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Revisiting Fracture via a Scaled Boundary Multiscale Approach

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Contents

1. Motivation
2. Salient Features of MsSBFEM
3. Calculation of Stress Intensity Factors
4. Numerical Examples
   1. Unit Cell with embedded slant crack subject to tension
   2. Plate with multiple cracks in tension
5. Conclusions
6. Questions

Motivation

1. **Motivation**
2. MsSBFEM theory
3. SIF calculation
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- Scaled Boundary Finite Element Method (SBFEM)
  - Computationally efficient
  - Semi-analytical
  - SIFs can be determined accurately and with ease

- Computational challenges for large domains remain
  - Extended multiscale finite element method (EMsFEM)
    - Coarse mesh: solve governing equations of the problem
    - Fine mesh: account for fracture phenomena

MsSBFEM Theory

1. Motivation
2. **MsSBFEM theory**
   i. **SBFEM**
   ii. MsFEM
3. SIF calculation
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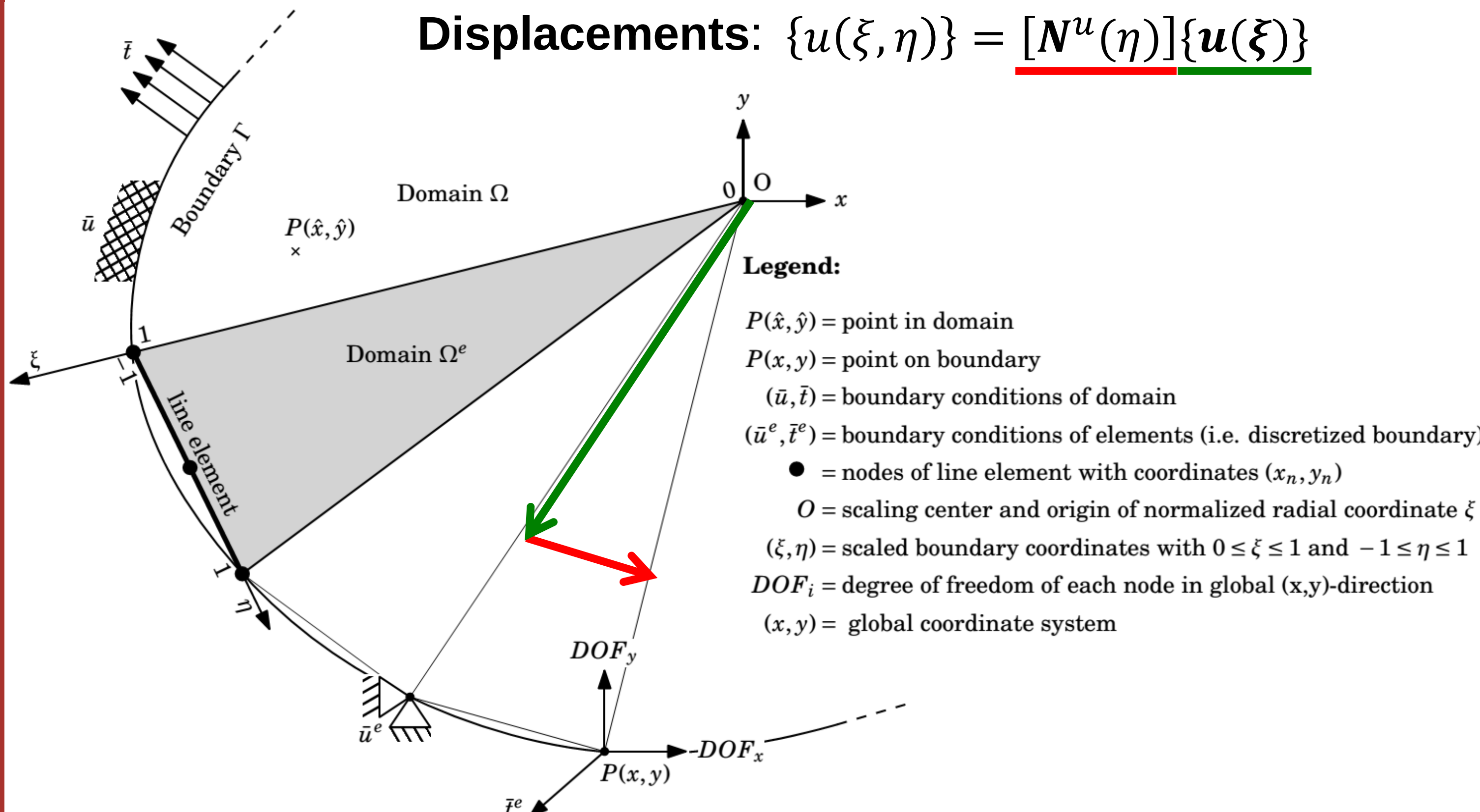
Coordinates:

$$x(\xi,\eta) = x_O + \xi x(\eta)$$
$$= x_O + \xi[\boldsymbol{N}(n)]\{\boldsymbol{x}\}$$

