Mechanics of Building Materials

Elasticity Theory

F. Wittel

LEVI-CIVITÀ-tensor

KRONECKER-symbol

Strain energy

Principal stress

invariants

NABLA-operator

Monoclinic material

MOHR's circle

Octahedral plane

CAUCHY's equation

Deviatoric stress

BOLTZMANN axiom

HIGH-WESTERGAARD coordinates

Transversal-isotropic

Compatibility conditions

DRUCKER-PRAGER criterion

Bulk modulus

CAUCHY-elasticity

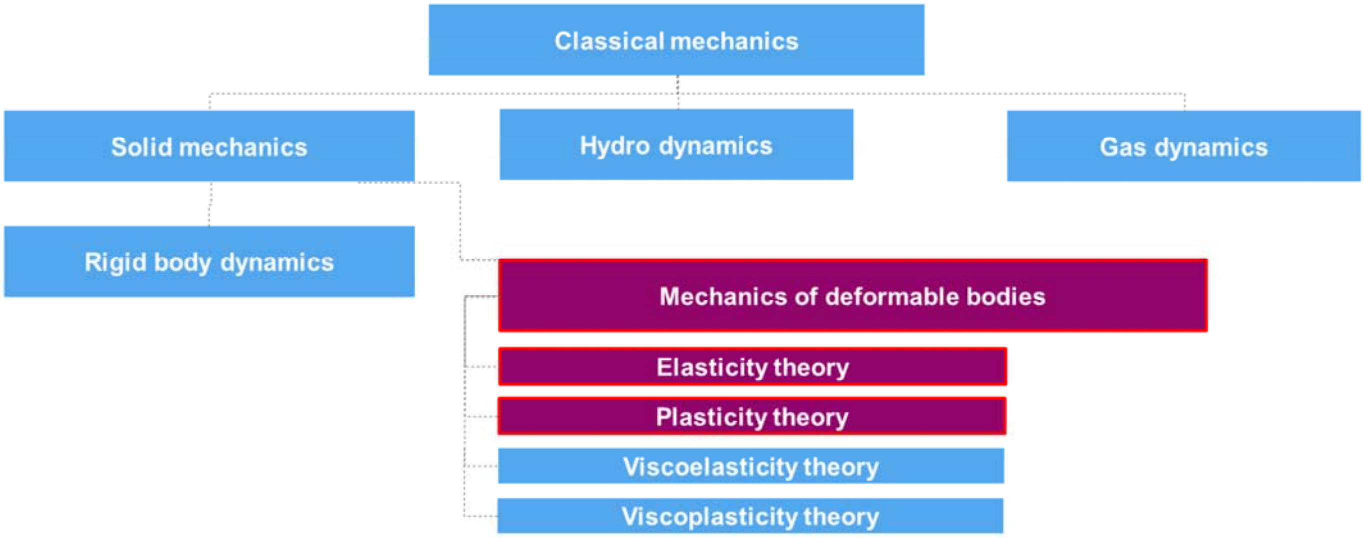
anisotropy

Distortion energy

LAMÉ's constants

E-book for download at library

Fields of classical mechanics

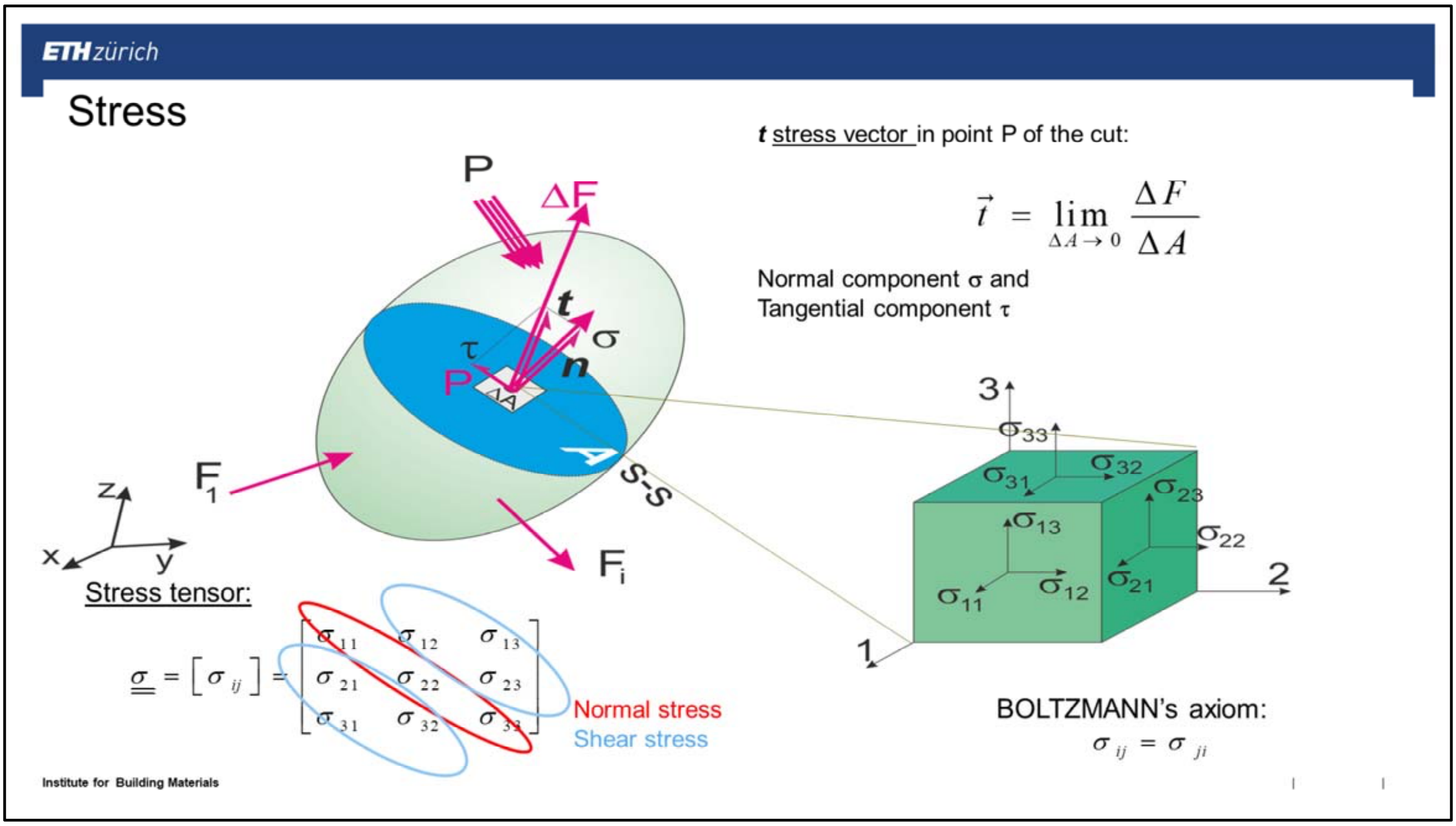


Elasticity theory (ET)

- Part of continuum mechanics
- Theory on stresses and deformations in elastic bodies
- Linear ET → linear elastic materials + infinitesimal small distortions

Red line:

Stresses (stress vector / Tensor, transformations, principal stresses, invariants, hydrostatic and deviatoric stress, equilibrium conditions)
Deformations and strains (Position and displacement vector, strain tensor, linear theory, compatibility conditions)
Material behavior (uni-axial), generalized HOOK's law, material symmetries



Let's start with a body, that is arbitrarily loaded by a singular force F and distributed load p.

- External loads cause internal forces.

2.

- Now we slice the body in s-s. The internal forces are distributed across the entire cutting area and are variable. On the area element dA, that contains point P, the internal force is DF, on the opposing face –DF. *Method of sections (Schnittprinzip): If a body is in equilibrium under external load, the cut body must as well be in equilibrium under external load + internal forces acting on the cutting are.* Following the principle of reactions, the opposed cut has to have inverted forces and moments of same magnitude on the same line of action.
- The ratio DF/DA gives the mean stress on the area element and with DA→0 we obtain the *stress vector* ***t.***
- The stress vector has a normal and tangential component with respect to the cutting plane. The normal component is called normal stress sigma and the tangential one shear stress tau.
- Hence there is no way to transmit a local moment what is the characteristic of a classical Boltzmann continuum.

3.

- The stress vector depends on the position of P and the orientation of the normal vector of the cutting plane through P. To fully characterize a stress state, one needs 3 perpendicular cuts through P, e.g. in direction 1,2,3.

- From the 3 cutting areas we obtain the stress tensor, that fully describes a stress state.

