RISK ANALYSIS OF COMPOSITE STRUCTURES BY SUBSET ESTIMATION USING THE HYSTERETIC MULTISCALE FINITE ELEMENT METHOD

S. P. Triantafyllou[1] and E. N. Chatzi[2]

[1]Department of Civil Engineering,School of Engineering and Design, Brunel University, UB8 3PH, Uxbridge, UK; email: savvas.triantafyllou@brunel.ac.uk
[2]Institute of Structural Engineering, School of Civil Engineering, ETH Zürich, Wolfgang-Pauli-Strasse 15, CH-8093 Zürich, Switzerland; email: chatzi@ibk.baug.ethz.ch

ABSTRACT

Composite materials are being implemented in numerous engineering applications including, though not limited to, the aerospace, auto-mobile, wind turbine industries and the retrofit and strengthening of existing buildings. Advancements in manufacturing processes enable the production of composites whose macroscopically observed properties are elaborately determined at the micro-scale. Composites are therefore inherently multiscale materials. Consequently, the reliability of structural systems being comprised of composites heavily depends on the micro-mechanical properties of the latter. In this work, a methodology is presented for the evaluation of failure probabilities of composite structures. The hysteretic multiscale finite element method (HMsFEM) is implemented for the modelling of composites while the subset simulation method is used to evaluate the corresponding probabilities of failure. In the HMsFE method, the nonlinear behaviour of the constituents is accurately modelled in the fine scale, while global solution of the structure is performed in a macro-scale thus significantly reducing the computational cost of the reliability analysis procedure.

INTRODUCTION

Composites are mixtures of two or more mechanically separable solid materials (Strong 2008). Composites are inherently multiscale materials, i.e. the scale of the constituents can be of lower order than the scale of the resulting material. The content, geometry, distribution and phase of the different constituents, that are can only be observed at the micro-scale, significantly affect the macroscopically measured behaviour of the composite (Mishnaevsky 2007). Furthermore, the resulting structure, that is an assemblage of composites, can be of an even larger scale than the scale of the constituents (e.g. a textile strengthened masonry structure (Fuggini et al. 2013), a bio-sensor consisting of several nano-wires (Park 2010)). Thus, the required modelling approach has to

account for such a level of detailing that spreads through scales of significantly different magnitude.

Instead of implementing the standard finite element method, upscaled or multiscale methods have been proposed to account for such types of problems, therefore significantly reducing the required computational resources (Kim et al. 2013). Amongst the various multiscale analysis procedures (Kanouté et al. 2009), the multiscale finite element method has been proven very efficient in terms of accuracy and computational complexity for the case of linear and nonlinear flow (Efendiev and Hou 2009; He and Ren 2005) as well as for the analysis of heterogeneous structures (Zhang et al. 2012). In the latter, a numerically derived mapping is interpolation scheme is implemented to map the micro-scale displacement field onto the macro-scale where the solution of the problem is actually performed. However, during a nonlinear analysis procedure the material properties of the micro-structure are updated due to damage progression, thus the material dependent numerical mapping needs also to be updated during the incremental solution procedure. In (Zhang et al. 2012) the initial stiffness approach is implemented for the solution of the incremental governing equations, thus avoiding the re-evaluation of the basis functions. Nevertheless, this method is known to face serious convergence problems and usually requires a large number of iterations to achieve convergence (Powell and Simons 1981).

In this work, a hysteretic multiscale finite element method (HMsFEM) is used for the nonlinear static and dynamic analysis of heterogeneous structures (Triantafyllou and Chatzi 2013). In this, the evaluation of the micro-scale basis functions is accomplished within the hysteretic finite element framework (Triantafyllou and Koumousis 2013). In the hysteretic finite element scheme, inelasticity is treated at the element level through properly defined evolution equations that control the evolution of the plastic part of the deformation component. Using the Principle of Virtual Work, the tangent stiffness matrix of the element is replaced by an elastic and a plastic state matrix both of which remain constant throughout the analysis.