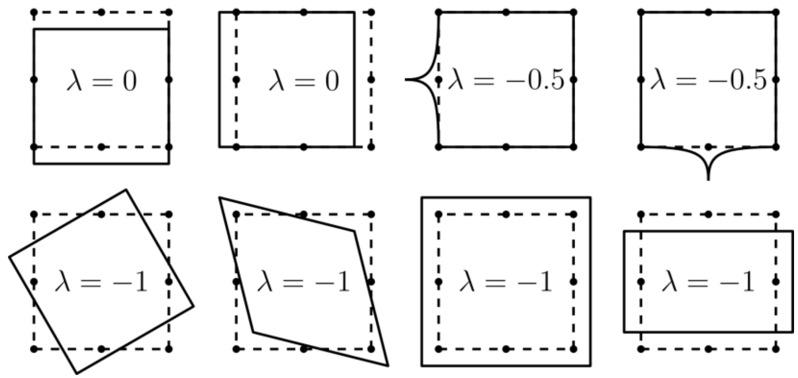
Displacements: $\{u(\xi,\eta)\} = [\boldsymbol{N}^u(\eta)]\{\boldsymbol{u}(\boldsymbol{\xi})\}$



1. Motivation
2. **MsSBFEM theory**
   i. **SBFEM**
   ii. MsFEM
3. SIF calculation
4. Num. Examples
5. Conclusion
6. Questions

MsSBFEM Theory

- General solution as power series



$$\{u(\xi)\} = \sum [\phi_i]\xi^{-[\lambda_i]}[c_i]$$

$[\phi_i]$ = eigenvector
λ_i = eigenvalue
$[c_i]$ = integration constant

- Having performed the eigen-decomposition

Displacements: $\{u(\xi)\} = [\phi_-^u]\xi^{-[\lambda_-]}[c_-]$

Forces: $\{q(\xi)\} = [\phi_-^q]\xi^{-[\lambda_-]}[c_-]$

- Equating displacement modes and force modes on the boundary $\{u(\xi = 1)\}$:

Stiffness Matrix: $[\boldsymbol{K}_{bounded}] = [\phi_-^q][\phi_-^u]^{-1}$

MsSBFEM Theory

1. Motivation
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   i. SBFEM
   ii. MsFEM
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- Difference FEM and MsFEM
  - Basis functions (G) map the response between fine (micro) and coarse (macro) mesh

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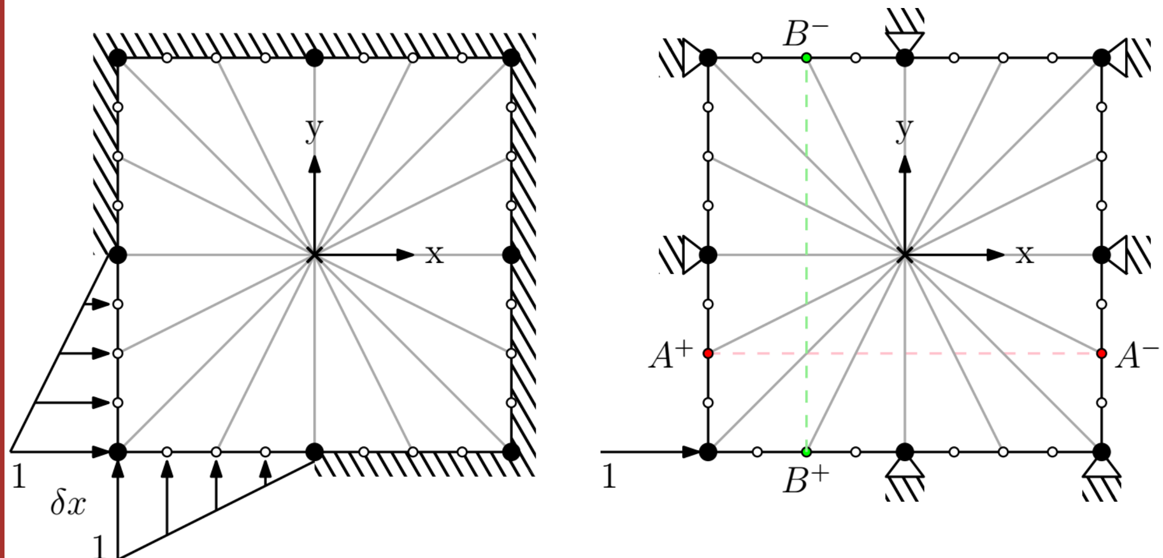
5. Conclusion
6. Questions