4.

- The indices show, weather we have a shear or normal stress. Identical index → normal stress, hence principal diagonal of the stress tensor, different index → shear stress.
- The first index gives the orientation of the area normal, while the 2nd one describes the orientation of the stress component.
- From the moment equilibrium follows the Boltzmann axiom, since shear stresses in two perpendicular cuts that intersect have to be identical.
- Also positive stresses point at positive cuts in the positive coordinate direction.
- The stress tensor is symmetric, hence sigmaT=sigma.

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Stress

Stress tensor known, stress vector with respect to cut with normal ***n*** wanted.

Projection of dA with normal vector:

$$dA_1 = dA\,n_1; \quad dA_2 = dA\,n_2; \quad dA_3 = dA\,n_3$$

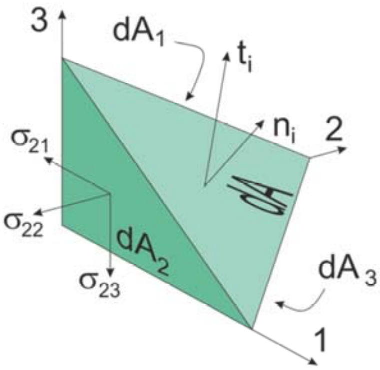
$$\Rightarrow \; dA_i = dA\,n_i$$

Equilibrium:

$$t_1 dA = \sigma_{11} dA_1 + \sigma_{21} dA_2 + \sigma_{31} dA_3 \qquad t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

$$t_2 dA = \sigma_{12} dA_1 + \sigma_{22} dA_2 + \sigma_{32} dA_3 \qquad t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

$$t_3 dA = \sigma_{13} dA_1 + \sigma_{23} dA_2 + \sigma_{33} dA_3 \qquad t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$



Cauchy's equation:

$$t_i = \sum_{j=1}^{3} \sigma_{ji} n_j = \underline{\underline{\sigma}}^T \bullet \vec{n}$$

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Now let’s make an arbitrary cut with the normal of the cutting plane n. By projection of dA with n one obtains…

The equilibrium conditions lead to the CAUCHY’s equation, that is used to project a normal vector by use of the stress tensor onto a stress vector. Hence the stress state is really completely defined by sigma_ij.

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Baron Augustin Louis Cauchy
Was born in 1789 in Paris and died in 1857 in Sceaux close to Paris. After his death he was honored by including his name in the row of the 72 on the Eiffel tower. At very young age he became engineer in Napolen's army. With 26 he became professor at the ecole polytechnique, where he quickly evolved to become the leading French mathematician of his time. There was the rumor, that his colleagues gave him the name «cochon», since he liked to take his ideas from publications he refereed. Instead of this, one must for sure acknowledge that the majority of his 789 publications must have been his idea. After the death of Euler, many people had the impression that in math there were no more significant problems to be solved. It was Gauss and Cauchy, that could oppose this impression. Cauchy was catholic and from the dynasty of the Bourbones. This brought him during the French revolution constantly in conflict with his fellow citizen and gave him an exciting academic life.

The young Cauchy studied at the Ecole Polytechnique road and bridge construction. At that time the classes were mathematically overloaded by his teachers called Lacroix, de Prony, Hachette, Ampère. After two years he was primus and allowed to move on to the more prestigious École Nationale des Ponts et Chaussées. In February 1810 he received the contract for the construction of the harbor in Cherbourg, at that time Europe's largest construction site with more than 3000 worker. It was the preparation of the invasion of England. Working hours were hard and in his little free time he continued with math. In 1813 he did get a position at the École Polytechnique and he developed to be a good teacher. He thought analysis to be the fundament of mechanics and all other important engineering disciplines. This was not liked by students, who found his lectures too abstract or let's say not engineering oriented enough. One time he was even booed by them. Anyway he was always a respected mathematician and in his late years remained influential by evaluating and refereeing many works.

Exercise stress vector:

In a material point the stress tensor σ with respect to a Cartesian coordinate system x_i is:

$$\sigma_{ij} = \begin{bmatrix} 54 & -27 & 9 \\ -27 & 18 & 0 \\ 9 & 0 & -36 \end{bmatrix}$$

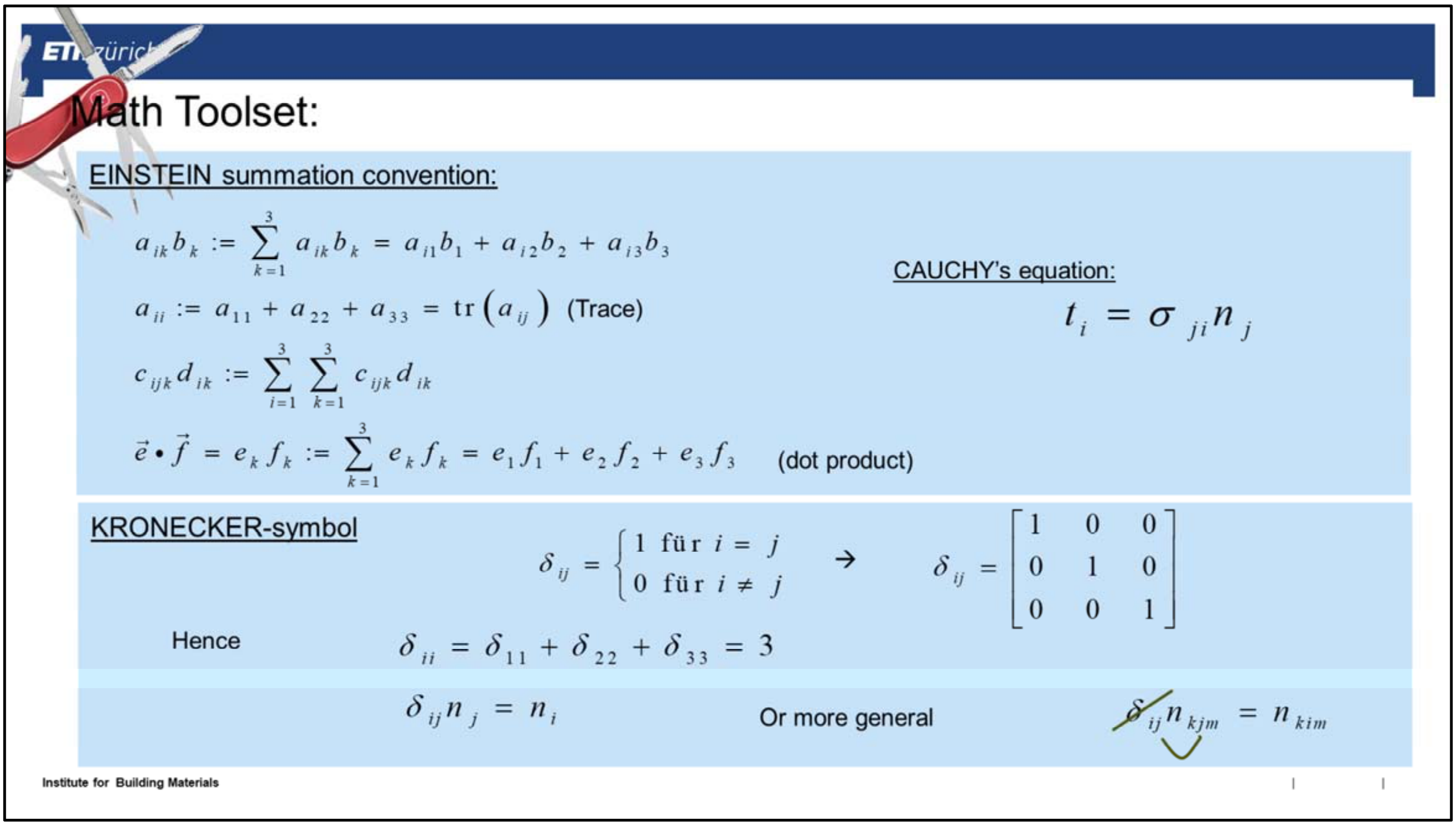
a. Calculate the three components of a stress vector $\boldsymbol{t}_n$ on a cutting plane with normal vector $\boldsymbol{n}=1/3[2,-2,1]$.

b. What magnitude does the stress vector have?

c. How large is the normal stress component σ perpendicular to the plane and ist shear counterpart τ in the cutting plane?

d. What is the angle between ***t*** and ***n***?

