Identification Methods for Structural Systems

Prof. Dr. Eleni Chatzi

Lecture 3 - 2 March, 2016
Overview

- General Harmonic Response
- Frequency Response Functions
- The Fourier and Laplace Transforms for SDOFs
- Bode plots for SDOFs
Introduction to the Frequency domain

General Harmonic Response of SDOF system

Let’s revisit the case of **Forced Damped Vibration** with a **Harmonic Excitation** $F_0 \cos(\omega t)$

\[
m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t
\]  

(1)

Then particular solution is of the type:

\[
x_p(t) = C \cos \omega t + D \sin \omega t
\]

and its derivatives are:

\[
\dot{x}_p(t) = \omega (-C \sin \omega t + D \cos \omega t)
\]

\[
\ddot{x}_p(t) = \omega^2 (C \cos \omega t - D \sin \omega t)
\]
Introduction to the Frequency domain

Let us plug these expressions in (??)

\[
-m\omega^2 C + c\omega D + kC \cos\omega t + \left[-m\omega^2 D - c\omega C + kD\right] \sin\omega t = F_0 \cos\omega t
\]

If we equate the coefficients of similar terms:

\[
-m\omega^2 C + c\omega D + kC = F_0 \quad \& \quad \left[-m\omega^2 D - c\omega C + kD\right] = 0
\]

\[
\Rightarrow C = \frac{k - m\omega^2}{c\omega} D \quad \& \quad (k - m\omega^2)\frac{k - m\omega^2}{c\omega} D + c\omega D = F_0
\]

which leads to the following solution for the two unknown coefficients \(C, D\).

\[
D = \frac{(c\omega)}{(k - m\omega^2)^2 + (c\omega)^2} F_0 \quad \& \quad C = \frac{(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2} F_0
\]
However, we know from trigonometry, that the expression

\[ x_p(t) = C\cos\omega t + D\sin\omega t \]

is equivalent to:

\[ x_p(t) = X_0 \cos(\omega t - \phi) = X_0 \cos\omega t \cos\phi + X_0 \sin\omega t \sin\phi \]

where \( X_0 = \sqrt{C^2 + D^2} \) \& \( \phi = \arctan \left[ \frac{D}{C} \right] \)

Therefore,

\[ X_0 = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \] \& \( \phi = \arctan \left[ \frac{c\omega}{k - m\omega^2} \right] \) (2)
However we have previously defined that:

\[ c = 2m\omega_n\zeta, \quad k = m\omega_n^2 \]

Using the above relationships we can rewrite (??) as:

\[
X_0 = \frac{F_0/k}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \& \quad \phi = \arctan \left[ \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}} \right]
\]
Summarizing, the particular solution then becomes:

$$x_p(t) = X_0 \cos(\omega t - \phi) \Rightarrow x_p(t) = \frac{F_0}{k} H(\omega) \cos(\omega t - \phi)$$

where $\frac{F_0}{k} = \delta_{st}$ is also known as static deflection

and $H(\omega)$ is what is known as:

**Frequency Response Function**

$$H(\omega) = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + (2\zeta \frac{\omega}{\omega_n})^2}}$$
The half power Method

The half power Method for the estimation of Damping

The amplitude of the FRF at resonance is called the **quality factor** ($Q$) of the system

For $\omega \approx \omega_n \Rightarrow H(\omega) \approx \frac{1}{2\zeta} = Q$

Points $R_1$, $R_2$ where it holds that $H(\omega) = \frac{Q}{\sqrt{2}}$ are called 1/2 power points. These are used for extracting the bandwidth of the system.
The **Bandwidth** is defined as: \( \Delta \omega = \omega_2 - \omega_1 \) where \( \omega_1, \omega_2 \) are calculated from:

\[
H(\omega) = \frac{Q}{\sqrt{2}} = \frac{1}{\zeta \sqrt{2}} \Rightarrow \frac{1}{\sqrt{(1 - \frac{\omega_1^2}{\omega_n^2})^2 + (2\zeta \frac{\omega_1}{\omega_n})^2}} = \frac{1}{\zeta \sqrt{2}}
\]

\[
\frac{\omega_{1,2}^2}{\omega_n^2} = (1 - 2\zeta^2 \pm 2\zeta \sqrt{1 + \zeta^2})\omega_n^2 \quad \Rightarrow \quad \omega_{1,2}^2 = (1 \pm 2\zeta)\omega_n^2
\]

and \( \omega_2^2 - \omega_1^2 = 4\zeta \omega_n^2 \Rightarrow (\omega_2 - \omega_1)(\omega_2 + \omega_1) = 4\zeta \omega_n^2 \)
Bandwidth - Identification of $\zeta$