MsSBFEM Theory

MsFEM construction of basis functions



$$x^{B+} = x^{B-} + \delta x$$
$$y^{B+} = y^{B-}$$

$$x^{A+} = x^{A-} + \delta x$$
$$y^{A+} = y^{A-}$$

	Linear	Periodic
+	• Simple implementation	• Local periodicity included • Softer behaviour
-	• Too stiff • Too restrained • No local variation	• Requires adjacent pairs • Computational effort

Calculating Stress Intensity Factors

1. Motivation
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4. Num. Examples
5. Conclusion
6. Questions

- SBFEM expression for the stresses

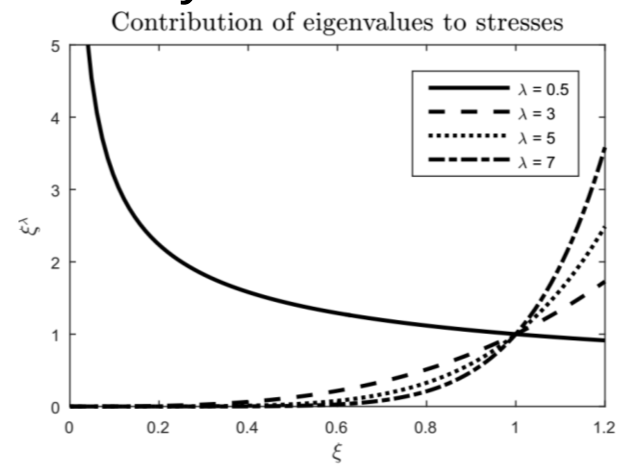
$$\{\sigma(\xi,\eta)\} = [\boldsymbol{D}]\left([\boldsymbol{B}^1(\eta)]\{u(\xi)_{,\xi}\} + \frac{1}{\xi}[\boldsymbol{B}^2(\eta)]\{u(\xi)\}\right)$$

- Inspection of modal representation yields
  - Singularity for $-1 < \lambda < 0$

$$\{\sigma^s(\xi,\eta)\} = [\boldsymbol{\Gamma}_i(\eta)]\xi^{-[\lambda_s]-[I]}\{\boldsymbol{c}^s\}$$

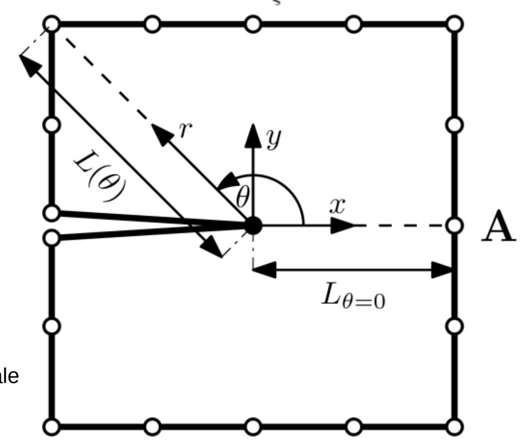
where:

$$\boldsymbol{\Gamma}_i = [\boldsymbol{D}](-\lambda_i[\boldsymbol{B}^1] + [\boldsymbol{B}^2])[\boldsymbol{\phi}_i]$$



- By matching expressions with the exact solution:

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \sqrt{2\pi L_0}\begin{Bmatrix} \sum_{i=I,II} c_i \Gamma_{yy}(\eta = \eta_A)_i \\ \sum_{i=I,II} c_i \Gamma_{xy}(\eta = \eta_A)_i \end{Bmatrix}$$



Numerical Examples:

