Overview

- Fundamentals of Signal Processing
  - Classification
  - The Fourier Transform
  - Common Problems
  - Filtering
Classification

Deterministic Signals
Uniquely defined and can be determined by a mathematical expression, rule or table

Example: $\sin(x)$

Random/Stochastic Signals
Cannot be predicted ahead of time.

Example: White noise
**Random Signals** are characterized by a rich frequency content and a rapidly varying amplitude over time.

We need to analyze the statistical characteristics across an ensemble of records in order to assess their properties.

The collection of signals is a **random process (r.p)** denoted by $X(t)$.

Each realization - **random variable (r.v)** in this collection is denoted by $x(t)$ or $x[n]$ in discrete time.
Statistical Properties

**Mean (1\textsuperscript{st} Moment):**  
\[ E[X] = \mu_x(t) = \int_{-\infty}^{\infty} xf(x) dx \]

\( f(x) \) is the associated probability density function with \( \int_{-\infty}^{\infty} f(x) dx = 1 \) and \( E \) stands for Expected Value

**Variance (2\textsuperscript{nd} Moment):**

\[ E[(X - \mu_x(t))^2] = Var(X) = \int_{-\infty}^{\infty} (x - \mu_x(t))^2 f(x) dx = E[X^2] - \mu_x^2(t) \]

and **Standard Deviation:**  
\[ \sigma_x(t) = \sqrt{Var(x)} \]
Statistical Properties

Time Averages

Usually we cannot view the entire ensemble \((-\infty, \infty)\), therefore we must use time averages.

This will generally give acceptable results only in the case of ergodic processes, where each signal seems to have the same statistical behavior as the entire process. Then,

\[
\bar{X} = E[X] = \frac{1}{T} \int_{0}^{T} X(t) dt
\]

and for the usual case of discrete signals

\[
\bar{X} = E[X] = \frac{1}{N} \sum_{n=1}^{N} X[n] dt
\]

then

\[
\sigma_{X}^{2} = \bar{X}^{2} - (\bar{X})^{2}
\]
**Statistical Properties**

**Autocorrelation**

In a probabilistic sense: It is the expected value of the product of a random variable with a time-shifted version of itself.

Assume we have two realizations $x_1 = X(t_1)$, $x_2 = X(t_2)$ of the same process $X$, then:

$$R_{xx}(t_1, t_2) = E[X_1X_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1x_2f(x_1, x_2)dx_1dx_2$$

In the signal processing sense: for **Stationary Processes**, that is processes whose average properties are independent of a time shift $\tau$ (with $E[X] = const$), this is rewritten as:

$$R_{xx}(t, t + d\tau) = R_{xx}(\tau) = E[X(t)X(t + \tau)]$$

In **discrete time** this becomes

$$R_{xx}(n, n + m) = \sum_{n=-\infty}^{\infty} x[n]x[n + m]$$
Cross-Correlation

In a probabilistic sense: Relates a random variable $x$ from a random process $X$ to a time shifted r.v. $y$ from a random process $Y$:

$$R_{xy}(t_1, t_2) = E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)\,dx\,dy$$

where $f(x, y)$ is the associated joint probability density function

In the signal processing sense: in **discrete time** this becomes

$$R_{xy}(n, n - m) = R_{xy}(m) = \sum_{n=-\infty}^{\infty} x[n] \, y[n - m]$$
Further Classifications

Transient Signals

They exist for a finite range of time.

Examples include hammer excited systems, explosions and shock loading.
Further Classifications

Periodic (vs. Aperiodic) Signals:

\[ x(t + T) = x(t) \]
Further Classifications

1D vs. 2D Signals: For instance, speech vs. image

Continuous vs. Discrete Signals:
Typical Operations

Typical Signal Processing Operations

These take place either in **time** or **frequency** domain

Time Domain Operations

- **Scaling:** $y(t) = \alpha x(t)$ with $\alpha > 1 \rightarrow$ gain, $\alpha < 1 \rightarrow$ attenuation

