The FDD method

The Frequency Domain Decomposition Method (FDD)

This method which is very similar to the peak picking method is used for the identification of the modal characteristics of the structure, given only ambient measurements.

Assume the standard input $X(t)$ - output $Y(t)$ relationship, based on the Fourier Transform:

$$Y(\omega) = H(\omega)X(\omega)$$

When assuming wide sense stationary processes one can equivalently write the following using PSDs:

$$G_{yy}(\omega) = H^*(\omega)G_{xx}(\omega)H^T(\omega)$$  \hspace{1cm} (1)

where $G_{xx}$ is the $(r \times r)$ power spectral density (PSD) matrix of the input, $r$ is the number of inputs, $G_{yy}$ is the $(m \times m)$ PSD matrix of the responses, $m$ is the number of responses, $H(\omega)$ is the $(m \times r)$ frequency response function (FRF) matrix and the overbar and superscript $T$ denote the complex conjugate and transpose, respectively.
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The method is applicable for cases where the Power Spectral Density of the input is constant, i.e. $G_{xx}(\omega) = C$. This applies for the case of broadband random input excitations like white noise (such as ambient excitations). In this way FDD is developed for output only modal identification where only measurements of the response are available.

Proof

Remember that the PSD $G_{xx}(\omega)$ is the FT of the autocorrelation function $R_{xx}(t)$. The autocorrelation function essentially indicates the correlation between values of the process at different points in time, as a function of the two times $t_1$, $t_2$ or of the time difference (lag - $\tau$).
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By definition:

A continuous time random process \( w(t) \) where \( t \in \mathbb{R} \) is a white noise process if and only if its mean function and autocorrelation function satisfy the following:

\[
\mu_w(t) = E\{w(t)\} = 0 \\
R_{ww}(t_1, t_2) = E\{w(t_1)w(t_2)\} = C\delta(t_1 - t_2)
\]

i.e. it is a zero mean process for all time and has infinite power at zero time shift since its autocorrelation function is the Dirac delta function. The above autocorrelation function implies the following power spectral density, since the FT of Dirac Delta is unity:

\[
G_{ww}(i\omega) = C
\]

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As already mentioned the FRF can be written in partial fraction expansion. (i.e poles/residues form)

\[
H(i\omega) = \sum_{m=1}^{N} \frac{A_m}{i\omega - \lambda_m} + \frac{A^*_m}{i\omega - \lambda^*_m}
\]

where \( A_m = \Phi_m \gamma^T_m \)

\( \Phi_m \): modal vector
\( \gamma_m \): modal participation vector

Suppose the input \( y(t) \) is white noise, then the PSD is a constant matrix
\( \Rightarrow G_{xx}(i\omega) = C \). Hence,

\[
(1) \Rightarrow G_{yy}(iw) = \sum_{m=1}^{N} \sum_{k=1}^{N} \left[ \frac{A_m}{i\omega - \lambda_m} + \frac{A^*_m}{i\omega - \lambda^*_m} \right] C \left[ \frac{A_k}{i\omega - \lambda_k} + \frac{A^*_k}{i\omega - \lambda^*_k} \right]^H
\]

where \(^H\) denotes the conjugate transpose (Hermitian transpose) matrix \((^T)^T\).
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After some math manipulations, this can be further simplified to:

\[ G_{yy}(\omega) = \sum_{m=1}^{N} \frac{A_m}{i\omega - \lambda_m} + \frac{A_m^*}{i\omega - \lambda_m^*} + \frac{B_m}{-i\omega - \lambda_m} + \frac{B_m^*}{-i\omega - \lambda_m^*} \]

where \( A_m \) is the mth residue matrix of the output PSD and it is an \( N \times N \) matrix itself:

\[ A_m = R_mC \left( \sum_{K=1}^{N} \frac{R_K^T}{-\lambda_m - \lambda_K} + \frac{R_K^T}{-\lambda_m - \lambda_K^*} \right) \]  

\[ (2) \]