Unit Cell with embedded slant crack in tension

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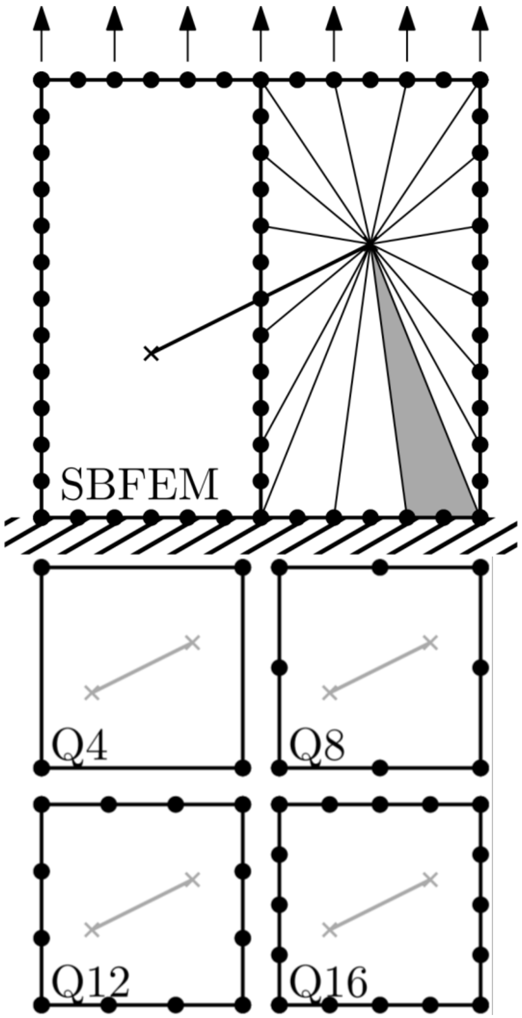
2. MsSBFEM theory
3. SIF calculation
4. **Num. Examples**
   i. **Unit cell intro**
   ii. Unit cell results
   iii. Plate intro
   iv. Plate results
5. Conclusion
6. Questions



Property	Value
E-modulus	200 [N/mm^2]
Poisson ratio	0.3
Side length L	1 [mm]
Crack angle	30°
Crack length a	variable
Tension force	0.1 [N/mm]

Quantities of interest:

- SIF K_1 and K_2 at right crack tip

Numerical Examples:

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a/L	SIF	SBFEM	Q4L	Q8L	Q8Q	Q12L	Q16L	Q4LP	Q8LP	Q8QP	Q12LP	Q16LP
0.1	K1	0.0281	0.53	0.66	1.21	0.26	0.04	0.79	0.47	0.66	0.06	0.07
	K2	0.0168	9.17	0.29	1.36	0.31	0.18	8.25	1.39	2.88	1.14	0.38
0.2	K1	0.0441	1.92	0.31	0.21	0.13	0.14	1.64	0.35	0.34	0.44	0.15
	K2	0.0241	10.49	0.06	1.35	0.15	0.22	10.00	1.70	2.77	1.11	0.47
0.3	K1	0.0573	5.21	2.35	1.82	0.80	0.50	3.74	0.10	0.31	0.95	0.28
	K2	0.0309	12.39	0.29	1.37	0.11	0.33	12.62	2.12	2.61	1.13	0.63
0.4	K1	0.0716	9.51	5.41	4.79	1.65	1.04	6.44	0.96	1.31	1.51	0.50
	K2	0.0370	14.38	0.61	1.34	0.35	0.59	15.43	2.43	2.32	1.28	0.82
0.5	K1	0.0881	14.19	9.51	8.60	2.62	1.77	9.27	2.23	2.48	1.99	0.82
	K2	0.0427	16.29	0.47	0.96	0.31	1.23	17.67	2.39	1.88	1.78	1.06
0.6	K1	0.1080	18.84	14.57	13.09	3.82	2.62	11.88	3.92	3.69	2.38	1.18
	K2	0.0484	18.03	0.93	0.32	0.58	2.70	18.08	1.75	1.48	3.07	1.32
0.7	K1	0.1327	23.31	20.23	18.08	5.96	3.27	14.31	6.19	5.13	3.20	1.51
	K2	0.0545	19.76	4.59	3.17	3.33	5.33	15.20	0.27	1.59	5.78	1.48
0.8	K1	0.1649	27.93	25.65	23.34	10.68	3.16	17.17	9.28	7.62	5.88	1.68
	K2	0.0613	21.96	10.98	8.04	8.80	7.94	8.29	2.35	2.41	10.02	1.65
0.9	K1	0.2102	33.67	29.75	28.54	20.12	2.87	21.64	13.26	12.25	12.30	2.12
	K2	0.0672	26.02	19.26	15.11	15.20	4.40	1.24	6.28	3.04	13.46	4.29

Error % linear (Q4L–Q16L) Error % periodic (Q4LP–Q16LP)

< 1	1-2	2-5	5-10	> 10

Numerical Examples:

Plate with multiple cracks in tension

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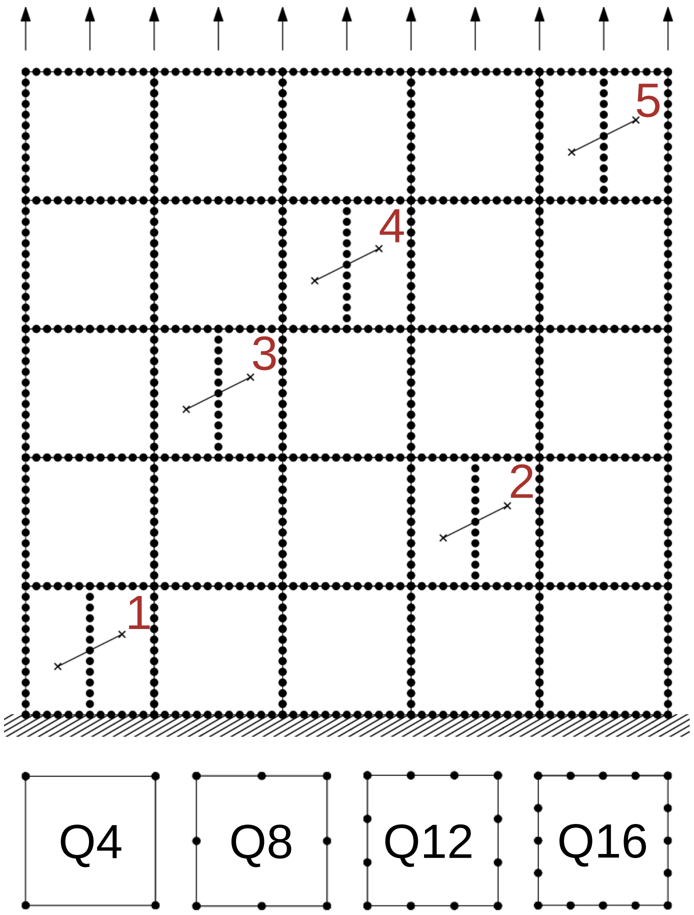
5. Conclusion
6. Questions



Property	Value
E-modulus	200 [N/mm^2]
Poisson ratio	0.3
Side length	5 [mm]
Crack angle	30°
Crack length	variable
Tension force	0.1 [N/mm]

Quantities of Interest:

- SIFs K_1 and K_2 at right crack tip of crack 1-5

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Plate with multiple cracks in tension

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2. MsSBFEM theory
3. SIF calculation
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6. Questions

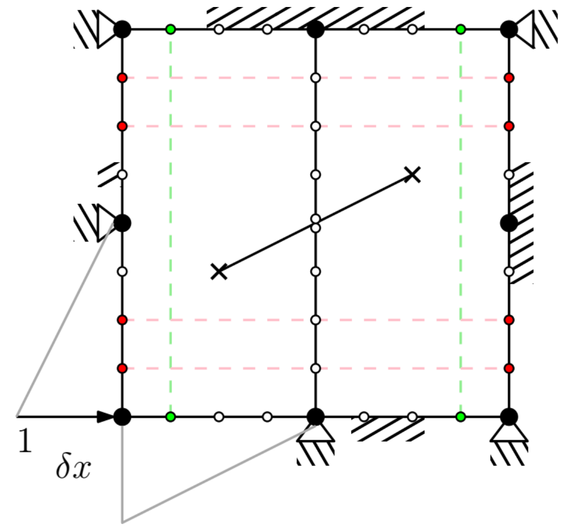
Conclusion and Outlook

MsSBFEM

- Cracks successfully incorporated at the microscale
- Testing of linear and periodic Q4, Q8, Q12 and Q16 elements
  - Q4 and Q8 elements not recommended when cracks present
  - No benefit from using quadratic BC
  - Q12 and Q16 elements deliver best performance
- Accurate for ratios of a/L ≤ 0.7
  - For larger ratios, boundary effects difficult to capture with just 12 or 16 coarse nodes and current choice of micro basis functions

Micro basis functions

- Linear BC better represent the physical behaviour of the crack
- Periodic BC better account for the uncracked behaviour of the UC
- Propose hybrid BCs as seen below



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Thank you for your attention!