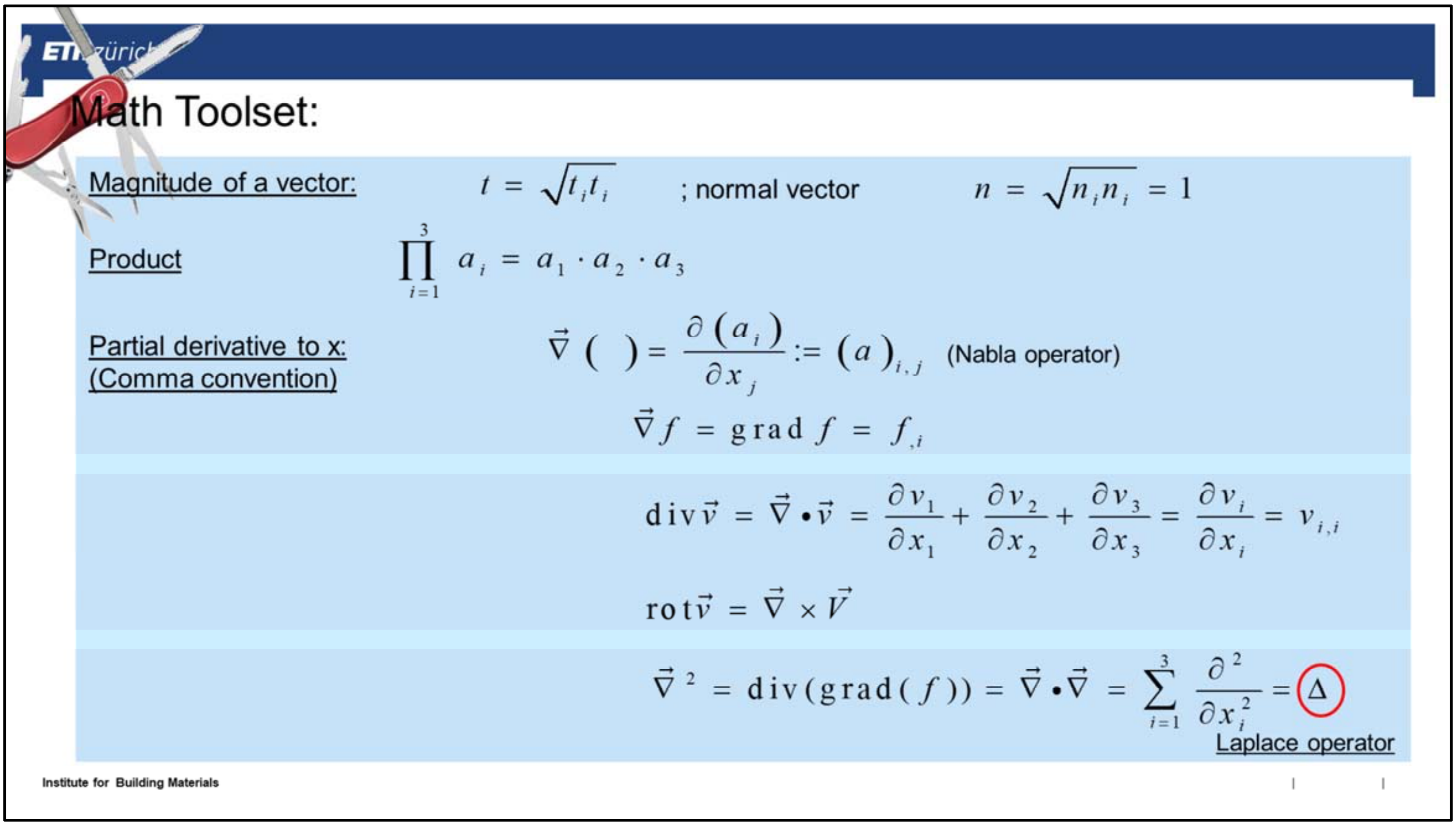
Lösung s. Beiblatt



There is a very handy method of writing very compact – the index notation. However one needs to summarize some math tools for the index notation like the Einstein summation convention or the Kronecker symbol (delta). It needs some practice to get into, but it is worth to use it.

The simplified way of writing sums is the Einstein summation convention. Hence we agree on summing, if in a term the identical index appears (runs form 1-3). This summation index can then be replaced by any other index, as we can see in the 1st equation. Of course our agreement is also valid for double sums, and when we look at the last equation we realize that this is nothing but the dot product. With the summation convention, we can also write the Cauchy's equation in a very compact form.

In this context also the Kronecker delta is used. It is the identity matrix I in index notation. Using the Kronecker-delta often strongly simplifies equations.



The index notation is a convincing compact way of writing, as we can demonstrate for various tensor operations

The nabla operator is a operation symbol used in vector analysis for the three differential operations: gradient, divergence, rotation. Formally the nabla operator is a vector, whose components are the partial derivative operators:

- The (formal) product of the nabla operator with the function f gives its gradient.
- The (formal) dot product with the vector field gives its divergence. If v is the velocity field of a fluid, div(v) can be seen as local source density of the field. (Div(v)>0 are sources, =0 is source free and <0 are sinks)
- The rotation of a vector field is given by the cross product with the vector field (gives vortex density of v in x)
- The dot product of the nabla operator with itself is called Laplace operator.

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Math Toolset:

LEVI-CIVITÀ-Tensor (permutation tensor) ε_{ijk}

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ for } i, j, k \text{ cyclic hence } 123, 231, 312 \\ -1 \text{ for } i, j, k \text{ anti cyclic hence } 321, 213, 132 \\ 0 \text{ otherwise} \end{cases}$$

Application:

Cross product:

$$\vec{M} = \vec{r} \times \vec{F} = \begin{bmatrix} r_2 F_3 - r_3 F_2 \\ r_3 F_1 - r_1 F_3 \\ r_1 F_2 - r_2 F_1 \end{bmatrix} = \varepsilon_{ijk} r_j F_k$$

Rotation of a vector:

$$\vec{\omega} = \frac{1}{2} \operatorname{rot} \vec{u} = \frac{1}{2} \vec{\nabla} \times \vec{u} = \frac{1}{2} \begin{bmatrix} u_{3,2} - u_{2,3} \\ u_{1,3} - u_{3,1} \\ u_{2,1} - u_{1,2} \end{bmatrix} = \frac{1}{2} \varepsilon_{ijk} u_{k,j}$$

Triple product:

$$\left[\vec{a}, \vec{b}, \vec{c} \right] := (\vec{a} \times \vec{b}) \bullet \vec{c} = \det \begin{bmatrix} \vec{a}^T \\ \vec{b}^T \\ \vec{c}^T \end{bmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j c_k$$

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The Levi-Civita tensor is very useful as well. It is a permutation tensor (3rd. Order) with 3*3*3=27 components, from whom only 6 are interesting. It’s application is manifold:

- Cross product of two vectors (a x b perpendicular to the plane defined by a and b, a, b lead to a right-handed trihedron, its magnitude ¦a x b¦=absinalpha is the area of the parallelogram, a x b=-b x a; a x b=0 → linear dependent vectors; lambda a x b= lambda (a x b); a x (b + c)=a x b +a x c)
- Rotation of a vector (gives vortex density of v in x)
- Triple dot product (Spatprodukt) is the oriented volume (magnitude is the volume of the parallelepiped = 6x of the tetrahedron, cyclic permutation does not change anything, >0 right handed, =0 Linear dependent vectors, <0 left handed)

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Math Toolset:

LEVI-CIVITÀ-tensor: ε_{ijk}

Application:

Symmetry of the stress tensor:

$$\varepsilon_{ijk}\sigma_{jk} = 0 \Rightarrow \sigma_{23} = \sigma_{32};\sigma_{12} = \sigma_{21};\sigma_{13} = \sigma_{31};$$

Determinant:

$$\det \underline{\underline{b}} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \varepsilon_{ijk} b_{1i} b_{2j} b_{3k} = \varepsilon_{ijk} b_{i1} b_{j2} b_{k3}$$

Calculation rule:

$$\varepsilon_{ijk}\delta_{ij} = \varepsilon_{iik} = 0$$

$$\varepsilon_{ikm}\varepsilon_{iln} = \delta_{kl}\delta_{mn} - \delta_{kn}\delta_{ml}$$