Therefore the contribution of only the mth mode in expression (2) is:

\[ A_m = \frac{R_mC R_m^*T}{-2\text{Re}(\lambda_m)} \]
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When the damping is light, the residue becomes proportional to the mode shape vector: (dominating term)

$$D_m \propto A_m C A_m^T = \Phi_m \gamma_m^T C \gamma_m \Phi_m^T = d_m \Phi_m \Phi_m^T$$

In fact, at a certain frequency $\omega$, only a limited number of modes will contribute significantly. Let this set be denoted by Sub($\omega$). This, the response spectral density can be written:

$$G_{yy}(\omega) = \sum_{m \in \text{Sub}(\omega)} \frac{d_m \Phi_m \Phi_m^T}{i\omega + \lambda_m} + \frac{d_m^* \Phi_m^* \Phi_m^{*T}}{i\omega - \lambda_m^*}$$

Steps

1. From a set of given output measurements $y(t), \in \mathbb{R}^{m \times 1}$ obtain the PSD $G_{yy}(\omega) \in \mathbb{R}^{m \times m}$. This can be done using Matlab’s function $cpsd(x, x)$

**Note:** $cpsd$ stands for cross spectral density. It is in general written for two separate signals $x(t), y(t)$ and it is equal to the power spectral density when $x = y$.

2. $G_{yy}(\omega)$ is “decomposed” by taking the Singular value Decomposition (SVD) of the $G_{yy}(\omega)$ matrix
Basics: Singular value Decomposition - SVD

Suppose \( A \) is an \( m \times n \) matrix whose entries \( \in \mathbb{C} \) (complex) or \( \mathbb{R} \) then:

\[
A = U \Sigma V^H
\]

where

- \( U \) is a \( m \times m \) unitary matrix: \( U^H U = U U^H = I_m \)
- \( \Sigma \) is a \( m \times n \) diagonal matrix with \( \Sigma_{ii} \geq 0 \)
- \( V^H \) is a \( n \times n \) unitary matrix
- \( \Sigma_{ii} \) are called the singular values of \( A \)

\(^H \) denotes conjugate transpose
Identification Algorithm

Properties:

\[ A^H A = (V \Sigma^H U^H)(U \Sigma V^H) = V(\Sigma^H \Sigma)V^H \]
\[ AA^H = U \Sigma V^H V \Sigma^H U^H = U(\Sigma \Sigma^H)U^H \]

Hence, the columns of \( V \) (right singular vectors) are eigenvectors of \( A^H A \)
So we can write at each discrete frequency \( \omega_i \):

\[ G_{yy}(\omega_i) = U_i \Sigma_i U_i^H \]
where \( U_i = [U_{i1}, \ldots, U_{im}] \)

Note that here, \( V = U \) because \( G_{yy} \) is a normal matrix (i.e. it is square and \( G_{yy}^H G_{yy} = G_{yy} G_{yy}^H \)
Near a peak corresponding to the k-th mode in the spectrum, that mode (or maybe a closely spaced mode) will be dominating. Assume the case of no closely spaced modes then the first singular vector is an estimate of that mode shape: \( \hat{\phi} = U_{i1} \) since only the k-th term dominates in Equation 1.

The corresponding singular value defines the PSD of the corresponding single degree of freedom (sdof) system. As a note, this (partial) PSD is identified by isolating that peak and comparing the mode shape estimate \( \hat{\phi} \) with the singular vectors obtained for frequency lines around the peak. Based on a modal criterion (MAC) we can assess the similarity of that vector to the ones around the peak and set an appropriate threshold, according to which the virtual “SDoF” area is obtained.
Using the **Modal Assurance Criterion (MAC)**. This essentially compares the relationship between 2 mode shapes by comparing their linearity.