BACKUP

SBFEM derivation I

The strong form of the governing equations in 2D elastostatics is given as follows:

equilibrium:	$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$	in Ω	(1a)
constitutive:	$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$	in Ω	(1b)
compatability:	$\boldsymbol{\varepsilon} = \nabla^T \mathbf{u}$	in Ω	(1c)
boundary conditions:	$\mathbf{u} = \hat{\mathbf{u}}$	on Γ	(1d)
	$\boldsymbol{\sigma} \cdot \mathbf{n} = \hat{\mathbf{t}}$	on Γ	(1e)

SBFEM derivation II

- Geometry transformation

$$x(\xi,\eta) = x_O + \xi x(\eta) = x_O + \xi [\mathbf{N}(n)]\{\mathbf{x}\} \quad (2a)$$

$$y(\xi,\eta) = y_O + \xi y(\eta) = y_O + \xi [\mathbf{N}(n)]\{\mathbf{y}\} \quad (2b)$$

- Jacobian

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \quad (3a)$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \quad (3b)$$

- Differential unit volumen

$$dV = |\mathbf{J}| \xi \, d\xi \, d\eta$$

SBFEM derivation III

- The linear differential operator L may thus be written as:

The derivatives may further be used to construct the linear operator $\mathbf{L}$ in scaled boundary coordinates. In a first step, it is split. Then the partial derivatives are substituted:

$$[\mathbf{L}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial y} = [\mathbf{L^1}] \frac{\partial}{\partial x} + [\mathbf{L^2}] \frac{\partial}{\partial y} \quad (5a)$$

$$= \frac{1}{|\mathbf{J}|} \left[[\mathbf{L^1}] \left(\frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) + [\mathbf{L^2}] \left(\frac{\partial x}{\partial \xi} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} \right) \right] \quad (5b)$$

$$= [\mathbf{b^1}(\eta)] \frac{\partial}{\partial \xi} + \frac{1}{\xi} [\mathbf{b^2}(\eta)] \frac{\partial}{\partial \eta} \quad (5c)$$

with $[\mathbf{b^1}(\eta)] = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & 0 \\ 0 & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$ $\quad [\mathbf{b^2}(\eta)] = \frac{1}{|\mathbf{J}|} \begin{bmatrix} -\frac{\partial y}{\partial \xi} & 0 \\ 0 & \frac{\partial x}{\partial \xi} \\ \frac{\partial x}{\partial \xi} & -\frac{\partial y}{\partial \xi} \end{bmatrix}$

SBFEM derivation IV

- Assuming an analytical solution in radial direction:

$$\{\mathbf{u}(\xi,\eta)\} = [\mathbf{N^u}(\eta)]\{\mathbf{u}(\xi)\}$$

$$[\mathbf{N^u}(\eta)] = [N^1(\eta)\mathbf{I_n}, N^2(\eta)\mathbf{I_n}, \ldots, N^2(\eta)\mathbf{I_n}]$$

- And therefore the strains and stresses become:

$$\{\boldsymbol{\varepsilon}(\xi,\eta)\} = [\mathbf{B^1}(\eta)]\{\mathbf{u}(\xi)\}_{,\xi} + \frac{1}{\xi}[\mathbf{B^2}(\eta)]\{\mathbf{u}(\xi)\} \qquad (9)$$

where

$$[\mathbf{B^1}(\eta)] = [\mathbf{b^1}(\eta)][\mathbf{N^u}(\eta)] \qquad (10a)$$

$$[\mathbf{B^2}(\eta)] = [\mathbf{b^2}(\eta)][\mathbf{N^u}(\eta)]_{,\eta} \qquad (10b)$$

And as a consequence the stresses follow as:

$$\{\boldsymbol{\sigma}(\xi,\eta)\} = [\mathbf{D}]\left([\mathbf{B^1}(\eta)]\{\mathbf{u}(\xi)\}_{,\xi} + \frac{1}{\xi}[\mathbf{B^2}(\eta)]\{\mathbf{u}(\xi)\}\right)$$

SBFEM derivation V

- Setting up the virtual work formulation:

Implementing the above in as a virtual work statement we consider $\{\delta\mathbf{u}(\xi,\eta)\}$ and $\{\delta\boldsymbol{\varepsilon}(\xi,\eta)\}$ as the virtual displacements and strains given as:

$$\{\delta\mathbf{u}(\xi,\eta)\} = [\mathbf{N}(\eta)]\{\delta\mathbf{u}(\xi)\} \tag{12a}$$

$$\{\delta\boldsymbol{\varepsilon}(\xi,\eta)\} = [\mathbf{L}]\{\delta\mathbf{u}(\xi,\eta)\} \tag{12b}$$

As such, the virtual work statement in scaled boundary coordinates becomes:

$$\int_V \{\delta\boldsymbol{\varepsilon}(\xi,\eta)\}^T\{\boldsymbol{\sigma}(\xi,\eta)\}dV - \int_{\partial\Omega}\{\delta\mathbf{u}(\eta)\}^T\{\mathbf{t}(\eta)\} = 0 \tag{13}$$