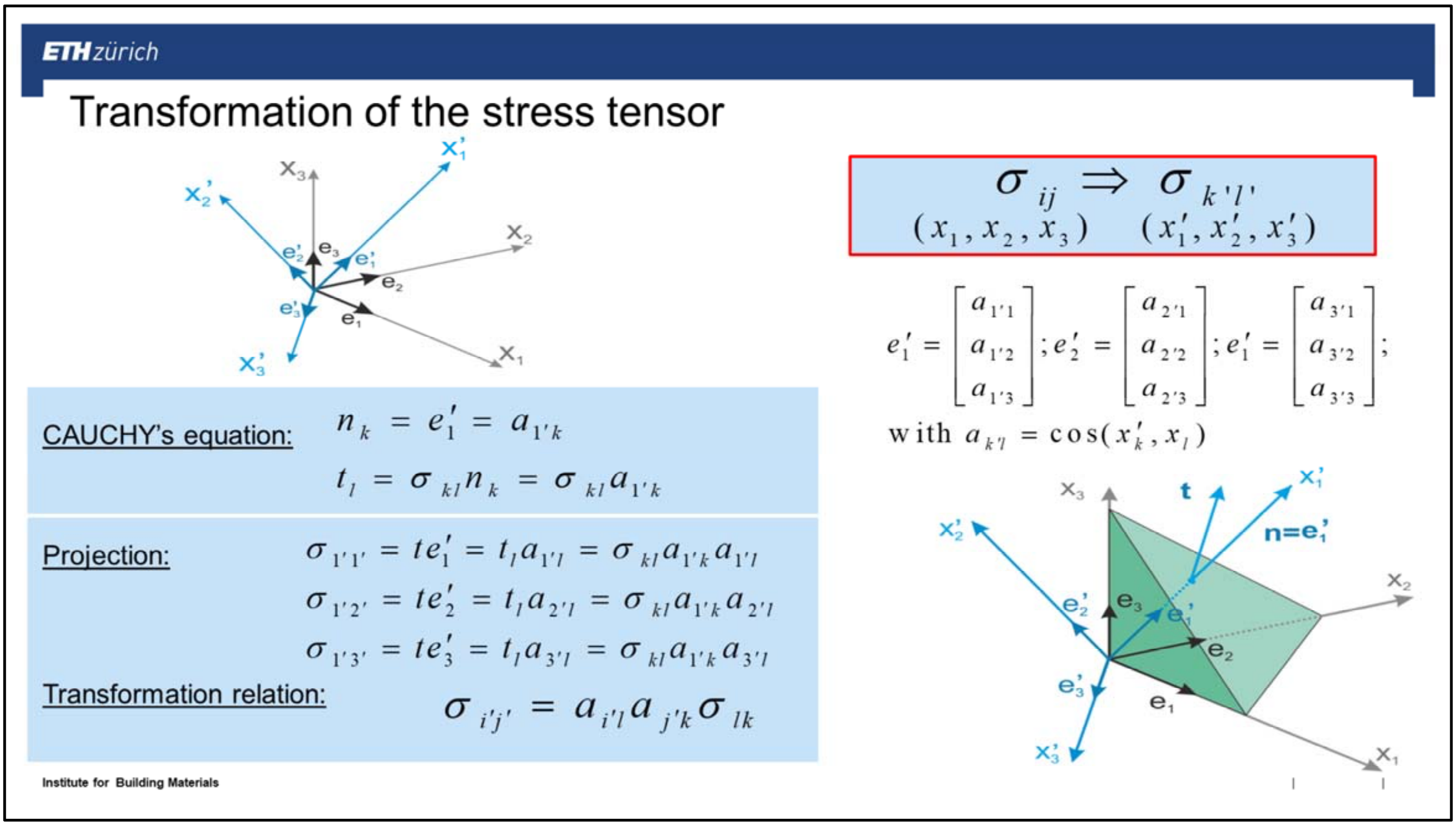
$$\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6$$

Further applications of the LEVI-CEVITA tensor are:

- Testing the symmetry of a tensor.
- Short form for calculating determinants developed after the 1st , resp. 2nd column.

Finally we look at some calculation rules of the tensors.



The change of coordinate systems and reference systems is a very common part of mechanics with high error potential. One starts off with a stress tensor sigma_ij that is formulated with respect to a coordinate system x_i . We are looking for the stress state sigma_k'l' in the rotated orthonormal coordinate system x'_i with the transformation coefficient a_k'l, also called directional cosine.

Let's look at the intersection plane whose normal vector is identical to the new coordinate x'_1. If we apply CAUCHY's equation, we obtain the components of the stress vector with respect to the 1-2-3 system. However to get the stress vectors in the 1'-2'-3' system, we have to project them onto the coordinate axes using a dot-product. The procedure now has to be repeated for the intersection planes with the normal vectors identical to x2' and x3' to obtain all 9 components of the Tensor. One can write it in the general transformation relation.

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Transformation of the stress tensor

Transformation-relation:

$$\sigma_{i'j'} = a_{i'l}\,a_{j'k}\,\sigma_{lk}$$

$$\begin{aligned}\sigma_{1'1'} &= a_{1'1}a_{1'1}\sigma_{11} + a_{1'1}a_{1'2}\sigma_{12} + a_{1'1}a_{1'3}\sigma_{13}\\ &+ a_{1'2}a_{1'1}\sigma_{21} + a_{1'2}a_{1'2}\sigma_{22} + a_{1'2}a_{1'3}\sigma_{23}\\ &+ a_{1'3}a_{1'1}\sigma_{31} + a_{1'3}a_{1'2}\sigma_{32} + a_{1'3}a_{1'3}\sigma_{13}\\ \sigma_{1'2'} &= a_{1'1}a_{2'1}\sigma_{11} + a_{1'1}a_{2'2}\sigma_{12} + a_{1'1}a_{2'3}\sigma_{13}\\ &+ a_{1'2}a_{2'1}\sigma_{21} + a_{1'2}a_{2'2}\sigma_{22} + a_{1'2}a_{2'3}\sigma_{23}\\ &+ a_{1'3}a_{2'1}\sigma_{31} + a_{1'3}a_{2'2}\sigma_{32} + a_{1'3}a_{2'3}\sigma_{13}\end{aligned}$$

Aso.

Transformation matrix:

$$\underline{\underline{a}} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix}$$

$$\underline{\underline{a}} \bullet \underline{\underline{a}}^T = \underline{\underline{I}} = \underline{\underline{a}} \bullet \underline{\underline{a}}^{-1}; \quad \det(\underline{\underline{a}}) = \pm 1$$

Transformation:

$$\underline{\underline{\sigma}}' = \underline{\underline{a}} \bullet \underline{\underline{\sigma}} \bullet \underline{\underline{a}}^T$$

Back transformation

$$\underline{\underline{a}}^T \bullet \underline{\underline{\sigma}}' \bullet \underline{\underline{a}} = \underline{\underline{a}}^T \bullet \underline{\underline{a}} \bullet \underline{\underline{\sigma}} \bullet \underline{\underline{a}}^T \bullet \underline{\underline{a}}$$

$$\underline{\underline{\sigma}} = \underline{\underline{a}}^T \bullet \underline{\underline{\sigma}}' \bullet \underline{\underline{a}}$$

$$\sigma_{ij} = a_{l'i}\,a_{k'j}\,\sigma_{l'k'}$$

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Since on the right side, k and l have to appear twice, one has to sum over both indices:
The transformation rule contains 9 equations with 9 summands (81 coefficients). As you can see, the index notation is very handy for such situations. One can also use the matrix notation with the transformation matrix a and get to the resulting back transformation.

Transformation of the stress tensor

A transformed coordinate system ($x_{i'}$) forms with the initial coordinate system (x_i) the angles $(x_{1'},x_1)=45°$; $(x_{1'},x_2)=60°$;$(x_{2'},x_2)=60°$, with the relation for the angle with the x_3 axis:
$0° \leq (x_{1'},x_3) \leq 90°$; $0° \leq (x_{2'},x_3) \leq 90°$;

a. Calculate the transformation matrix.

$$\sigma_{ij} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

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Tensor transformations: Principal stress

Principal axis system: Cutting planes oriented in principal directions have not shear stress components. The respective normal stresses are called **principal stress** and coincide with the normal of the cut.

$$t_i = \sigma n_i^* \qquad \rightarrow \sigma \text{ proportionality factor}$$

$$t_i = \sigma_{ij} n_i^* \quad \text{with } n_i = \delta_{ij} n_j \Rightarrow \left(\sigma_{ij} - \sigma\delta_{ij}\right) n_j = 0$$

In full:

$$\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} \begin{bmatrix} n_1^* \\ n_2^* \\ n_3^* \end{bmatrix} = 0$$

Eigenvalue equation:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

I_i first, second and third invariant

$$\Rightarrow \sigma_I, \sigma_{II}, \sigma_{III}$$

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = tr\,\underline{\underline{\sigma}}$$

$$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right) = tr^2\underline{\underline{\sigma}} = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$I_3 = \det\left[\sigma_{ij}\right] = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