- **Time Shift:** $y(t) = x(t - t_0)$ with $t_0 > 0 \rightarrow$ delay, with $t_0 < 0 \rightarrow$ advance

- **Addition:** $y(t) = x_1(t) + x_2(t) + x_3(t)\ldots$
The Fourier Series

It represents general periodic functions or periodic signals in terms of simple oscillating functions (sines and cosines):

Assuming \( x(t + T) = x(t) \), this can be approximated by:

\[
x(t) = a_0 + \sum_{n} \left\{ a_n \cos(n\omega t) + b_n \sin(n\omega t) \right\}
\]

where \( \omega = \frac{2\pi}{T} \), and \( \cos(n\omega t), \sin(n\omega t) \) constitute an orthogonal basis:

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(n\omega t) \sin(m\omega t) = 0, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(n\omega t) \cos(m\omega t) = 0, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(n\omega t) \sin(m\omega t) = 0
\]

**Fourier Coefficients:**

\[
a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt,
\]

\[
a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(n\omega t) \, dt, \quad b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(n\omega t) \, dt, \quad n = 1, 2, \ldots
\]
The Fourier Series

Using the Euler formula this can eventually be rewritten as:

\[ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega t} \]

where \( C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-in\omega t} \, dt \)

Relationship to Fourier Coefficients:

\[ c_0 = 0, \quad n = 0 \quad C_n = \frac{a_n - ib_n}{2}, \quad n > 0 \quad C_n = \frac{a_n + ib_n}{2} = c_n^*, \quad n < 0 \]
The Fourier Transform (FT)

The FT essentially decomposes a signal into sinusoidal components. The transform is generally applied to signals that extend to $\pm \infty$. However, we usually only have a finite number of samples. Obviously, one cannot use a group of sine and cosine waves extended from $\pm \infty$ to synthesize something finite in length. The way around this is to make the finite data look like an infinite in length signal. There are two ways of doing this:

**Duplicating** Duplicated of the original are added to the left and right of the finite signal (becomes periodic)

**Zero Padding** Zeros are added to the left and right of the finite signal (remains aperiodic)
The Fourier Transform

We can then rewrite:

\[
x(t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau) e^{-i\omega \tau} \, d\tau \right\} e^{i\omega t}
\]

as \( T \) increases to infinity, \( \omega \ll 1 \Rightarrow \omega = \Delta \omega \) and \( n\Delta \omega = \omega_n \Rightarrow \frac{1}{T} = \frac{\Delta \omega}{2\pi} \) hence

\[
x(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left\{ \int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau) e^{-i\omega_n \tau} \, d\tau \right\} e^{i\omega_n t} \Delta \omega
\]
The Fourier Transform

For $\Delta \omega \to 0 \Rightarrow T \to \infty$ the summation turns to an integral:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau) e^{-i\omega \tau} \, d\tau \right\} e^{i\omega t} \, d\omega$$

So we define the Fourier Transform (FT)

$$\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} \, dt$$

and the Inverse Fourier Transform (IFT)

$$\mathcal{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \, d\omega$$
The Fourier Transform

The Discrete Fourier Transform

Actual signals usually occur through sampling. Therefore, instead of continuous $t$ we use $t_k = k\Delta t$ and $T = N\Delta t$ (total time). The discretized equivalent of the continuous Fourier Transform becomes:

$$X_n = \sum_{k=0}^{N-1} x_k e^{-i2\pi \frac{n}{N} k}$$

This is obtained using $\omega_n = \frac{2\pi}{T} n = \frac{2\pi}{N\Delta t} n \Rightarrow \omega t = \frac{2\pi}{N} nk$.

Also, $\Delta f = \frac{\Delta \omega}{2\pi} = \frac{1}{N\Delta t}$ (in Hz) while

the **sampling frequency** is defined as $f_s = \frac{1}{\Delta t}$. 

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The Fourier Transform

The Discrete Fourier Transform

Important Note: $X_0 = \frac{1}{N} \sum_{k=0}^{N-1} x_k$ corresponds to $\omega = 0$.