The reliability analysis of structures consisting of composite materials can turn into an arduous task as it includes a number of Monte Carlo iterations over a computational expensive model (Kimiaeifar et al. 2013). Many methods have been proposed to address the problem of efficiently sampling the design space, thus avoiding exhaustive Monte Carlo iterations, such as Stratified Sampling, Latin-Hyper Cube sampling, Importance Sampling and Subset Simulation (Au and Beck 2001) amongst many. An illustrative review of the different methods can be found in (Mackay 1998). In this, work the Subset Sampling (or Subset Simulation) method is used that has been extensively implemented and tested in structural analysis problems and has proven to be very efficient both in terms of computational cost and robustness (Au and Beck 2003; Zio 2010). Surrogate models are also implemented that significantly reduce the computational cost by substituting the detailed computational model with a simplified but probabilistically equivalent model (Dimitrov et al. 2013). However, the accuracy of the method in problems involving when material non-linearities across multiple scales has not yet been examined. Rather than reverting to surrogate models, the HMsFE method is used

in conjunction with the subset simulation approach, yielding a cost effective reliability analysis procedure.

PROBABILITY OF FAILURE ESTIMATION

Problem Statement The performance of a structure is described by a cost function $f : \mathbb{X}^n \rightarrow \mathbb{R}^n$ where $\mathbb{X}^n$ denotes the n-dimensional space of structure related design variables and $\mathbb{R}^n$ is the n-dimensional real space. Given a design requirement $D \in \mathbb{R}^n$, a failure event is mathematically defined as the set

$$\mathbb{S}^n = \{\mathbf{x} \in \mathbb{X}^n : f(\mathbf{x}) > D\} \tag{1}$$

Based on equation (1), the probability of occurrence of the failure event is defined as

$$P_f = Prob(\mathbb{S}^n) = \int_{\mathbb{S}^n} p_f(\mathbf{x})\, d\mathbb{S}^n = \int_{\mathbb{X}^n} \mathbf{I}_{\mathbb{S}^n} p_f(\mathbf{x})\, d\mathbb{X}^n \tag{2}$$

where p_f is a probability density function, while $\mathbf{I}_{\mathbb{S}}$ is a Heaviside step function that is equal to unity when $\mathbf{x} \in \mathbb{S}^n$ and zero otherwise.

The evaluation of the integral defined in equation (2) is not trivial, especially in the case of complex and high-dimensional problems (e.g. problems that involve a large number of random variables). The most common computational procedure for the evaluation of p_f is the Monte-Carlo simulation method. This is based on the simulation of a large number of randomly derived individual samples $\mathbf{x}_i, i = 1 \ldots Nsamples$ so that the probability of failure can be statistically evaluated through the following relation:

$$P_f = \frac{\text{Number of performed samples that lead to failure}}{\text{Total number of samples}} \tag{3}$$

However, the accuracy of equation (3) greatly depends on distribution of the individual samples within the MC sample space. If the individual samples $\mathbf{x}_i$ lay all outside of the failure region (e.g. $f(\mathbf{x}_i) < D$), then the corresponding probability will be evaluated as to be exactly zero. As a result, a large number of individual samples needs to be processed, thus considerably increasing the required amount of computational time.

The subset sampling method The basic idea of Subset Sampling is the subdivision of the failure event into a sequence of N_sub partial failure events (subsets) $\mathbb{S}^n_m, m = 1 \ldots N_{sub}$ that adhere to the following relation

$$\mathbb{S}^n_1 \supset \mathbb{S}^n_2 \cdots \supset \mathbb{S}^n_{N_{sub}} \tag{4}$$

Two conclusions can be drawn from the sequential determination of the failure events defined in equation (4):

