+1} & \ldots & y_{n+k}
\end{bmatrix} = H_1
\]

\[
\begin{bmatrix}
CB & CAB & \ldots & CA^nB \\
CAB & CA^2B & \ldots & CA^{n+1}B \\
CA^2B & CA^3B & \ldots & CA^{n+2}B \\
\vdots & \vdots & \ddots & \vdots \\
CA^nB & CA^{n+1}B & \ldots & CA^{n+k}B
\end{bmatrix} = H_1
\]
Decomposition

By constructing the Hankel matrix of the Markov Parameters $y_i$:

\[
\begin{bmatrix}
y_1 & y_2 & \ldots & y_n \\
y_2 & y_3 & \ldots & y_{n+1} \\
y_3 & y_4 & \ldots & y_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
y_n & y_{n+1} & \ldots & y_{n+k}
\end{bmatrix}
= H_1
\]

Which is equivalent to the matrix product:

\[
\begin{bmatrix}
CB & CAB & \ldots & CA^n B \\
CAB & CA^2 B & \ldots & CA^{n+1} B \\
CA^2 B & CA^3 B & \ldots & CA^{n+2} B \\
\vdots & \vdots & \ddots & \vdots \\
CA^n B & CA^{n+1} B & \ldots & CA^{n+k} B
\end{bmatrix}
= H_1
\]

\[
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^n
\end{bmatrix}
\begin{bmatrix}
B & AB & A^2 B & \ldots & A^n B
\end{bmatrix}
= H_1
\]
Decomposition

By constructing the **Hankel matrix** of the **Markov Parameters** $y_i$:

$$
\begin{bmatrix}
y_1 & y_2 & \cdots & y_n \\
y_2 & y_3 & \cdots & y_{n+1} \\
y_3 & y_4 & \cdots & y_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
y_n & y_{n+1} & \cdots & y_{n+k}
\end{bmatrix} = H_1
$$

$$
\begin{bmatrix}
CB & CAB & \cdots & CA^n B \\
CAB & CA^2 B & \cdots & CA^{n+1} B \\
CA^2 B & CA^3 B & \cdots & CA^{n+2} B \\
\cdots & \cdots & \ddots & \cdots \\
CA^n B & CA^{n+1} B & \cdots & CA^{n+k} B
\end{bmatrix} = H_1
$$

**Controllability matrix**

**Observability matrix**
By constructing the Hankel matrix of the Markov Parameters $y_i$:

$$
\begin{bmatrix}
y_1 & y_2 & \ldots & y_n \\
y_2 & y_3 & \ldots & y_{n+1} \\
y_3 & y_4 & \ldots & y_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
y_n & y_{n+1} & \ldots & y_{n+k}
\end{bmatrix} = H_1
$$

$$
\begin{bmatrix}
CB & CAB & \ldots & CA^nB \\
CAB & CA^2B & \ldots & CA^{n+1}B \\
CA^2B & CA^3B & \ldots & CA^{n+2}B \\
\vdots & \vdots & \ddots & \vdots \\
CA^nB & CA^{n+1}B & \ldots & CA^{n+k}B
\end{bmatrix} = H_1
$$

Controllability matrix

Observability matrix

$$
H_1 = O_p C_q
$$
In order to obtain these matrices we perform Singular Value Decomposition for H1:

\[ H_1 = U \Gamma^2 V^T \]
Matrix Realization

Product of
Singular Value Decomposition:

\[ H_1 = U \Gamma^2 V^T \]

\[ P = U \Gamma \]
\[ Q = \Gamma V^T \]

TIP:

\[ H_1 = O_p C_q \]
\[ H_1 = PQ \]

New observability matrix
New controllability matrix
Note: The Decomposition \( H_1 = PQ \) is not unique!

In fact by using a different number of time shifts \( k \), and total measurements \( n \), different alternatives can occur.

This is due to the fact that the set of system matrices \((A, B, C)\) are only one (out of many) realizations of the system that connects the inputs \( u \) (loads) to the output \( y \) (measurements).