In addition, it can be shown that $X_{N-k} = X_k^*$

whereas for FT: $X(\omega) = X^*(-\omega)$

Hence, $X_{N-k}$ corresponds to a frequency of $-k\Delta\omega$.

This means that the sequence $X_n$ stores the frequency representation of the signal $x(t)$ as follows:
The Fourier Transform

The Discrete Fourier Transform

\[ \omega = 0 \quad \Delta \omega \quad 2 \Delta \omega \quad 2 \Delta \omega \quad \Delta \omega \]

\[ n = 0 \quad 1 \quad 2 \quad N - 2 \quad N - 1 \]

Only half of the representation is useful

\[ X_{N/2}^* = X_{N/2} \] which means that the maximum frequency we can represent is

\[ N \frac{\Delta \omega}{2} = \frac{F_s}{2} \]
The Fourier Transform

Definitions - Discrete Fourier Transform (DFT)

\[ X_n = \sum_{k=0}^{N-1} x_k e^{-i2\pi \frac{n}{N} k} \]

Inverse Discrete Fourier Transform (IDFT)

\[ x_k = \frac{1}{N} \sum_{n=0}^{N-1} X_n e^{i2\pi \frac{k}{N} n} \]

*MATLAB* uses the fft and ifft intrinsic functions, based on the Fast FT, a method that essentially breaks DFT into pairs of smaller size DFTs and is much faster.
Discrete Fourier Transform (DFT)

Although in the above formulations we have stated that both $n$ and $k$ range over $0 \ldots N - 1$, both definitions have a periodicity of $N$:

$$X_{n+N} = X_n \quad x_{k+N} = x_k$$

Therefore $X_n$ and $x_k$ are defined for all (integer) $n$ and $k$ respectively, but we only need to calculate values in the range $0 \ldots N - 1$. Any other point can be obtained using the above periodicity property.
So how to obtain system output?

For a given input \( u(t) \) and a system with a known impulse response \( h(t) \) we know that the response \( y(t) \) in the frequency domain is:

\[
Y(\omega) = H(\omega)U(\omega)
\]

Hence, we can perform the multiplication of the Fourier Transform series: \( U(\omega) = \mathcal{F}\{u(t)\} \), \( H(\omega) = \mathcal{F}\{h(t)\} \) where \( H(\omega) \) is the Frequency Response function and obtain \( Y(\omega) \).

Next, by applying the bf IFT we can obtain the actual response time history \( y(t) \). (see uploaded FT Demo)

In the time domain, the equivalent operation is convolution: \( (f \ast g)(t) = \int_{-\infty}^{\infty} f(t - \tau)g(\tau) \, d\tau \).
Closely Spaced Peaks

A factor that limits the frequency resolution with which closely spaced peaks can be is the length of the DFT. The frequency spectrum produced by an N point DFT consists of \( \frac{N}{2} + 1 \) samples equally spaced between zero and one half of the sampling frequency. To separate two closely spaced frequencies, the sample spacing must be smaller than the distance between the two peaks. For example, a 512-point DFT is sufficient to separate the peaks in the figure, while a 128-point DFT is not.
**Typical FT Associated Problems**

**Leakage**

Spectral leakage is an effect in the frequency analysis of finite-length signals or finite-length segments of infinite signals where it appears as if some energy has “leaked” out of the original signal spectrum into other frequencies. Typically the leakage shows up as a series of “lobes”.

![Graph showing leakage from a sinusoid (rectangular window)](image)
Typical FT Associated Problems

Leakage

- The Fast Fourier Transform is commonly used because it requires much less processing power than the Fourier Transform. Like all shortcuts, there are some compromises involved in the FFT.
- The signal must be periodic in the sample window or leakage will occur.
- The signal must start and end at the same point in its cycle.
- Leakage is the smearing of energy from the true frequency of the signal into adjacent frequencies.
- Leakage also causes the amplitude representation of the signal to be less than the true amplitude of the signal.
Leakage is tackled using appropriate window functions (other than rectangular ones)
The Gibbs phenomenon

When only some points (frequencies) are used in the reconstruction, Fourier sums overshoot at a jump discontinuity, and this overshoot does not die out (ringing) as the frequency increases.