1. The events are sorted in a descending order of probability of occurrence.

2. Each next event can only occur on the condition of occurrence all the the preceded events.

Thus, the probability of occurrence of the rare event $\mathbb{S}^n_{N_{sub}}$ (e.g. the solution of the reliability problem) can be readily evaluated through the following relation

$$\begin{aligned} P_f &= P_{\mathbb{S}^n_{N_{sub}}} \\ &= P\left(\mathbb{S}^n_{N_{sub}} \middle| \mathbb{S}^n_{N_{sub}-1}\right) P\left(\mathbb{S}^n_{N_{sub}-1} \middle| \mathbb{S}^n_{N_{sub}-2}\right) ....P\left(\mathbb{S}^n_2 \middle| \mathbb{S}^n_1\right) P\left(\mathbb{S}^n_1\right) \\ &= P\left(\mathbb{S}^n_1\right) \prod_1^{N_{sub-1}} P\left(\mathbb{S}^n_{i+1} \middle| \mathbb{S}^n_i\right) \end{aligned} \tag{5}$$

Using the definition of the failure event presented in equation (1), each subset is defined accordingly as:

$$\mathbb{S}^n_m = \left\{\mathbf{x} \in \mathbb{X}^n : f\left(\mathbf{x}\right) > D_m\right\}, m = 1 \ldots N_{sub} \tag{6}$$

where D_m is the limit value corresponding to $m_t h$ probability of failure. Relation (6) is more conveniently defined as

$$\mathbb{S}^n_m = \left\{\mathbf{x} \in \mathbb{X}^n : g\left(\mathbf{x}\right) < g_m\right\}, m = 1 \ldots N_{sub} \tag{7}$$

where $g\left(\mathbf{x}\right) = D_m - f\left(\mathbf{x}\right)$. The formulation introduced in equation (7) bares some implementation advantages, as the least rare event $\mathbb{S}^n_1$ can be associated with an extremely large value of the limit value g_1, while the limit value $g_m = 0$ accounts for the failure probability that the method searches for.

In the formulation introduced in (Au and Beck 2001), the limit values g_m are adaptively determined during the analysis procedure in such a way so that the conditional probabilities introduced in equation (5) are equal to a predefined value. This is achieved through the implementation of a Markov chain Monte Carlo scheme in conjunction with a modified version of the Metropolis-Hastings algorithm (Robert and Casella 2010).

THE HYSTERETIC MULTISCALE FINITE ELEMENT METHOD

Equilibrium in the fine scale Based on the hysteretic finite element method (Triantafyllou and Koumousis 2013), equilibrium at the fine scale is established on the grounds of the following relation

$$\left[k^{el}\right]_{m(i)} \left\{d\right\}_{m(i)} - \left[k^h\right]_{m(i)} \left\{\varepsilon^{pl}_{cq}\right\}_{(i)} = \left\{f\right\}_{m(i)} \tag{8}$$

where $\left[k^{el}\right]_{m(i)}$ is the elastic stiffness matrix, $\left\{d\right\}_{m(i)}$ is the nodal displacement vector, $\left[k^h\right]_{m(i)}$ is the plastic state matrix, $\left\{\varepsilon^{pl}_{cq}\right\}_{(i)}$ is the set of plastic strains measured at predefined collocation points, while index $m\left(i\right)$ denotes the corresponding measure of the i_{th} micro-element. The plastic deformation components are considered to evolve