Therefore, if \((A,B,C)\) are one realization, leading to the following system:

\[
\begin{align*}
  x_{i+1} &= Ax_i + Bu_i \\
  y_i &= Cx_i + Du_i
\end{align*}
\]
System «Realization» – What does it mean?

Note: The Decomposition $H_1 = PQ$ is not unique!

Then, under the transform $\bar{x} = Tx \Rightarrow x = T^{-1}\bar{x}$

the set of matrices, $TAT^{-1}$, $TB$, $CT^{-1}$ is also a further, equivalent Realization of the system that links inputs $u$ to outputs $y$, with:

$$
\begin{align*}
\bar{x}_{i+1} &= TAT^{-1}\bar{x}_i + TBu_i \\
y_i &= CT^{-1}\bar{x}_i + Du_i
\end{align*}
\Rightarrow
\begin{align*}
\hat{x}_{i+1} &= \hat{A}\bar{x}_i + \hat{B}u_i \\
\hat{y}_i &= \hat{C}\bar{x}_i + \hat{D}u_i
\end{align*}
$$

Therefore the internal state $\bar{x}$ of the ERA identified system is not necessarily the one that corresponds to the structural dofs but some transformation of it.
Note: The ERA-identified system is NOT the original physical coordinates system!

The matrices \((A_d, B_d, C_d)\) that are defined via discretization of the original differential equation of the structural dynamics problem:

\[
M \ddot{u} + \Gamma \dot{u} + Ku = f \Rightarrow \begin{cases}
\dot{x} = A_c x + B_c u \\
y = Cx + Du
\end{cases} \Rightarrow \begin{cases}
x_{i+1} = A_d x_i + B_d u_i \\
y_i = Cx_i + Du_i
\end{cases}
\]

with \(A_d = e^{A_c \Delta t}, B_d = \left(\int_0^{\Delta t} e^{A_c \tau} d\tau\right) B = A_c^{-1} (A_d - I) B\)

are only one (out of many) realizations of the system that connects the inputs \(u\) (loads) to the output \(y\) (measurements).

The standard implementation of the ERA will NOT return these physical coordinate system matrices, but rather a transformation of them using matrix \(T\).
Matrix Realization

Then, using the *Shifted Hankel Matrix*:

\[ H_2 = O_p A C_q \]

By using the new observability and the new controllability matrices:

- \( O_p \) replaced by \( P \)
- \( C_q \) replaced by \( Q \)

Then, using the new observability and the new controllability matrices:

\[ A = O_p^{-1} H_2 C_q^{-1} \]

How many singular values do we keep? 2 \( \times \) the expected modes. This defines the dimension of \( A \)

\[ \hat{A} = P^{-1} H_2 Q^{-1} \]

Realization of \( A \)

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Matrix Realization

Realization of C
"Output matrix"

\[ P = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} \]

Realization of B
"Control matrix"

\[ Q = \begin{bmatrix} B & AB & A^2B & \ldots & A^nB \end{bmatrix} \]
Eigenvalue problem solving

\[ x_{i+1} = \hat{A}x_i + \hat{B}u_i \]
\[ y_i = \hat{C}x_i \]

By solving the eigenvalue problem

\[ \hat{A}v = \lambda v \]
\[ \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \]
\[ V = \{v_1, v_2, \ldots, v_n\} \]
Eigenvalue problem solving

How are the eigenvalues of $\hat{A}$ linked to the eigenvalues of other realizations of other realizations $A = T\hat{A}T^{-1}$?

The above linear transformation preserves the eigenvalues, i.e.,

$$eig(A) = eig(T\hat{A}T^{-1}) = eig(\hat{A}) = \Lambda$$

Additionally, the mode shapes in terms of the measured variable $y$ (which is physical) will be preserved:

$$eigenvectors(CA) = eigenvectors(\hat{C}\hat{A}) = \hat{C}V$$
Furthermore, we know the relationship between the eigenvalues $\lambda$ of the discrete and the original continuous system, $\lambda_c$:

Conversion for discrete time to continuous time representation

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \rightarrow \lambda_{c(i)} = \frac{\ln(\lambda_i)}{dt}$$