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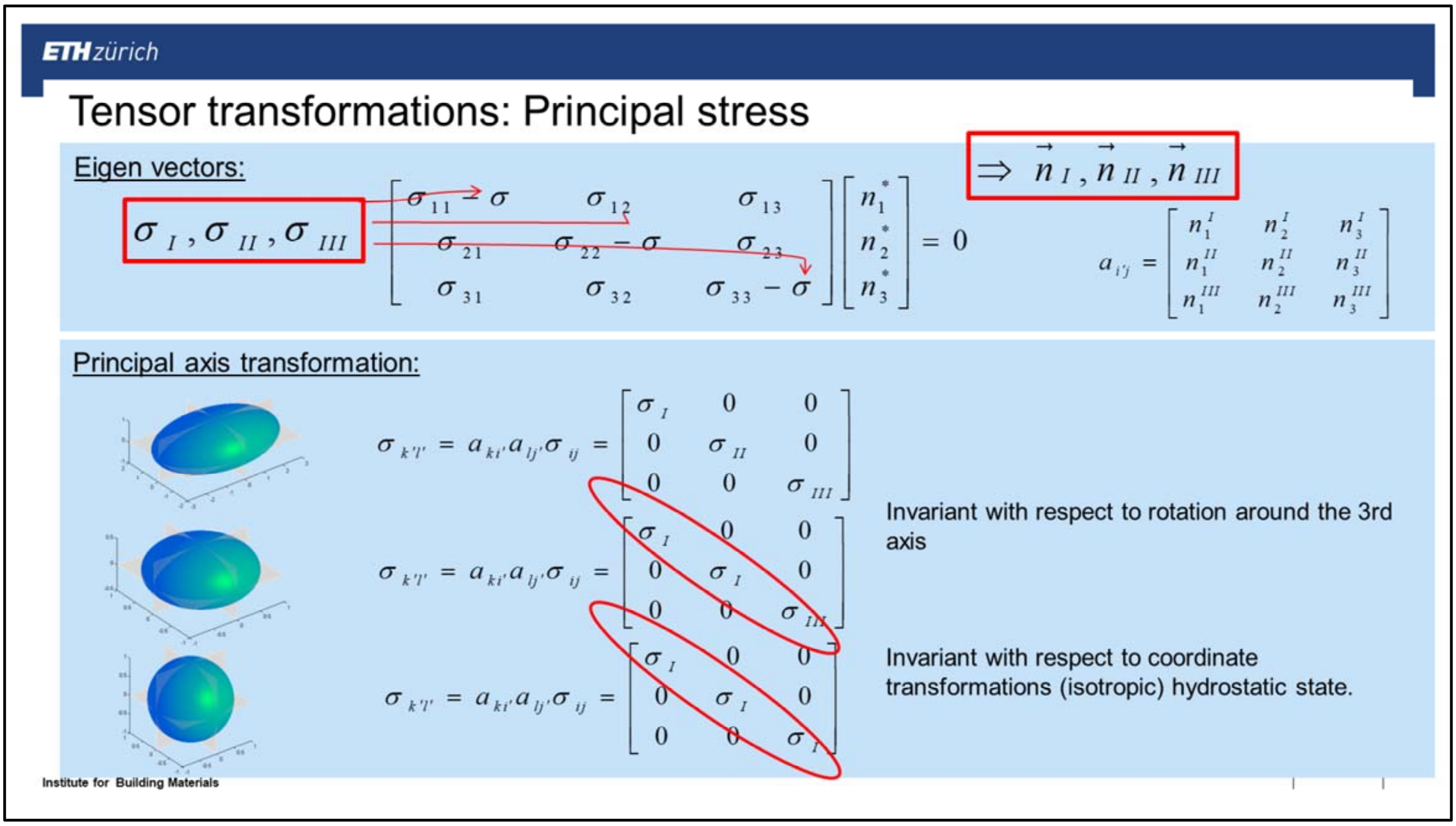
Of course we can now calculate the stress tensor for an infinite number of orthonormal coordinate systems. However there is one among them, called principal stress system that is peculiar. It is defined by the fact that in the cutting planes called principal direction or axis, only normal forces act and all shear stresses are zero. The stress vector is hence the normal vector of the cut multiplied by a proportionality factor sigma. If we use the Cauchy equation and take the just formulated condition for principal stress, we obtain the equation system.

This is called Eigen value problem and we are looking for the Eigen values (principal stresses) and Eigen vectors (principal axes). The determinant ordered after powers of sigma gives the characteristic equation or eigenvalue equation with the invariants. The fist invariant is the trace of the stress tensor. The second one is the sum of sub determinants for the principal diagonal and the third one is the determinate of the stress tensor itself. The characteristic equation gives three real solution sigma_1-3 for the principal stresses. These are stationary values of the normal stresses: The maximum one is called sigma_1, the minimum one simga_3 and sigma_2 is in between.



When the values are known, one can insert them into the equation above to obtain the normal directions. In principle one has to solve a linear system of equations with 3 unknowns. To start with one can first calculate the non-normalized version by setting the value of the first Eigen vector to 1 and then solving the system for 2 unknowns. In a second step the result can be normalized. When 2 vectors are calculated the 3rd one can be calculated with the cross product. When the principal axes are known, one can set up a transformation matrix for the coordinate transformation. In general sigma_I>sigma_II>sigma_III and there exists but one principle axis system. We can visualize the components of the stress vector in all possible cuts in form of the stress ellipsoid. If three principal stresses are identical, all coordinate systems are principle axis systems.

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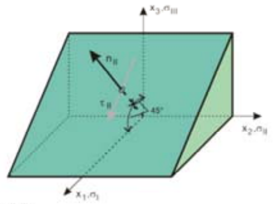
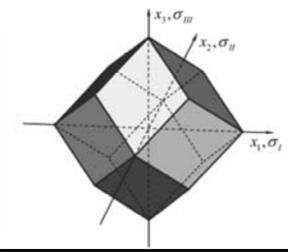
Tensor transformations: Principal shear stress

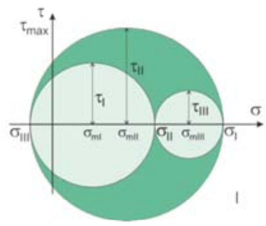
Principle shear stress:

	$\tau_1 = \frac{\lvert\sigma_2 - \sigma_3\rvert}{2}$	$n_i^I = \sqrt{2}/2\,[0 \quad 1 \quad 1]^T$	$\sigma_{mI} = \frac{1}{2}(\sigma_{II} + \sigma_{III})$
τ_{max}	$\tau_2 = \frac{\lvert\sigma_3 - \sigma_1\rvert}{2}$	$n_i^{II} = \sqrt{2}/2\,[1 \quad 0 \quad 1]^T$	$\sigma_{mII} = \frac{1}{2}(\sigma_I + \sigma_{III})$
	$\tau_3 = \frac{\lvert\sigma_1 - \sigma_2\rvert}{2};$	$n_i^{III} = \sqrt{2}/2\,[1 \quad 1 \quad 0]^T$	$\sigma_{mIII} = \frac{1}{2}(\sigma_{II} + \sigma_I)$

- Act in cuts, whose normal is perpendicular to a principal axis and has an angle of 45° with the other ones. (→rhombic dodecahedron)
- In cuts of maximal shear stress respective normal stresses do not vanish (MOHRs circle)







Of course one can also calculate the dependence of shear stresses with respect to cutting orientations. To obtain extremal values, the derivatives with respect to components of the normal vector have to be set to 0. After finite long calculations one obtains the principal shear stresses tau_i that can be formed by the principal stresses. The maximum is located at the difference between the 1st and 3rd principal stress. When we look at the orientation of the cutting plane for maximum shear stress, we realize, that those act in cuts, whose normal is perpendicular to a principal axis but forms an angle of 45° with the other two. The resulting cutting body is a rhombic dodecahedron (12 planes), whose area are of rhombic shape. The normal vectors of these bodies form an 60° angle.

We can also have a look at this in the MOHRs circle. We realize the 45° of the cutting plane for max. shear stress. The normal stress and shear stress can only be located in the intense green region, that is limited by the circle. The circles themselves represent cuts, whose normal is perpendicular to one of the three principal axes. As we can observe, in those cuts, normal stresses in general do not vanish.

So for a given stress filed, we can now calculate the stress trajectories of the principal stresses and principal shear stresses. Those are lines tangential to the respective principal stresses that are used to visualize the stress flow. Densifications of trajectories resemble stress concentrations. Materials mainly failing under tension (cleavage fracture) will fail along lines perpendicular to the principal stress trajectories. Materials whose failure is dominated by the shear criterion, will fail along critical shear stress trajectories, that can be visualized by photo elasticity.