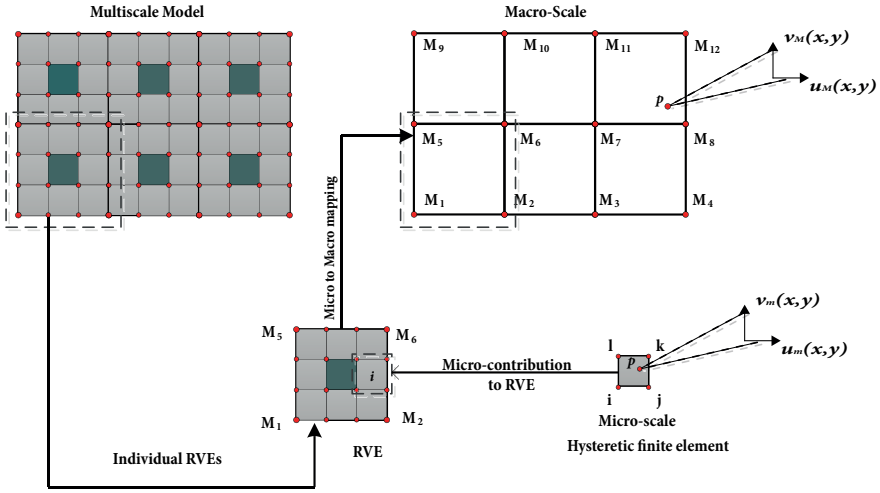


Figure 1. The hysteretic multiscale finite element method

according to a pre-defined set of rate equations of the following form

$$\left\{\dot{\varepsilon}^{pl}\right\}_{m(i)} = \mathcal{F}\left(\left\{\varepsilon^{el}\right\}_{m(i)}, \left\{\dot{\varepsilon}^{el}\right\}_{m(i)}, \{\sigma\}_{m(i)}\right). \tag{9}$$

According to MsFEM the following interpolation scheme is introduced:

$$\{d\}_{m(i)} = [N]_{m(i)} \{d\}_M \tag{10}$$

where $[N]_{m(i)}$ is the micro-to-macro numerical interpolation functions and $\{d\}_M$ is the vector of nodal macro-displacements.

Pre-multiplying equation (8) with $[N]^T_{m(i)}$ the following relation is derived:

$$\left[k^{el}\right]^M_{m(i)} \{d\}_M - \left[k^h\right]^M_{m(i)} \left\{\varepsilon^{pl}_{cq}\right\}_{(i)} = \{f\}^M_{m(i)} \tag{11}$$

where

$$\left[k^{el}\right]^M_{m(i)} = [N]^T_{m(i)} \left[k^{el}\right]_{m(i)} [N]_{m(i)} \tag{12}$$

is the elastic stiffness matrix of the i_{th} micro-element mapped onto the macro-element degrees of freedom while $\left[k^h\right]^M_{m(i)}$ is corresponding the hysteretic matrix of the i_{th} micro-element, evaluated by the following relation:

$$\left[k^h\right]^M_{m(i)} = [N]^T_{m(i)} \left[k^h\right]_{m(i)} \tag{13}$$

Finally, $\{P\}^M_{m(i)}$ in equation (11) is the equivalent nodal force vector of the micro-element mapped onto the macro-nodes of the coarse element.

$$\{f\}^M_{m(i)} = [N]^T_{m(i)} \{f\}_{m(i)} \tag{14}$$

Equation (11) is a multiscale equilibrium equation involving the displacement vector evaluated at the coarse-element nodes and the plastic part of the strain tensor evaluated at collocation points within the micro-scale element mesh. The derived multiscale elastic stiffness and hysteretic matrices are constant and need only be evaluated once during the analysis procedure.

Micro to Macro scale transition Having established the micro-element equilibrium in equation (11) in terms of macro-displacements a procedure is required to formulate the global equilibrium equations in terms of the macro-quantities. The potential energy at the coarse element level is evaluated using the following relation:

$$\int_{V_M} \{\varepsilon\}_M^T \{\dot{\sigma}\}_M \, dV_M = \left[K^{el}\right]_{CR(j)}^M \left\{\dot{d}\right\}_M - \left[K^h\right]_{CR(j)}^M \left\{\dot{\varepsilon}_{cq}^{pl}\right\}_M \tag{15}$$

where $\left[K^{el}\right]_{CR(j)}^M$, $\left[K^h\right]_{CR(j)}^M$ are the equivalent elastic stiffness and hysteretic matrix of the j_{th} coarse element respectively while $\left\{\dot{\varepsilon}_{cq}^{pl}\right\}_M$ is the vector of plastic strains defined at the collocation points. Within the multiscale finite element framework, these quantities are not known a priori and need to be expressed in terms of micro-scale measures, thus accounting for the micro-scale effect upon the macro-scale mesh. This is accomplished by demanding that the strain energy of the coarse element is additively decomposed into the contributions of each micro-element within the coarse-element. Thus, the following relation is established:

$$\int_V \{\varepsilon\}_M^T \{\sigma\}_M \, dV = \sum_{i=1}^{m_{el}} \int_{V_{mi}} \{\varepsilon\}_{mi}^T \{\sigma\}_{mi} \, dV_i \tag{16}$$