For obtaining the mode shapes:

$$V = \{v_1, v_2, \ldots, v_n\}$$

$$y_k = \hat{C}v_k$$

Original continuous system

$$\dot{x} = A_c x + B_c u$$

$$y = Cx + Du$$
For the continuous state-space system, describing the equation of motion

\[
\dot{x} = A_c x + B_c u \quad \text{with} \quad A_c = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}
\]

it holds that:

\[
\lambda_{c(i)} = \text{eig}_i (A_c) = -\zeta_i \omega_i \pm i \sqrt{1 - \zeta^2} \omega_i
\]

Therefore:

\[
\lambda_{c(i)} = \frac{\ln(\lambda_i)}{dt}
\]

\[
\omega_i = \left| \lambda_{c(i)} \right|
\]

\[
\zeta_i = \frac{\text{Re}(\lambda_{c(i)})}{\left| \lambda_{c(i)} \right|}
\]

*the eigenvectors of $A$, $\Phi_A$ are also linked to the structural system’s mode shapes $\Phi$, as

\[
\text{Re}(\Phi_A) = \Phi
\]
The ERA as an input-output Id method

It has already been mentioned that the ERA operates using output measurements of impulse response data. However, it possible to appropriately extend the method so as to account for response to a measured input loading.

Assuming measurements of the input $f(t)$ and output of the system $x(t)$ are available from $m$ measurement locations.

The Frequency Response Function (FRF) may be extracted as:

$$H_i(j\omega) = \frac{S_{xf}(j\omega)}{S_{ff}(j\omega)}, \quad i = 1 \ldots m$$
The ERA as an input-output Id method

The **Frequency Response Function (FRF)** may be extracted as\(^1\):

\[
H_i(j\omega) = \frac{S_{xf}(j\omega)}{S_{ff}(j\omega)}, \quad i = 1 \ldots m
\]

Then by applying the **Inverse Fourier Transform**, the **Impulse Response Functions (IRF)** per measurement channel (usually this implies per dof) are obtained.

The ERA method, as described previously can then be implemented on the **IRFs** which essentially simulate the system’s response to impulse.

\(^1\)See the Appendix for the proof
The Natural Excitation Technique (NExT)

For the case of ambient (operational) loads, it is assumed that the excitation and responses are stationary random processes. Assuming that the structural parameter matrices are deterministic, post-multiplying the Eq. of motion by a reference scalar response process \( X_i(t_2) \) and taking the expected value of each side yields:

\[
ME\left[ \ddot{X}(t_1)X_i(t_2) \right] + CE\left[ \dot{X}(t_1)X_i(t_2) \right] + KE\left[ X(t_1)X_i(t_2) \right] = E\left[ F(t_1)X_i(t_2) \right] \\
\Rightarrow MR_{\dddot{X}X_i}(t_1,t_2) + CR_{\dddot{X}X_i}(t_1,t_2) + KR_{XX_i}(t_1,t_2) = R_{FX_i}(t_1,t_2)
\]

where \( X(t), F(t) \) denote the displacement and excitation stochastic vector process respectively. Additionally, for weakly (or strongly) stationary processes, we know that:

\[
R_{A^{(m)}B}(\tau) = R_{AB}^{(m)}(\tau), \quad \tau = t_2 - t_1, \text{ where } m \text{ denotes the } m^{th} \text{ derivative.}
\]
The Natural Excitation Technique (NExT)

Recognizing that the responses of the system are uncorrelated to the disturbance for $t>0$, and assuming that the random vector processes $X, \dot{X}, \ddot{X}$ are \textbf{weakly stationary}, we can eliminate the term on the right-hand side:

$$M\ddot{\mathbf{R}}_{XX_i}(\tau) + C\dot{\mathbf{R}}_{XX_i}(\tau) + K\mathbf{R}_{XX_i}(\tau) = 0$$

Thus, the vector of displacement cross-correlation functions (wrt to a reference $X_i$), satisfies the homogeneous differential equation of motion.