Principle stress

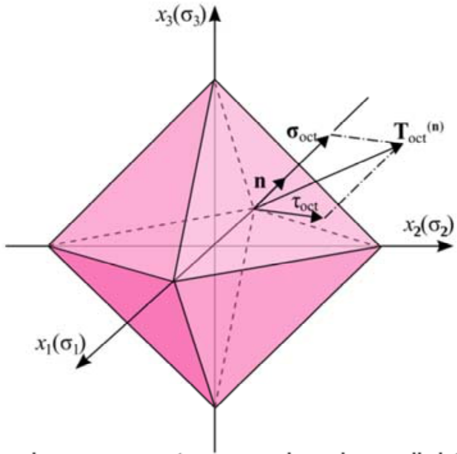
In point P of a body the stress state is given by the tensor:

$$\sigma_{ij} = \begin{bmatrix} \frac{5}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{7}{4} & -\frac{3}{4} \\ \frac{\sqrt{2}}{4} & -\frac{3}{4} & \frac{7}{4} \end{bmatrix}$$

a. Give the values of invariants.
a. Calculate principle stresses and orientations
b. Draw a Mohr’s circle
c. Calculate the transformation matrix $a_{i'j}$.
d. Make a principal axis transformation of the stress tensor.
e. How large is the principal shear stress and in what direction ***n*** does it act?

Principle stress / octahedral stresses

The cutting plane intersecting the space diagonal of the principal axis system is called octahedral plane.



a. Calculate the stress vector τ_{okt} with magnitude and components normal and parallel to the octahedral plane.
b. Show the relation of invariants to the stress tensor.

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Decomposition of the stress tensor

Decomposition by separation of the hydrostatic tensor:

$$\sigma^H = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{kk} = \frac{1}{3}I_1$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma^H & 0 & 0 \\ 0 & \sigma^H & 0 \\ 0 & 0 & \sigma^H \end{bmatrix} + \begin{bmatrix} \sigma_{11} - \sigma^H & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma^H & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma^H \end{bmatrix}$$

$$\sigma_{ij} = \sigma_{ij}^H + \sigma_{ij}^D$$

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad s_{ij}$$

Hydrostatic stress

Deviatoric stress

→Volume change

→shape change

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We saw the hydrostatic stress state before, where all principle stresses are identical. Well often it is quite handy to separate from an arbitrary stress tensor the hydrostatic, isotropic one. One can use the mean hydrostatic stress sigma^H for this. The difference from the hydrostatic tensor is called deviatoric stress tensor. We will see later on how the decomposition is used in plasticity theory and for limit analysis. By decomposing the stress tensor a particularly simple representation of the elasticity law and the strain energy becomes possible. But let's first look at the invariants of both tensors.

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Decomposition of the stress tensor

Hydrostatic stress

$$I_1^H = 3\sigma^H = I_1$$

$$I_2^H = 3\sigma^{H^2} = \tfrac{1}{3} I_1^2$$

$$I_3^H = \sigma^{H^3} = \tfrac{1}{27} I_1^3$$

Deviatoric stress

$$J_1 = I_1^D = \sigma_{ii}^D = \sigma_{ii} - \tfrac{1}{3}\sigma_{kk}\delta_{ii} = 0$$

$$J_2 = I_2^D = -\tfrac{1}{2}\sigma_{ij}^D \sigma_{ij}^D = -\tfrac{1}{2}\left(\sigma_I^{D^2} + \sigma_{II}^{D^2} + \sigma_{III}^{D^2}\right)$$

$$= -\tfrac{1}{6}\begin{bmatrix} (\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 \\ +6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \end{bmatrix}$$

$$= -\tfrac{1}{6}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2)\right]$$

$$= -\tfrac{1}{6}\left[2I_1^2 - 6I_2\right] = I_2 - \tfrac{1}{3} I_1^2$$

$$J_3 = I_3^D = \det \sigma_{ij}^D = I_3 - \tfrac{1}{3} I_2 I_1 + \tfrac{2}{27} I_1^3$$

$$J_2 = -\frac{1}{3}\sigma_v^2 = -\frac{3}{2}\tau_{okt}^2 = -\frac{E}{1+\nu} U_G$$

Deformation energy density

$$\sigma_I^D = \sigma_I - \tfrac{1}{3} I_1; \sigma_{II}^D = \sigma_{II} - \tfrac{1}{3} I_1; \sigma_{III}^D = \sigma_{III} - \tfrac{1}{3} I_1;$$

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Since the hydrostatic stress tensor is isotropic, and has no unique principle axis system, the principal axes of sigma_ij and sigma^D_ij have to be identical. The invariants of sigma^H_ij can be expressed using the 1st Invariant of sigma_ij. The invariants of the deviatoric stress tensor are named J_i.
J_1 vanishes.
J_2 is a measure for the mean quadratic deviation from the hydrostatic stress state. It can be expressed by the invariants of the initial stress tensor. Note that the equation is similar to the equivalent stress sigma_v following HUBER, v.Mises, Hencky (HMH), as well as the octahedral shear stress (exercise 4). We want to use here the deformation energy density Ug without definition, that can be used to express the invariant J_2 very easily. Of course also the principal stresses of the deviatoric stress tensor can be calculated like before, but the< can also be obtained from the ones of the stress tensor just by subtracting the volumetric part 1/3*I_1.

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Decomposition of the stress tensor

p-q-r invariants:

$$p = \frac{1}{3} I_1; \quad q = \sqrt{3 J_2} = \sigma_{eq}; \quad r = 3 \sqrt[3]{\tfrac{1}{2} J_3}$$

Equivalent stress:

$$\sigma_{eq} = \frac{1}{\sqrt{2}} \left\{ \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + 6 (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right\}^{1/2}$$

$$= \sqrt{\tfrac{3}{2} \sigma_{ij}^D \sigma_{ij}^D}$$

ξ–ρ–θ invariants (Haigh-Westergaard coordinates) :

$$\xi = \frac{1}{\sqrt{3}} I_1 = \sqrt{3} p; \quad \rho = \sqrt{2 J_2} = \sqrt{\tfrac{2}{3}} q; \quad \cos(3\theta) = \left(\frac{r}{q} \right)^3 = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \xi \\ \xi \\ \xi \end{bmatrix} + \sqrt{\frac{2}{3}} \rho \begin{bmatrix} \cos\theta \\ \cos(\theta - 2\pi/3) \\ \cos(\theta + 2\pi/3) \end{bmatrix}$$