**Example** Approximation of square wave

- using 5 steps
- using 25 steps
- using 125 steps
The Gibbs phenomenon

It is obvious that the width of the overshoot becomes smaller as more sinusoids are included. The overshoot is still present with an infinite number of sinusoids, but it has zero width. Exactly at the discontinuity the value of reconstructed signal converges to the midpoint of the step.

Problems related to the Gibbs effect are frequently encountered in digital signal processing. For instance, a low-pass filter is a truncation of the higher frequencies, resulting in overshoot and ringing at the edges in the time domain. Another common procedure is the truncation of the ends of a time domain signal. This in turn distorts the edges in the frequency domain.
Typical FT Associated Problems

Aliasing

In signal processing and related disciplines, aliasing refers to an effect that causes different signals to become indistinguishable (or aliases of one another) when sampled.

The sampling frequency is equal to \( f_s = \frac{1}{\Delta t} \).

For \( N \) equally spaced samples, the total time is \( T = (N - 1)\Delta t \).

Hence: \( f_s = \frac{N - 1}{T} = (N - 1)\Delta f \).

The sampling theorem asserts that the uniformly spaced discrete samples are a complete representation of the signal if the bandwidth \( f_{\text{max}} \) is less than half the sampling rate. The sufficient condition for exact reconstructability from samples at a uniform sampling rate \( f_s \) (in samples per unit time) \((f_s \geq 2f_{\text{max}} \implies \text{Nyquist Frequency: } f_N \geq f_{\text{max}})\).
Typical FT Associated Problems

Aliasing

Aliasing occurs when there is an overlap in the shifted, periodic (as we already noted) copies of our original signal’s FT, i.e. spectrum. In the frequency domain, one will notice that part of the signal will overlap with the periodic signals next to it. In this overlap the values of the frequency will be added together and the shape of the signal’s spectrum will be altered.
Aliasing

A typical case of aliasing is displayed below where the sampled (blue) signal corresponds to both a frequency change and a phase shift with respect to the original signal (red).
Assume that we are trying to build a Fourier transforming device which can give us the spectrum of a given time signal. Suppose that we have a maximum sampling frequency of 1000 Hz i.e. a Nyquist frequency of 500 Hz.

If the time signal has a broadband spectrum which is flat up to 750 Hz, what will the estimated spectrum look like?
Aliasing Effect

If the time signal has a broadband spectrum which is **flat up to 750 Hz**, what will the estimated spectrum look like?

Parseval’s theorem states that the FT is unitary, i.e., the total energy of $x(t)$ equals the Fourier transform $X_n$ energy. Therefore, the excess energy from the unsampled range of 500-750 Hz is aliased (folds) into the range 250-500 Hz and we obtain a distorted spectrum (falsely indicating more energy in the higher frequencies).

**Diagram:**
- Sampling at 1000Hz $\rightarrow$
- Nyquist Frequency $N_Y = 500$ Hz

**Graph:**
- Aliased Spectrum
- Frequency (Hz)
- Amplitude
- **250** 500 750
Antialiasing Filter

How can we help this? Suppose we had a device which removed the part of the signal at frequencies between 500 and 750 Hz. Then we would have changed the signal admittedly but the FFT would at least give us an accurate spectrum all the way up to 500 Hz.

A device which passes parts of a signals frequency content and suppresses others is called a filter. The filter described previously is called an antialiasing filter.
Filtering

**Ideal Filter**

**Actual Filter**
Low - Pass

![Diagram of a low-pass filter with passband ripple, transition band, and stopband.](image-url)
Filter Classifications

High - Pass

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Pass - Band

Diagram showing the classification of filters with transition bands and stop bands.
Filter Classifications

Stop - Band

Gain

Pass Band

Stop Band

Pass Band

Transition Bands

\( \omega_{p_1} \)

\( \omega_{s_h} \)

\( \omega_{s_h} \)

\( \omega_{p_1} \)