SBFEM derivation VI

Considering the term representing the internal virtual work first it can be rearranged

$$
\begin{aligned}
&\int_V \{\delta \boldsymbol{\varepsilon}(\xi,\eta)\}^T \{\boldsymbol{\sigma}(\xi,\eta)\} dV \\
&= \int_V \left[[\mathbf{B^1}(\eta)]\{\delta \mathbf{u}(\xi)\}_{,\xi} + \frac{1}{\xi}[\mathbf{B^2}(\eta)]\{\delta \mathbf{u}(\xi)\} \right]^T \\
&\times [\mathbf{D}] \left([\mathbf{B^1}(\eta)]\{\mathbf{u}(\xi)\}_{,\xi} + \frac{1}{\xi}[\mathbf{B^2}(\eta)]\{\mathbf{u}(\xi)\} \right) dV \\
&= \int_{\partial\Omega} \int_{\xi=0}^{\xi=1} \{\delta \mathbf{u}(\xi)\}_{,\xi}^T [\mathbf{B}^1(\eta)]^T [\mathbf{D}][\mathbf{B}^1(\eta)] \xi \{\mathbf{u}(\xi)\}_{,\xi} |\mathbf{J}| d\xi d\eta \\
&+ \int_{\partial\Omega} \int_{\xi=0}^{\xi=1} \{\delta \mathbf{u}(\xi)\}_{,\xi}^T [\mathbf{B}^1(\eta)]^T [\mathbf{D}][\mathbf{B}^2(\eta)] \{\mathbf{u}(\xi)\} |\mathbf{J}| d\xi d\eta \\
&+ \int_{\partial\Omega} \int_{\xi=0}^{\xi=1} \{\delta \mathbf{u}(\xi)\}^T [\mathbf{B}^2(\eta)]^T [\mathbf{D}][\mathbf{B}^1(\eta)] \{\mathbf{u}(\xi)\}_{,\xi} |\mathbf{J}| d\xi d\eta \\
&+ \int_{\partial\Omega} \int_{\xi=0}^{\xi=1} \{\delta \mathbf{u}(\xi)\}^T [\mathbf{B}^2(\eta)]^T [\mathbf{D}][\mathbf{B}^2(\eta)] \frac{1}{\xi} \{\mathbf{u}(\xi)\} |\mathbf{J}| d\xi d\eta
\end{aligned}
\tag{14}
$$

SBFEM derivation VII

By applying Green's theorem, i.e. integration by parts, the integrals containing $\{\delta\mathbf{u}(\xi)\}^T_{,\xi}$ are converted, which leads to the following formulation:

$$\begin{aligned}
&\int_V \{\delta\boldsymbol{\varepsilon}(\xi,\eta)\}^T\{\boldsymbol{\sigma}(\xi,\eta)\}dV \\
&= \int_{\partial\Omega} \{\delta\mathbf{u}(\xi)\}^T[\mathbf{B}^1(\eta)]^T[\mathbf{D}][\mathbf{B}^1(\eta)]\xi\{\mathbf{u}(\xi)\}_{,\xi}|\mathbf{J}|d\eta\Big|_{\xi=1} \\
&- \int_{\partial\Omega} \{\delta\mathbf{u}(\xi)\}^T[\mathbf{B}^1(\eta)]^T[\mathbf{D}][\mathbf{B}^1(\eta)] \\
&\times \{\{\mathbf{u}(\xi)\}_{\xi} + \{\mathbf{u}(\xi)\}_{\xi\xi}\}|\mathbf{J}|d\xi d\eta \\
&+ \int_{\partial\Omega} \{\delta\mathbf{u}(\xi)\}^T[\mathbf{B}^1(\eta)]^T[\mathbf{D}][\mathbf{B}^2(\eta)]\{\mathbf{u}(\xi)\}|\mathbf{J}|d\eta\Big|_{\xi=1} \\
&- \int_{\partial\Omega}\int_{\xi=0}^{\xi=1} \{\delta\mathbf{u}(\xi)\}^T[\mathbf{B}^1(\eta)]^T[\mathbf{D}][\mathbf{B}^2(\eta)]\{\mathbf{u}(\xi)\}_{,\xi}|\mathbf{J}|d\xi d\eta \\
&+ \int_{\partial\Omega}\int_{\xi=0}^{\xi=1} \{\delta\mathbf{u}(\xi)\}^T[\mathbf{B}^2(\eta)]^T[\mathbf{D}][\mathbf{B}^1(\eta)]\{\mathbf{u}(\xi)\}_{,\xi}|\mathbf{J}|d\xi d\eta \\
&+ \int_{\partial\Omega}\int_{\xi=0}^{\xi=1} \{\delta\mathbf{u}(\xi)\}^T[\mathbf{B}^2(\eta)]^T[\mathbf{D}][\mathbf{B}^2(\eta)]\frac{1}{\xi}\{\mathbf{u}(\xi)\}|\mathbf{J}|d\xi d\eta
\end{aligned}$$