Since, $\frac{\omega_1 + \omega_2}{2} = \omega_n$, we have that

$$\text{Bandwidth} = \Delta \omega = \omega_n^2 = \omega_2 - \omega_1 \approx 2\zeta \omega_n$$

Therefore, from experimental response we can evaluate the damping coefficient:

$$\zeta = \frac{\omega_2 - \omega_1}{2\omega_n}$$
Structural Systems excited at the base - ground motion (earthquake)
Two alternatives exist for formulating the Equation of Motion

**Absolute Motion** \( x(t) \):

\[
m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \implies m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky
\]

**Relative Motion** \( z(t) = x(t) - y(t) \)

\[
m(\ddot{z} + \dot{y}) + c\dot{z} + kz = 0 \implies m\ddot{z} + c\dot{z} + kz = -m\dot{y}
\]
Base Excited Systems-Absolute Motion

\[ m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \]

**Total solution** \( x_{tot} = x_h + x_p \).

The homogeneous solution is already explored. Assuming that the base excitation is of harmonic type, i.e., \( y(t) = Y_0 \cos \omega t \), the particular solution will also be harmonic.

In accordance with the harmonic force excited system shown earlier, we now obtain:

\[
x_p(t) = Y_0 \sqrt{\frac{(k^2 + (c\omega)^2)}{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \phi) \Rightarrow \]

\[
x_p(t) = Y_0 H(\omega) \cos(\omega t - \phi) \]
We therefore can define the following terms.

**Gain Function or Transmissibility of Displacement:**

\[
T_d = H(\omega) = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}
\]

and **Phase:**

\[
\phi = \tan^{-1} \left( \frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2} \right), \quad r = \frac{\omega}{\omega_n}
\]
Base Excited Systems-Absolute Motion

Transmissibility of Displacement

\[ T_d = \frac{X}{Y} \]

Frequency ratio: \( r = \frac{\omega}{\omega_n} \)

- \( \zeta = 0.05 \)
- \( \zeta = 0.10 \)
- \( \zeta = 0.20 \)
- \( \zeta = 0.25 \)
- \( \zeta = 1.0 \)

\( \zeta = 0.50 \)
\( \zeta = 1.0 \)
Transmissibility of Displacement

This signifies how larger the maximum displacement of the system $X_0$ is with respect to the maximum amplitude of the input harmonic force $Y_0$.

- The value of $T_d$ is unity at $r=0$ and close to unity for small values of $r$.
- For an undamped system $\zeta=0$, $T_d \to \infty$ at resonance ($r=1$).
- The value of $T_d$ is less than unity ($T_d < 1$) for values of $r > \sqrt{2}$ (for any amount of damping $\zeta$)
- The value of $T_d$ is equal to unity ($T_d=1$) for all values of $\zeta$ at $r=\sqrt{2}$

\[
\frac{X_o}{Y_o} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}}
\]
Relative Motion: \( z(t) = x(t) - y(t) \Rightarrow m\ddot{z} + c\dot{z} + kx = -m\ddot{y} \)

- In nondimensionless form, \( \frac{Z_o}{Y_o} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \)

- The gain function for the relative motion for the base-excited system is shown in the figure:
Response of SDOF System to Arbitrary Excitation

Let's consider the impulse response:

**Newton:** \[
\int_{t}^{t+\Delta t} F\,dt = (\text{change in momentum}) = \int_{t}^{t+\Delta t} m\ddot{x}\,dt = \int_{t}^{t+\Delta t} m\frac{\dot{x}}{dx}\,dt \Rightarrow F\,dt = m\dot{x}_{t+\Delta t} - m\dot{x}_{t}
\]

Assuming \( m\dot{x}_{t} = 0 \) \( \Rightarrow \dot{x}_{t+\Delta t} = \frac{F\,dt}{m} \).