Using a similar approach it can be shown that the acceleration correlation functions also satisfy this equation (Beck et al. 1994).

\textbf{We can therefore employ the ERA for the correlation signals!}
Main Hyperparameters – as defined by Caicedo 2011

ERA
The ERA has three main parameters to be determined: the number of rows and columns of the Hankel matrix (m and n), and the number of poles (which is 2×modes) to identify.

NEXT
The number of points of the FFT, and the reference channel used to calculate the cross-correlation function are the two main parameters for NExT.

*The length and sampling frequency of the time domain records affect the modal identification procedure.

Usually, this is an iterative procedure where the analyst performs the identification with a set of parameters, analyzes the output of this process and changes the parameters depending on the results obtained.
Preprocessing – as defined by Caicedo 2011

- Before starting the modal identification procedure, it is useful to create a few cross-spectral density functions and determine if clean peaks are shown within the spectrum. These peaks indicate possible natural frequencies.

- NEXT assumes white noise as the driving excitation. This implies that the record to be used should be stationary, and that therefore the mean and variance should be constant over time.

- Longer records are preferred for noisy data, while relatively short records can be used for relatively clean data.

- The record length should be selected based on the expected frequency of the structure. Lower frequency structures, such as cable-stayed bridges, require longer records to capture the same number of cycles than higher frequency structures.
Preprocessing – as defined by Caicedo 2011

• The sample frequency should be at least twice the higher frequency of interest to comply with the Nyquist criterion but it is recommended to select a sampling frequency not much higher than that.

• Anti-aliasing filters should be used when resampling the data. However, it is important to consider that anti-aliasing filters could appear as poles on the signals due to filter’s dynamic characteristics.

• NEXT requires a reference channel to calculate the cross-correlation function. One option is to choose a reference channel with high amplitude, low noise to-signal ratio, and far from a node of vibration of the modes of interest. Using
Hankel Matrix Dimension – as defined by Caicedo 2011

- The number of rows and columns is chosen based on the number of expected natural frequencies. A first approximation to the number of modes can be obtained by counting the peaks of the cross-spectral density function.

- **A rule of thumb for the number of columns of the Hankel matrix is to use four times the number of expected modes** (twice the number of expected poles. A low-rank Hankel matrix could lead to missing some physical modes of vibration but this is usually a good starting point.

- The number of rows is a set based on the number of points available in the cross-spectral density function. The goal is to use as much data from the spectral density function as possible without including noisy signals found at the end of the cross-correlation function.
Practical Guidelines to Using ERA, ERA-NEXT

Stabilization Diagram

Specifying the correct number of poles, or model order, is probably the most important step in the modal identification procedure:

- If the model order is too high, fictitious modes of vibration occur.
- If the model order is too small, some of the modal parameters might not be identified.

Stabilization diagrams are an effective tool to determine the correct number of poles. The idea behind stabilization diagrams is to repeat the identification process with a different number of poles (or modes) each time.

- Stable poles should remain constant for all or most of the iterations.
Stabilization Diagram

Example: stabilization diagram for a 3 degree of freedom system:
Appendix

Derivation of the FRF formula:

As mentioned in Lecture 1, the system’s response to a random input can be obtained via discrete convolution with the IRF:

\[ x[t] = \sum_{\tau=0}^{\infty} h[t-\tau] f[\tau] \]  

(1)

On the other hand, the \textit{cross-correlation} of two discrete time signals is defined as:

\[ R_{xf}[\tau] = \sum_{t=-\infty}^{\infty} x[t] f[t-\tau] \]  

(2)

\[
\sum_{t=-\infty}^{\infty} x[t] f[t-\tau] = \sum_{t=-\infty}^{\infty} \left\{ \sum_{\tau=0}^{\infty} h[t-\tau] f[\tau] \right\} f[t-\tau] \Rightarrow R_{xf}[\tau] = R_{ff}[\tau] * h[\tau]
\]

However, convolution in the time domain is multiplication in the frequency domain. Thus, by taking the Fourier Transform we obtain:

\[ S_{xf}(j\omega) = S_{xx}(j\omega) H(j\omega) \]