Alternative Modal ID Methods

Several additional Modal ID methods exist for identifying structural properties.

- Identification of Structural System Parameters (ISSPA) method
- Ewins & Gleeson method
- Rational fraction Polynomial
Modal Analysis techniques

Identification of structural system parameters (ISSPA)

This is a MIMO, direct method.

Assume a set of harmonic excitations at discrete frequencies 
$[\omega_1...\omega_{nf}]^T$. Since the FT of a sin pulse is a single spike at the given
$\omega_i$, we can write the corresponding force FT amplitudes as $F_1, \ldots F_{nf}$

The response displacement $x(t)$ of the system at each dof $1 \ldots N$, which is also harmonic, is experimentally measured for each $F_i$.

Hence, its Fourier Transform $X(\omega_i)$ is also known and equal to
$X_i = [X_{1i} \ldots X_{Ni}]^T$, $i = 1 \ldots nf$. Therefore we have the following
system of equations, for $N_f$ discrete frequencies:

$[-\omega_1^2 M + i\omega_1 C + K]X_1 = F_1$

$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$

$[-\omega_{nf}^2 M + i\omega_{nf} C + K]X_{nf} = F_{nf}$
Modal Analysis techniques

The result is an overdetermined system of equations from where \( C, M, K \) are computed (system matrices).

**Note concerning the frequency response function definition:**

If the measurement is displacement, the receptance FRF is:

\[
H(\omega) = [-\omega^2 M + i\omega C + K]^{-1}
\]

If the measurement is acceleration, the accelerance FRF is:

\[
H(\omega) = -\omega^2 [-\omega^2 M + i\omega C + K]^{-1}
\]

If the measurement is velocity the mobility FRF is instead:

\[
H(\omega) = j\omega [-\omega^2 M + i\omega C + K]^{-1}
\]
Ewins & Gleeson method

This is an indirect, SISO method designed for lightly damped structures. Like the peak picking method, it is based on experimentally measuring the components of the FRF matrix (also called receptance: $H_{ij}(\omega)$). This is done by measuring the response at node i when applying a unit force at j.

In modal testing, FRF measurements are usually made under controlled conditions, where the test structure is artificially excited by using either an impact hammer, or one or more shakers driven by broadband signals.

A multi-channel FFT analyzer (instrument) is then used to make FRF measurements between input and output DOF pairs on the test structure.
Ewins & Gleeson method

Similarly to the peak picking method this approach relies on the FRF peaks being usually sharp and easy to identify by inspection (equivalent “SDoF” area). Peaks correspond to natural frequencies $\omega_r$, $r = 1, ..., N$.

Assuming the damping is light enough to be neglected, we have the standard expression for the FRF of an N dof system:

$$H_{ij}(\omega) = \sum_{k=1}^{N} \frac{k A_{ij}}{\omega_k^2 - \omega^2}$$

Taking $M$ measured frequencies $\Omega_1, \ldots, \Omega_M$ and the corresponding FRF matrices $H_{ij}(\Omega_1), \ldots, H_{ij}(\Omega_M)$, we can formulate the following system in order to determine the modal coefficients $k A_{ij}$.

$$
\begin{bmatrix}
A_{ij} \\
\vdots \\
A_{ij}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\omega_1^2 - \Omega_1^2} & \frac{1}{\omega_2^2 - \Omega_1^2} & \cdots & \frac{1}{\omega_N^2 - \Omega_1^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\omega_1^2 - \Omega_N^2} & \frac{1}{\omega_2^2 - \Omega_N^2} & \cdots & \frac{1}{\omega_N^2 - \Omega_N^2}
\end{bmatrix}^{-1}
\begin{bmatrix}
H_{ij}(\Omega_1) \\
\vdots \\
H_{ij}(\Omega_M)
\end{bmatrix}
$$
The Rational fraction Polynomial method

This is an mdof frequency domain method based on curve fitting. We can in general write the FRF $H(\omega)$ as a ratio of polynomials. The roots of the denominator polynomial (poles) are related as shown before to the modal characteristics of the mdof system.

\[
H(\omega) = \frac{\sum_{K=0}^{2N-1} a_K (i\omega)^K}{\sum_{K=0}^{2N} b_K (i\omega)^K}
\]

We can use an optimization method to minimize the error between the calculated $H(\omega)$ and the measured values $H(\omega_f)$, obtained from experiment, at different frequencies $\omega_f$.

\[
\Rightarrow \sum_{K=0}^{2N-1} a_K (i\omega_f)^K - \sum_{K=0}^{2N} H(\omega_f) \cdot b_K (i\omega) = 0
\]

$\Rightarrow a_K, b_K$ are determined $= 0 \Rightarrow$ using Least Squares Estimation (LSE) methods
Damage Detection

Once again, let us assume the model of a dynamical system.

\[ M\ddot{x} + C\dot{x} + Kx = f \]

Then in frequency domain:

\[ Z(\omega)X(\omega) = F(\omega) \]

where the Impedance Matrix is defined as:

\[ Z(\omega) = -\omega^2 M + i\omega C + K, \quad i = \sqrt{-1} \]

In case of damage, the natural frequencies of the structure will be shifted and \( Z \) will change as \( K \) decreases.