where $\{\varepsilon\}_{mi}$, $\{\sigma\}_{mi}$ are the micro-strain and micro-stress field defined over the volume V_{mi} of the i_{th} micro-element. Implementing the hysteretic finite element method the r.h.s of equation (16) is cast onto the following form

$$\begin{aligned} \sum_{i=1}^{m_{el}} \int_{V_{mi}} \{\varepsilon\}_{mi}^T \{\sigma\}_{mi} \, dV_i = \sum_{i=1}^{m_{el}} \Big(& \{d\}_{mi}^T \left[k^{el}\right]_{m(i)} \{d\}_{m(i)} \\ & - \{d\}_{mi}^T \left[k^h\right]_{m(i)} \left\{\varepsilon_{cq}^{pl}\right\}_{m(i)} \Big) \end{aligned} \tag{17}$$

Substituting relation (10) into relation (17) the following expression is derived

$$\begin{aligned} & \sum_{i=1}^{m_{el}} \int_{V_{mi}} \{\varepsilon\}_{mi}^T \{\sigma\}_{mi} \, dV_i = \{d\}_M^T \cdot \\ & \sum_{i=1}^{m_{el}} \left([N]_{Mi}^T \left[k^{el}\right]_{m(i)} [N]_{Mi} \{d\}_M - [N]_{Mi}^T \left[k^h\right]_{m(i)} \left\{\varepsilon_{cq}^{pl}\right\}_{m(i)} \right) \end{aligned} \tag{18}$$

Substituting equations (15) and (18) into equation (16), the following expression is derived:

$$\begin{aligned}&\left[K^{el}\right]^{M}_{CR(j)}\left\{d\right\}_{M}-\left[K^{h}\right]_{CR}\left\{\varepsilon^{pl}\right\}_{cq}=\\&\sum_{i=1}^{m_{el}}\left[k^{el}\right]^{M}_{m(i)}\left\{d\right\}_{M}-\sum_{i=1}^{m_{el}}\left[k^{h}\right]^{M}_{m(i)}\left\{\varepsilon^{pl}_{cq}\right\}_{m(i)}\end{aligned}\tag{19}$$

Relation (19) holds for every compatible vector of nodal displacements $\{d\}_M$ as long as:

$$\left[K^{el}\right]^{M}_{CR(j)}=\sum_{i=1}^{m_{el}}\left[k^{el}\right]^{M}_{m(i)}\tag{20}$$

and

$$\left[K^{h}\right]^{M}_{CR(j)}\left\{\varepsilon^{pl}_{cq}\right\}_{M}=\sum_{i=1}^{m_{el}}\left[k^{h}\right]^{M}_{m(i)}\left\{\varepsilon^{pl}_{cq}\right\}_{m(i)}\tag{21}$$

thus, the following multiscale equilibrium equation is derived for the coarse element:

$$\left[K^{el}\right]^{M}_{CR(j)}\left\{d\right\}_{M}=\left\{f\right\}_{M}-\left\{f_{h}\right\}_{M}\tag{22}$$

where $\{f_h\}_M$ is a nonlinear correction to the external force vector arising from the evolution of the plastic strains within the micro-structure

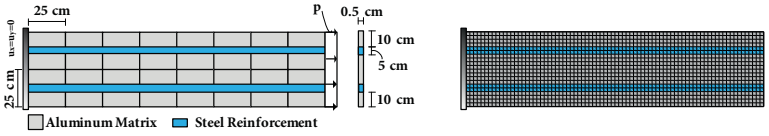
$$\left\{f_{h}\right\}_{M}=\sum_{i=1}^{m_{el}}\left[k^{h}\right]^{M}_{m(i)}\left\{\varepsilon^{pl}_{cq}\right\}_{m(i)}\tag{23}$$

while the plastic strain vectors $\left\{\varepsilon^{pl}_{cq}\right\}_{m(i)}$ are considered to evolve according to relation (9). Equations (22) and (23) are used to derive the equilibrium equation at the structural level as will be described in the next Section.