SBFEM derivation VIII

- Introducing some substitutions

$$[E^0] = \int_{\partial\Omega} [\mathbf{B}^1(\eta)]^T [\mathbf{D}][\mathbf{B}^1(\eta)]|\mathbf{J}|d\eta \qquad (16a)$$

$$[E^1] = \int_{\partial\Omega} [\mathbf{B}^1(\eta)]^T [\mathbf{D}][\mathbf{B}^2(\eta)]|\mathbf{J}|d\eta \qquad (16b)$$

$$[E^2] = \int_{\partial\Omega} [\mathbf{B}^2(\eta)]^T [\mathbf{D}][\mathbf{B}^2(\eta)]|\mathbf{J}|d\eta \qquad (16c)$$

- Leads to some significant simplifications

$$\int_V \{\delta\boldsymbol{\varepsilon}(\xi,\eta)\}^T \{\boldsymbol{\sigma}(\xi,\eta)\} dV$$
$$= \{\delta\mathbf{u}\}^T \{[\mathbf{E^0}]\{\mathbf{u}\}_{,\xi} + [\mathbf{E^1}]^T \{\mathbf{u}\}\}$$
$$- \int_{xi=0}^{\xi=1} \{\delta\mathbf{u}(\xi)\}^T \Big\{ [\mathbf{E^0}]\xi \{\mathbf{u}(\xi)\}_{,\xi\xi}$$
$$+ [[\mathbf{E^0}] + [\mathbf{E^1}]^T - [\mathbf{E^1}]]\{\mathbf{u}(\xi)\}_{,\xi} - [\mathbf{E^2}]\frac{1}{\xi}\{\mathbf{u}(\xi)\}\Big\} d\xi \qquad (17)$$

SBFEM derivation IX

Having performed the necessary derivations for the internal virtual work, let us shift our focus to the external virtual work term:

$$\int_{\partial\Omega} \{\delta\mathbf{u}(\eta)\}^T \{\mathbf{t}(\eta)\} d\eta = \{\delta\mathbf{u}\}^T \int_{\partial\Omega} \{\mathbf{N}(\eta)\}^T \{\mathbf{t}(\eta)\} d\eta \tag{18}$$

Assuming that no other forces are acting on the domain other than the traction on the domain boundary, these may be identified as the equivalent nodal forces due to boundary tractions also termed $\{\mathbf{P}\}$. Equating the internal virtual work to the external virtual work statements we can formulate the complete virtual work equation:

$$\{\delta\mathbf{u}\}^T \{[\mathbf{E^0}]\{\mathbf{u}\}_{,\xi} + [\mathbf{E^1}]^T \{\mathbf{u}\} - \{\mathbf{P}\}\}$$
$$- \int_{xi=0}^{\xi=1} \{\delta\mathbf{u}(\xi)\}^T \Big\{ [\mathbf{E^0}]\xi \{\mathbf{u}(\xi)\}_{,\xi\xi}$$
$$+ [[\mathbf{E^0}] + [\mathbf{E^1}]^T - [\mathbf{E^1}]]\{\mathbf{u}(\xi)\}_{,\xi} - [\mathbf{E^2}]\frac{1}{\xi}\{\mathbf{u}(\xi)\} \Big\} d\xi = \{0\}$$

SBFEM derivation X

In order for this equation to hold for all ξ, which implies continuously satisfied in radial direction and only compliant in the finite element sense in the tangential direction, both of the following conditions must be met:

$$\{\mathbf{P}\} = [\mathbf{E^0}]\{\mathbf{u}\}_{,\xi} + [\mathbf{E^1}]^T\{\mathbf{u}\} \tag{20}$$

$$[\mathbf{E^0}]\xi^2\{\mathbf{u}(\xi)\}_{,\xi\xi} + [[\mathbf{E^0}] + [\mathbf{E^1}]^T - [\mathbf{E^1}]]\xi\{\mathbf{u}(\xi)\}_{,\xi} - [\mathbf{E^2}]\{\mathbf{u}(\xi)\} = \{\mathbf{0}\} \tag{21}$$

The above equation is termed the scaled boundary finite element equation in displacement.