Then, we can rewrite:

\[ [Z(\omega) + \Delta Z(\omega)]X(\omega) = F(\omega), \text{quad } \Delta Z(\omega): \text{ effect of damage} \]

And, a “damage vector” can be defined as:

\[ d\omega = F(\omega) - Z(\omega) \cdot X(\omega) = \Delta Z(\omega)X(\omega) \]

where \( F \) is known as the system input, \( X \) is known as the measured response and the original \( Z \) is evaluated from models or experiments on the undamaged system.
Inspection of $d(\omega)$ in terms of the above equation reveals that the $j$-th element of $d(\omega)$ will be 0 when the $j$-th row of $\Delta Z(\omega)$ is 0.

- The original model for the $j$-th DOF is not really affected by the damage.
- Inspecting $d(\omega)$ can give information on dofs affected by damage.
- Due to the presence of noise, a perfect zero/non-zero pattern of $d(\omega)$ would rarely occur.
Change in Flexibility Method

The flexibility Matrix is defined as:

\[ F = K^{-1} = \Phi \Omega^{-2} \Phi^T = \sum_{i=1}^{n} \frac{1}{\omega_i^2} \Phi_i \Phi_i^T \]

For the damaged structure:

\[ F^* = \Phi^* \Omega^*^{-2} \Phi^* T \]

So a measure of damage is:

\[ \Delta F = F - F^* \]

The columns of the flexibility matrix corresponding to the largest change is indicative of the degree of freedom where damage is located.
Change in Stiffness Method

For the **undamaged structure**, the eigenvalue problem states that:

\[(\lambda_i M + K)\phi_i = 0\]

Assume perturbations \(\Delta M_d, \Delta K_d\) for the **damaged case**:

\[(\lambda^*_i [M - \Delta M_d] + [K - \Delta K_d])\phi_i^* = 0\]

\[\Rightarrow D_i = (\lambda^*_i M + K)\phi_i^* = (\lambda^*_i \Delta M_d + \Delta K_d)\phi_i^*\]

(usually we have \(\Delta M_d = 0\) \(\Rightarrow D_i = \Delta K_d\phi_i^*)\)

Again, both these methods involve looking for **zero/non zero** damage patterns.
Damage Detection Notes

In general: $K \downarrow \Rightarrow w \downarrow$
However, in early stages of damage changes in model parameters might not be apparent.

Alternatives:

- The mode shape curvature can actually be more sensitive to damage.
- Damage Index measure decrease in modal strain energy Curvatures/strain energy require the denuatres of displacement mode shapes

$\Rightarrow$ These methods are prone to numerical errors.
Change in Uniform load Surface Curvature

The coefficients of the i-th column of the flexibility matrix $\mathbf{F}$, represent the deflected shape assumed by the structure when a unit load is applied at the i-th degree of freedom. Therefore, the sum of all columns yields deflection due to a uniform load.

This is defined as the **Uniform Load Surface**.

Since commonly curvature (second derivative) is more sensitive to changes than the surface itself we use the following measure for flexibility change/damage:

$$\Delta F'' = \left| F^{*''} - F'' \right|$$

The $\Delta F''$ vector can indicate damage at location i.
Damage Index Method based on the Strain Energy

Quite often changes in F, K or $\phi, \lambda$ are not apparent unless the level of damages is significant. It has been found though that derivatives of displacement mode shapes are more sensitive than the mode shapes themselves.

Assume the case of a beam element:

Strain energy for bending is defined as:

$$dU = \frac{1}{2} Md\theta$$
Also, \( ds = Rd\theta = dx \) (for \( dx \ll \))

where \( \frac{1}{R} \) is the curvature also signified as \( w'' \). Therefore,

\[
M = Elw'' = \frac{EI}{R} \quad \text{and} \quad dx = Rd\theta \\
\Rightarrow \quad d\theta = \frac{M}{EI} \, dx \quad \Rightarrow \quad dU = \frac{1}{\theta} \frac{M^2}{EI} \, dx
\]

Hence, \( U \propto \int (w''(x))^2 \, dx \)
Damage Index Method based on Strain Energy

Then, looking at mode shapes (which for a beam are continuous $\phi''_i(x)$) the related strain energy for mode $i$ is:

$$\int_0^L (\phi''_i(x))^2 \, dx$$

A damage index can therefore be defined for mode $i$ at location $j$ as:

$$\beta_{ij} = \frac{(\int_a^b \phi'''_i \phi''_i \, dx + \int_0^L \phi'''_i \phi''_i \, dx) \ast \int_0^L \phi''_i^2 \, dx}{(\int_0^L \phi''_i^2 \, dx + \int_a^b \phi'''_i \phi''_i \, dx) \ast \int_0^L \phi'''_i \phi''_i \, dx}$$

$[a, b]$ signifies an interval around location $j$ and $\ast$ denotes damaged quantities.