Assume the unit impulse at \( t = 0 \), where

\( F\,dt = \hat{F} = 1 \Rightarrow F = \frac{1}{\Delta t} \). \( F \) behaves like the Dirac \( \delta \) function \( \int_{t}^{t+\Delta t} \delta(t)\,dt = 1 \Rightarrow m\dot{x}_{0} = 1 \Rightarrow \dot{x}_{0} = 1 = \frac{1}{m} \) and \( x_{0} = 0 \) (no move yet). Hence, the solution is the free response with I.C. \( x_{0} = 0, \dot{x}_{0} = \frac{1}{m} \).
Unit Impulse Response for impulse at $t = 0$:

$$x(t) = \left( x_0 \cos \omega_d t + \left( \frac{\dot{x}_0}{\omega_d} + x_0 \zeta \omega_n \right) \sin \omega_d t \right) e^{-\zeta \omega_n t}$$

with $x_0 = 0, \dot{x}_0 = \frac{1}{m}$ \Rightarrow $x(t) = \frac{1}{m \omega_d} \sin \omega_d t e^{-\zeta \omega_n t}, t > 0$

For a unit impulse of magnitude $Fd\tau$, applied at $t = \tau$:

$$x(t) = \frac{Fd\tau}{m \omega_d} \sin \omega_d (t - \tau) e^{-\zeta \omega_n (t-\tau)}, t > \tau$$
For 2 unit impulses of magnitude $F_1 d\tau$, $F_2 d\tau$ applied at $t = \tau_1$, $t = \tau_2$:

$$x(t) = \frac{F_1 d\tau}{m\omega_d} \sin\omega_d(t - \tau_1) e^{-\zeta\omega_n(t-\tau_1)} + \frac{F_2 d\tau}{m\omega_d} \sin\omega_d(t - \tau_2) e^{-\zeta\omega_n(t-\tau_2)}$$
**Duhamels’s (or convolution) Integral** Similarly, for \( n \) finite impulses:

\[
    x(t) = \sum_{i=1}^{n} \frac{F_i d\tau}{m\omega_d} \sin\omega_d(t - \tau_i) e^{-\zeta\omega_n(t-\tau_i)}
\]

Adding up, for \( d\tau \to 0 \), we obtain the continuous expression:

\[
    x(t) = \int_{0}^{t} \frac{F(\tau)}{m\omega_d} \sin\omega_d(t - \tau) e^{-\zeta\omega_n(t-\tau)} d\tau
\]
**SDOF systems: From Time to Frequency Domain**

**The Fourier Transform**

**Definition:**

\[ F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt \]

**Inverse:**

\[ f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} \, d\omega \]

**Fourier Transform of a cosine**
**SDOF systems: From Time to Frequency Domain**

**Fourier Transform of a cosine - Note**

The Fourier Transform (FT) of a cosine simply consists in two symmetric spikes at values corresponding to the specific frequency of that cosine (symmetric transformation).

Mathematically this can be written in the form of two delta functions $\delta(\omega \pm \omega_0)$ or $\delta(p \pm p_0)$ in Hz.

This reflects the fact that the frequency content of a perfect cosine (or sine) function contains a single frequency component.

see Laplace and Fourier transform pdfs uploaded on the website
The Laplace Transform

Definition: \[ F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \quad s = \sigma + i\omega \]

Inverse: \[ f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma-iT}^{\gamma+iT} F(s)e^{st}ds \]

Basic Laplace Transform Property

\[ \mathcal{L}\left[ \frac{df}{dt} \right] = \int_0^\infty e^{-st} \frac{df(t)}{dt}dt \]

Applying integration by parts: \[ \int udv = uv - \int vdu \], for \( u = e^{-st} \), \( v = f \), we obtain:

\[ \mathcal{L}\left[ \frac{df}{dt} \right] = e^{-st}f(t)\bigg|_0^\infty - \int_0^\infty (-s)e^{-st}f(t)dt \]

\[ = -f(0) + s \int_0^\infty f(t)e^{-st}dt \Rightarrow \mathcal{L}\left[ \frac{df}{dt} \right] = sF(s) - f(0) \]
Similarly,

$$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - \frac{df}{dt}(0) - sf(0)$$

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \frac{d^{n-1} f}{dt^{n-1}}(0) - \ldots - sn - 2\frac{df}{dt}(0) - s^{n-1}f(0)$$