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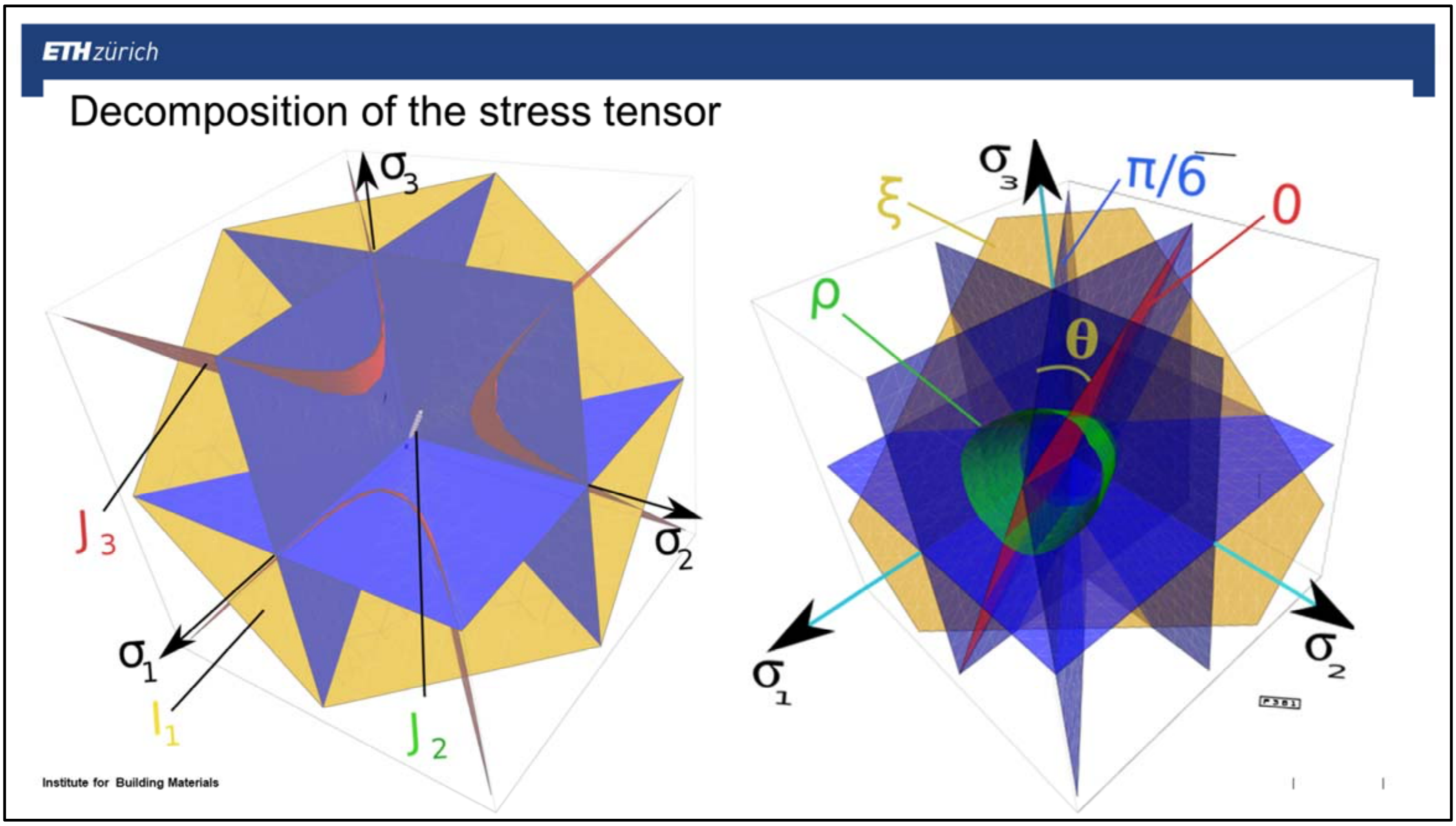
For cohesive frictional materials (like concrete) the scaled p-q-r invariant space is important. Formulating equations in invariant spaces makes the formulation automatically invariant too. P and q are scale versions of I1 and J2 and r is function of J3. The threat of negative J3 and resulting imaginary r limits the use of this set of invariants for engineering practice.

The equivalent stress is used for strain hardening like we will see in future lectures on plasticity theory.

An other important set of invariants are the HAIGH-WESTERGAARD coordinates, given in a cylindrical coordinate system. The angle Theta is the LODE angle being function of the difference of second principal stress with respect to the other two. Note that if simga2=sigma3 the Lode angle=60°, while for sigma1=sigma2 the angel=0°. Hence the lode angel is an indicator of the magnitude of the middle principal stress with respect to min and max.



In principle stress space, the areas of constant invariants I_1, J_2 and J_3 can nicely be displayed. The I_1-J_2-J_3 stress space is important for yield surfaces, like we saw in the last lecture.

Deviatoric stress

For the stress state of uniaxial tension in 1 direction one should calculate:

a. The deviatoric stress.
b. The invariants of the deviatoric stress.
c. The Eigen values of the stress tensor and deviator.

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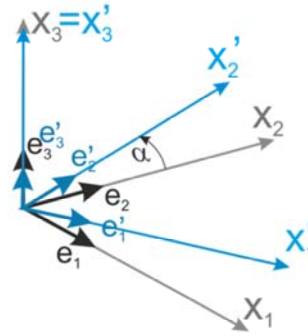
Remember… plane stress state (PS)



Disc and plate:

$$\sigma_{3i} = \sigma_{i3} = 0 \qquad \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

Transformation matrix:

$$a_{i'j} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Principal stress:

$$\sigma_{I/II} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2}$$

Angle of principal stress:

$$\tan\alpha^*_{I,II} = -\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}}\right)^2 + 1}$$

Angle of principal stress for principal shear stress:

$$\tan\alpha^*_{I,II} = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} \pm \sqrt{\left(\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right)^2 + 1}$$

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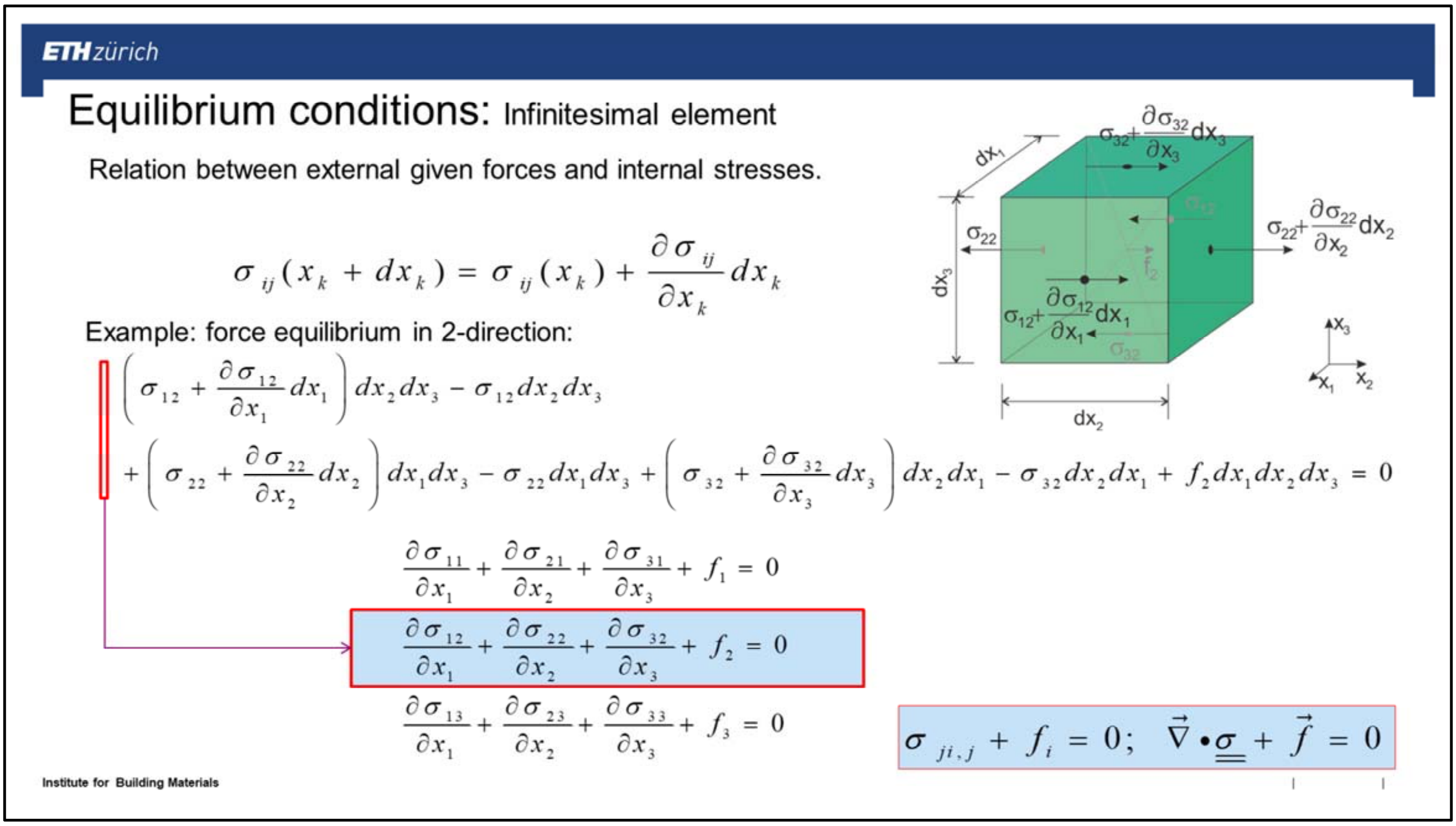
Many situations allow for a two-dimensional perspective, where the 3rd dimension is reduced by additional assumptions like plane strain or plane stress. In principle you remember and this is just for completeness.

For PS all stress components in the 3rd direction are zero, while for plain strain sigma_33 is not zero but linearly dependent on sigma_11+sigma_22. Hence we look at the stress tensor including sigma_33. The derivation of the transformation tensor is relatively simple, since it can only rotate about the x3-axis. The principle axis transformation has to consider that the 3-direction is already a principle stress, and consequently only the angle alpha has to be calculated for which the shear stress vanishes. We obtain the Eigen stress, insert it into the characteristic equation to get the Eigen vector and finally alpha after selecting the one for a right hand system.