Note: This is not a straightforward estimation method. It requires use of statistical methods in order to examine potential damage scenarios and associate them with relevant changes in $\beta_{ij}$. 
Mode Shape Curvature Method

Again assume a beam subjected to bending moment:

\[ M(x) = EIw''(x) \]

Therefore, when \( K \downarrow \) then \( (EI \downarrow) \Rightarrow w'' \uparrow \)
i.e. reduction in stiffness will lead to an increase in curvature

Differences in the undamaged and damaged curvature mode shapes will in theory be larger in the damaged region.

For the case of multiple mode shapes, the process is modified into adding the absolute values of the mode shapes in order to obtain a DI for a particular location.

Note: The above 3 methods rely on differentiation (double). This leads to numerical errors which can affect the calculations.
The above methods explore damage in terms of dofs. What if we look at it at the element level?

**Damage Equations - Santos et al 1990**

For an undamaged structure, the modal characteristics can be obtained by the eigenvalue equations:

\[ K \Phi_i = \omega_i^2 M \Phi_i \quad \text{for } i = 1, \ldots, n \quad (1) \]

after damage:  
\[ \tilde{K} \tilde{\Phi}_j = \tilde{\omega}_j^2 \tilde{M} \tilde{\Phi}_j \quad \text{for } j = 1, \ldots, m \quad (2) \]

**Element damage Index**

Many methods have attempted to evaluate structural damage based solely on changes of the eigenfrequencies of a structure. In order to properly reflect damage though, it is useful to utilize models of the structure in question such as FE based ones. Then, damage can be located at the element level.
Damage Equations

Element damage Index

Assuming a structural element \( e \), the fractional change in stiffness can be expressed using

\[
\Delta a_e(DI) \Rightarrow \Delta K_e - \tilde{K}_e = \Delta a_e K_e \quad \in [0, 1]
\]

\[
\Rightarrow \tilde{K}_e = K_e(I - \Delta a_e)
\]

Example 1 - Truss Element

\[
K_e = \frac{EA}{l} \begin{bmatrix}
l & -l \\
-l & l
\end{bmatrix} \Rightarrow \Delta a_e = 1 - \frac{\tilde{E}\tilde{A}}{EA}
\]

Example 2 - Truss Element

\[
K_e = -\frac{2EI}{l^3} \begin{bmatrix}
6 & 3l & -6 & 3l \\
3l & 2l^2 & -3l & l^2 \\
-6 & -3l & 6 & -3l \\
3l & l^2 & -3l & 2l^2
\end{bmatrix} \Rightarrow \Delta a_e = 1 - \frac{\tilde{E}\tilde{l}}{EI}
\]
Damage Equations

Assume the damaged structure:

\[
\tilde{\Phi}_j^T \tilde{K} \tilde{\Phi}_j = \tilde{\omega}_j^2 \tilde{\Phi}_j^T M \tilde{\Phi}_j \quad \text{(premultiply by } \tilde{\Phi}_j^T) \]

but global: \[ K = \sum_{e=1}^{N} K_e \text{ where } \sum : \text{Assemblage} \]

\[ \Rightarrow \quad \tilde{K} = \sum_{e=1}^{N} K_e (I - \Delta a_e) \]

Similarly, the i-th global mode shape can be written as:

\[
\tilde{\Phi}_i = \sum_{e=1}^{N} \tilde{\Phi}_{ie} \quad \Rightarrow \sum_{e=1}^{N} \tilde{\Phi}_{ie}^T K_e \tilde{\Phi}_{je} = \tilde{\omega}_j^2 \tilde{\Phi}_i^T \tilde{N} \Phi_j
\]
Using the orthogonality conditions and \( \tilde{K}_e = K_e (I - \Delta a_e) \) ⇒

\[
\sum_e \Phi_{ie}^T K_e \Delta a_e \Phi_{je} = \Phi_i^T K \Phi_j - \tilde{\omega}^2 \delta_{ij}
\]

for \( i = 1, \ldots, n \), \( j = 1, \ldots, m \)

In compact matrix notation:

\[
[S_1] \{\Delta a\} = \{\Delta R_1\}
\]
On the other hand, pre-multiplying equation (2) by the undamaged mode shape $\Phi_i^T \quad (i = 1, \ldots, n)$ yields:

$$\Phi_i^T \tilde{K} \tilde{\Phi}_j = \tilde{\omega}_j^2 \Phi_{ij} \tilde{M} \tilde{\Phi}_j$$

Let's now also assume $\tilde{M} = M$ since usually mass remains unchanged during damage.

Then:

$$\sum_e \Phi_{je} K_e \Phi_{ie} \Delta a_e = \left(1 - \frac{\tilde{\omega}_j^2}{\omega_i^2}\right) \tilde{\Phi}_j^T K \Phi_i$$

again $\Rightarrow [S_2]\{\Delta a\} = \{\Delta R_2\}$

Therefore, two sets of damage Equations are derived. These can be used separately or combined to solve for $\Delta a$