The coarse element stiffness matrices are independent and therefore their evaluation can be performed in parallel. Furthermore the coarse element stiffness matrices $\left[K^{el}\right]^{M}_{CR(j)}$ can be assembled at the structural level using the direct stiffness approach. Likewise, the coarse element hysteretic load matrices $\{f_h\}_M$ can be also assembled at the structural level, accompanied by a set of hysteretic equations (10). The solution of the coarse structure is then performed using the hysteretic finite element solution procedure introduced in (Triantafyllou and Koumousis 2013).

APPLICATION

The FERUM reliability toolbox (**?**) was used for the purpose of implementing and verifying the methodology presented in this work. The toolbox was properly modified so as to cooperate with the developed hysteretic multiscale analysis software.



(a) (b)

Figure 2. (a) Model Definition (b) Finite Element mesh

Steel reinforced aluminum panel In this example, an aluminum sheet is considered, reinforced with two steel strips (Fig. 2(a)). The length, width and height of the beam are $L_m = 200cm$, $b_m = 0.5cm$ and $h_m = 50cm$ respectively. The height of the steel strips is $h_f = 5cm$. The constituents are assumed to be elastic-perfectly plastic with deterministic Poisson ratios $\nu_a = 0.33$ and $\nu_s = 0.3$ for the aluminum and steel respectively. The elastic moduli and the corresponding yield stresses of the materials are considered to be random variables. A uniform distribution is considered all random variables with corresponding mean values $E_{ma} = 70GPa$ and $f_{ya} = 214MPa$ for the aluminum and $E_{ms} = 200GPa$ and $f_{ys} = 235MPa$ respectively. The fine meshed finite element model, presented in 2(b), consists of 1600 linear quadrilateral plane stress elements with a total of 3358 free degrees of freedom. The multiscale finite element model is formulated by 16 plane stress coarse elements. The corresponding representative RVE consists of 100 plane stress elements.

Three analysis cases are considered. In the first, a crude MC method is implemented with the multiscale model (MCMs). The second involves the subset simulation MC with the multiscale model (ssMCMs) while in the third a finite element model is used in conjuction with the subset simulation method (ssMCFEM) for verification. First a varying amplitude sinusoidal deterministic pressure load is considered at the free end $p(t) = 20000sin(\pi t)kPa$. Failure is defined as the maximum axial displacement at the cantilever tip exceeding the value of 100 mm (e.g. where the maximum aluminum normal deformation is $\varepsilon_{xx} > 5\%$ during the first ten seconds of the response. The average acceleration Newmark scheme is implemented for the solution of the equations of motion with a constant time step $T_step = 0.01sec$. For the crude MC approach, 10000 model realizations are considered. The initial population of the subset simulation method is set equal to $N_sub = 500$ while the target probability of failure of the intermediate failure events is set to $P(|) = 0.1$.

The derived propability of failure for each analysis case is presented in Table 1.

Next, the case of random loading is also considered by multiplying the load amplitude with a white noise factor of spectral intensity $S = 1$. The corresponding results are presented in Table 2.

Table 1. Probability of failure and function evaluations

Method	P_f	Function Evaluations	Time (sec)
MCMs	0.0002	10000	219192.48
ssMCMs	0.000187	1700	42760.05
ssMCFEM	0.000184	1700	152343.34

Table 2. Probability of failure and function evaluations - Random Loading

Method	P_f	Function Evaluations	Time (sec)
MCMs	0.00015	10000	219194.48
ssMCMs	0.000152	1700	42762.05
ssMCFEM	0.000156	1700	152341.34

CONCLUSION

In this work, a method is presented for the reliability analysis of composite structures. The method makes use of the hysteretic multiscale finite element formulation (HMsFEM) for structural modeling in conjunction with the subset simulation method for the estimation of the probability failure. Compared to a fine meshed finite element model, HMsFEM provides a faster solution since the order of the underlying governing equations is significantly reduced. Likewise, the subset simulation method effectively evaluates small failure probabilities by significantly reducing the number of required numerical experiments. An example is presented, where both the computational efficiency and the accuracy of the proposed scheme are examined as opposed to the classical reliability analysis procedure.

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