**Example - Obtaining the Transfer Function (TF):**

Apply the Laplace Transform on a 2nd order ODE:

$$\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + 3y = 4u \quad \overset{\mathcal{L}}{\Rightarrow} \quad s^2 Y(s) - \frac{dy}{dt}(0) - sY(0) + 2sY(s) - 2Y(0) + 3Y(s) = 4U(s)$$

$$0 \overset{\mathcal{L}}{\Rightarrow} \quad Y(s) = \frac{4}{s^2 + 2s + 3} U(s) : \quad \text{Transfer Function}$$
Transfer Function
A transfer function (also known as the system function) is a mathematical representation, in terms of the system frequency, of the relation between the input and output of a linear time-invariant system with zero initial conditions and zero-point equilibrium. A linear time-invariant (LTI) system is characterized by two properties:

Linearity which means that the relationship between the input and the output of the system is a linear map. If input $x_1(t)$ produces response $y_1(t)$ and input $x_2(t)$ produces response $y_1(t)$ then the scaled and summed input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ produces the scaled and summed response $\alpha_1 y_1(t) + \alpha_2 x_2(t)$ where $\alpha_1$, $\alpha_2$ are real scalars.

Time invariance which means that whether we apply an input to the system now or $T$ seconds from now, the output will be identical except for a time delay of the $T$ seconds. That is, if the output due to input $x(t)$ is $y(t)$, then the output due to input $x(t - T)$ is $y(t - T)$. Hence, the system is time invariant because the output does not depend on the particular time the input is applied.
The Laplace Transform

More Laplace Transform Properties

- \( \mathcal{L}\{\delta(t)\} = 1 \), \( \delta(t) \): Dirac
- \( \mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F \left( \frac{s}{\alpha} \right) \)
- \( \mathcal{L}\{e^{\alpha t} f(t)\} = \frac{1}{\alpha} F \left( s - \alpha \right) \) (Frequency Shift)
- \( \mathcal{L}\{f(t - \alpha)H(t - \alpha)\} = e^{-\alpha s} F(s) \) (Time Shift) where \( H \) is the Heaviside step function \( H(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \)
- \( \mathcal{L}\{f \ast g(t)\} = \int_0^t f(\tau)g(t - \tau) d\tau = F(s)G(s) \) (Convolution)
Example: Equation of Motion

\[ m\ddot{x} + c\dot{x} + kx = f(t) \quad \overset{L,0 \rightarrow C.}{\Rightarrow} \quad ms^2 X(s) + csX(s) + kX(s) = F(s) \Rightarrow \]

\[ X(s) = \frac{1}{ms^2 + cs + k} F(s) \Rightarrow \quad H(s) = \frac{1}{ms^2 + cs + k} \]

Transfer Function

Assuming \( s = i\omega \) we obtain the complex **Frequency Response Function (FRF)**:

\[ H(i\omega) = \frac{1}{m\omega^2 + ci\omega + k} \]

We can then use what we define as the **Inverse Laplace Transform** in order to determine \( x(t) \)
Basic Inverse Transform Properties (also look at given tables)

- \( \mathcal{L}^{-1} \left\{ \frac{1}{s + \alpha} \right\} = e^{-\alpha t} \)

- \( \mathcal{L}^{-1} \left\{ \frac{1}{(s + \alpha)^2} \right\} = te^{-\alpha t} \)

- \( \mathcal{L}^{-1} \left\{ \frac{1}{(s + \alpha)(s + \beta)} \right\} = \frac{1}{\beta - \alpha} [e^{-\alpha t} - e^{-\beta t}] \)

Hence, \( x(t)\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{ms^2 + cs + k} F(s) \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{m(s + \alpha)(s + \beta)} F(s) \right\} \)

where, \( \alpha, \beta = \frac{c \mp i\sqrt{4mk - c^2}}{2m} \) assuming an underdamped system.
The Laplace Transform

Also for \( f(t) = \delta(t) \) (unit impulse) \( \Rightarrow F(s) = 1 \)

Then, \( x(t) = \frac{1}{m(\beta - \alpha)} \left[ e^{-\alpha t} - e^{-\beta t} \right] \)

Using \( k = m\omega_n^2 \), \( c = 2m\omega_n\zeta \), \( \omega_d = \omega_n\sqrt{1 - \zeta^2} \)

\[ \Rightarrow x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega nt} \left[ \sin \omega_d t \right] \]

Obviously this agrees with the time domain derived SDOF impulse response
The Frequency Response Function (FRF)
Assume a linear system characterized by its Transfer Function:

$$Y(s) = G(s)U(s), \quad \text{where } s \in \mathbb{C} \text{ (Laplace Domain)}$$

Evaluated at the imaginary axis $s = i\omega$, the TF yields the FRF, $G(i\omega)$:

**Case of a SISO System**

\[
\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \frac{du}{dt} + 2u \quad \mathcal{L}, \quad 0 \text{ ICs}
\]

\[
(s^2 + 4s + 3)Y(s) = (st2)U(s) \Rightarrow G(s) = \frac{s + 2}{(s + 3)(s + 1)}
\]

Hence the FRF is,

$$G(i\omega) = \frac{i\omega + 2}{(i\omega + 3)(i\omega + 1)}$$
The FRF

As shown FRF describes the response of the system for sinusoidal inputs. It is also the FT of the system’s impulse response.