Assume two modal vectors $\phi, \psi$. One expression for the MAC criterion is:

$$MAC = \frac{|\psi^T \phi|^2}{(\psi^T \psi)(\phi^T \phi)}$$

for $MAC = 1 \Rightarrow$ the vectors are consistent (similar)

$MAC = 0 \Rightarrow$ not consistent

As long as the SVDs around that peak singular value give high MAC values $\Rightarrow$ that singular vector belongs to the SDoF PSD (same mode).
After we define the limits of the sdof, we can either apply the $1/2$ power method to find $\zeta$ or we can take the signal back to the frequency domain. There, we define $f_d$ (damped frequency) as $1/T_d$ from the zero crossings (or peak distance) and $\zeta$ from the logarithmic decrement method.

In case of closely spaced modes, the first singular vector will always be a good estimate of the strongest mode. In the case though that these two closely spaced modes are orthogonal, the first two singular vectors can give unbiased estimates of the corresponding mode shapes.
Variations of the method

- Enhanced Frequency Domain Decomposition (EFDD)
  This method uses the transformation in time domain for the derivation of $\omega_n$, $\zeta_n$

- Curve Fit frequency Domain Decomposition (CFDD)
  This method does not go to time domain but uses a SDoF curve fitter in the frequency domain in order to obtain, $\omega_n$, $\zeta_n$
Consideration

Selection of model order - Stabilization Chart

Estimated poles corresponding to physically relevant system modes tend to appear, for all estimation orders (i.e. assumed frequency peaks), at nearly identical locations.

While the so-called mathematical poles, i.e. poles resulting from the mathematical solution of the normal equations but meaningless with respect to the physical interpretation, tend to vary in location.

The occurrence of this last class of poles is mainly due to the presence of noise in the measurements.
The size of the square matrix $M$ is $n+1$, and thus much smaller than the original normal equation (29). To remove the parameter redundancy of transfer function model (20) (and to avoid the trivial solution with all coefficients equal to zero), a constraint has to be imposed on the coefficients of the transfer functions. This can be done, for instance, by imposing that one of the coefficients is equal to a non-zero constant value. Assume, for instance, that the last coefficient of $D$ is constrained to 1 (i.e. coefficient $n+1$). In that case, the Least Squares (LS) estimate of $D$ is given by

$$
\hat{D}_{LS} = \left[ (1) (n+1) \right]^{-1} \left[ \begin{array}{c} \sum_{i=1}^{n} w_i \end{array} \right].
$$

Once $\hat{D}_{LS}$ is known, (32) can be used to derive all LS estimates. This approach is more time efficient than solving (29) directly (approximately 22 times faster). The mode shape vectors are derived using (14) and (15).

3.1.5 Stabilization chart

In modal analysis, a stabilization chart is an important tool that is often used to assist the user in separating physical poles from mathematical ones. A stabilization chart is obtained by repeating the analysis for increasing model order $n$. For each model order, the poles are calculated from the estimated denominator coefficients. The stable poles (i.e. the poles with a negative real part) are then presented graphically in ascending model order in so-called "stabilization chart" (see Figure 6). Estimated poles corresponding to physically relevant system modes tend to appear for each estimation order at nearly identical locations, while the so-called mathematical poles, i.e. poles resulting from the mathematical solution of the normal equations but meaningless with respect to the physical interpretation, tend to jump around. These mathematical poles are mainly due to the presence of noise on the measurements.

Figure 6. Stabilization chart obtained with (a) a time-domain estimator (LSCE) and (b) the frequency-domain least-squares estimator. The LSCE estimator (Least Squares Complex Exponential) is probably the most frequently used technique in industry. The LSCE estimator is a time-domain technique that makes use of impulse response functions (16) to derive the modal parameters. In Figure 6 the stabilization chart of the LSCE estimator is compared with the proposed frequency-domain least squares estimator. It turns out that in many applications, the frequency-domain estimator is able to generate quite clear stabilization compared to the LSCE approach.