The FRF can be visualized using Bode Plots

Assume \( G(i\omega) = \frac{1}{i\omega + 1} \Rightarrow G(i\omega) = \frac{1}{1 + \omega^2} - \frac{i\omega}{1 + \omega^2} \)

Bode Magnitude Plot

For \( \omega = 1 \Rightarrow |G(i\omega)| = \left| \frac{1 - i}{2} \right| = \frac{1}{\sqrt{2}} \Rightarrow -20\log |G| = -3.02\,dB \)
The FRF

Bode Magnitude Plot - Asymptotes

For \( \omega \ll 1 \Rightarrow \frac{1}{i\omega + 1} \approx 1 \) and \(-20\log |G| = -20\log(1) = 0\)

For \( \omega \gg 1 \Rightarrow \frac{1}{i\omega + 1} \approx \frac{1}{i\omega} \) and \(\left| \frac{1}{i\omega} \right| = \frac{1}{\omega}\)

Then assuming \(\omega_2 = 10\omega_1\) \(\Rightarrow\)

\[
20\log \left( \frac{1}{\omega_2} \right) = 20\log \left( \frac{1}{10\omega_1} \right) = 20\log \left( \frac{1}{10} \right) + 20\log \left( \frac{1}{\omega_1} \right)
\]

Hence we can approximate the decline per “decade”

\[
20\log \left| \frac{1}{i\omega_2 + 1} \right| - 20\log \left| \frac{1}{i\omega_1 + 1} \right| \approx 20\log \left( \frac{1}{\omega_2} \right) - 20\log \left( \frac{1}{\omega_1} \right) \Rightarrow
\]

\[-20(dB/\text{decade})\]
The FRF

\[ G(i\omega) = \frac{1}{1 + \omega^2} - \frac{i\omega}{1 + \omega^2} \Rightarrow \phi(\omega) = \tan^{-1}(-\omega) \]

Bode Phase Plot

For \( \omega = 1 \) \( \Rightarrow \phi(\omega) = \tan^{-1}(-1) = -45^\circ \)
For \( \omega = \frac{1}{10} \) \( \Rightarrow \phi(\omega) = \tan^{-1}\left(\frac{1}{10}\right) = -5^\circ \)
For \( \omega = 10 \) \( \Rightarrow \phi(\omega) = \tan^{-1}(-10) = -90^\circ \)
Generalizing:

Suppose we had \( G(i\omega) = \frac{\alpha}{\alpha + i\omega} = \frac{1}{i \left(\frac{\omega}{\alpha}\right) + 1} \)

Then, the same plots apply for \( \bar{\omega} = \frac{\omega}{\alpha} \)
Returning to the SISO Example

We had \( G(i\omega) = \frac{i\omega + 2}{(i\omega + 3)(i\omega + 1)} \) \( \Rightarrow \)

\[
20\log |G(i\omega)| = 20\log |i\omega + 2| + 20\log \left| \frac{1}{(i\omega + 3)} \right| + 20\log \left| \frac{1}{(i\omega + 1)} \right|
\]

\[
= 20\log \left| 2i\frac{\omega}{2} + 1 \right| + 20\log \left| \frac{1/3}{(i\frac{\omega}{3} + 1)} \right| + 20\log \left| \frac{1}{(i\omega + 1)} \right| \Rightarrow \]

Superposition of Plots

\[
20\log |G(i\omega)| = 20\log \left( \frac{2}{3} \right) + 20\log \left| i\frac{\omega}{2} + 1 \right| + 20\log \left| \frac{1}{(i\frac{\omega}{3} + 1)} \right| + 20\log \left| \frac{1}{(i\omega + 1)} \right|
\]
The FRF

Plot Superposition

Institute of Structural Engineering
Identification Methods for Structural Systems