Up to now we always started from a given stress state at a point. So up to now we did not contribute to the calculation of the components of the stress tensor. To get the relation between external given forces and internal stresses equilibrium relations are used. One has the choice to either make a local consideration using the infinitesimal element or one can make a global consideration of the body.

The equilibrium at the infinitesimal element of size dx1*dx2*dx3 is given here for the stress component in the x2 direction only. Note that also the volumetric force f2 only is considered with its component in the 2 direction. If we move away from the position dx_k, the stress changes following the differential representation. This engineering representation corresponds to a Taylor-series expansion with only 1 element, only valid for infinitesimal dimensions.

Equilibrium conditions must hold for forces and for moments, but not for stresses. Consequently, we must consider the area the forces act on. For the 2-direction this is:

Some terms cancel out, others can be combined with the volume V=dx1*dx2*dx3 and give the final equilibrium condition:… that can be given in index or symbolic notation for all directions. As we can see there are 3 equations with 6 unknown stress components sigma_ij; hence 3 times statically indeterminate. To progress from here, additional displacement-distortion equations and the constitutive relation is needed.

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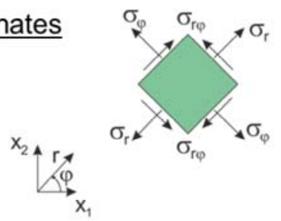
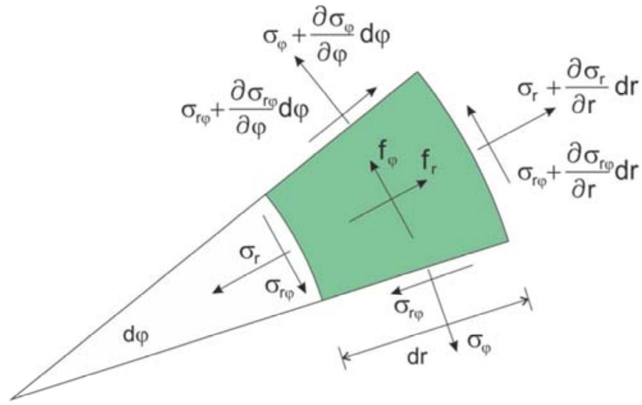
Equilibrium conditions: Infinitesimal element

Polar coordinates



Radial stress

circumferential stress



Force equilibrium in radial direction:

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right) t(r + dr) d\varphi - \sigma_r t r d\varphi - \sigma_{r\varphi} t dr$$

$$+ \left(\sigma_{r\varphi} + \frac{\partial \sigma_{r\varphi}}{\partial \varphi} d\varphi\right) t dr - \left(\sigma_\varphi d\varphi\right) t dr + f_r t r d\varphi dr = 0$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{1}{r}\left(\sigma_r - \sigma_\varphi\right) + f_r = 0$$

$$\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{2}{r}\sigma_{r\varphi} + f_\varphi = 0$$

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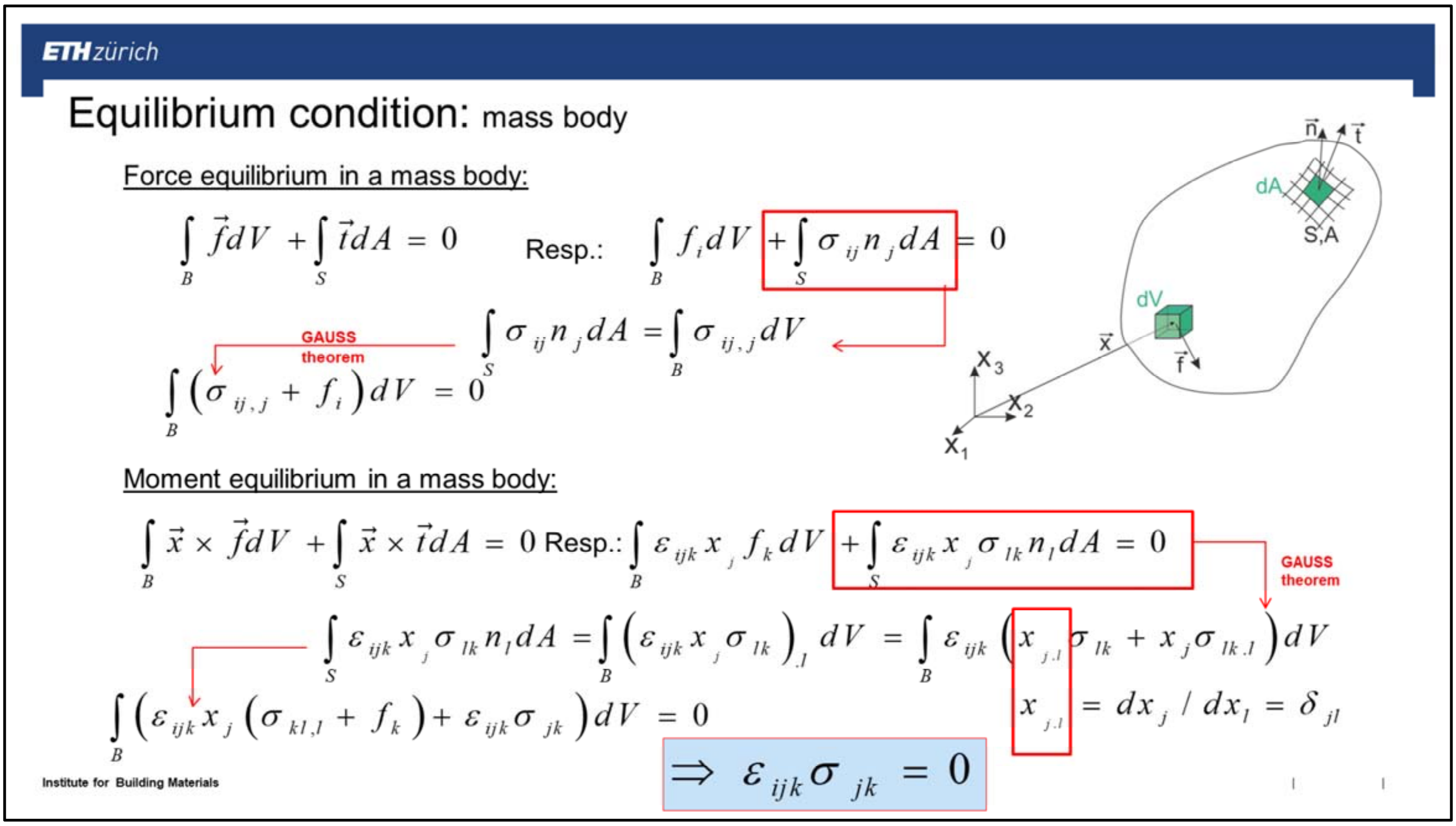
Extension: Often it is useful to formulate equation conditions not in Cartesian but directly in polar coordinates with radial stress sigma_r and circumferential stress sigma_phi. We look here at the force equilibrium of a thin disk of thickness t. Using approximations for small angles, namely sin(a)=a, cos(a)=1 one obtains the force equilibrium in radial direction.



If we take instead of the infinitesimal small volume the entire body, or an arbitrarily cut part from it with finite volume V and surface S with area A and the outside pointing normal vector n under the action of the volume force f and surface tension t.

The body is in equilibrium if the resultant from volume force and surface tension disappears. This can be written in vector or component notation with the Cauchy equation. The Gauss theorem can be applied with the 2nd summand to be able to merge the two parts into a volume integral. The equation is only fulfilled for arbitrary volumes, if the integrand disappears, what leads to the differential element seen before. The same can be made for the moment equilibrium, but the solution reveals nothing but the symmetry of the stress tensor. For the integration of course boundary conditions are needed like the stress vector used here.

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Summary: Stress states

Normal stress	$\sigma_{11}, \sigma_{22}, \sigma_{33}$
Shear stress	$\sigma_{12} = \sigma_{21}; \sigma_{23} = \sigma_{32}; \sigma_{13} = \sigma_{31}$
Principal stress	$\sigma_{I}; \sigma_{II}; \sigma_{III}$
Invariants	$I_1; I_2; I_3;\quad J_1; J_2; J_3;$
Equilibrium condition	$\sigma_{ij,j} + f_i = 0$
Boundary condition	$t_i = \sigma_{ji} n_j \big\|_S$

Let’s summarize: The stress state describes the internal loading and the edge loading of a body due to external loads. We introduced the stress tenors, discussed its transformation behavior, in particular principle stress transformation. We derived equilibrium conditions and it became evident, that if one want to calculate the components of the stress tensor, one runs into a statically indetermined problem. The stresses can thus not be solely obtained from equilibrium conditions.

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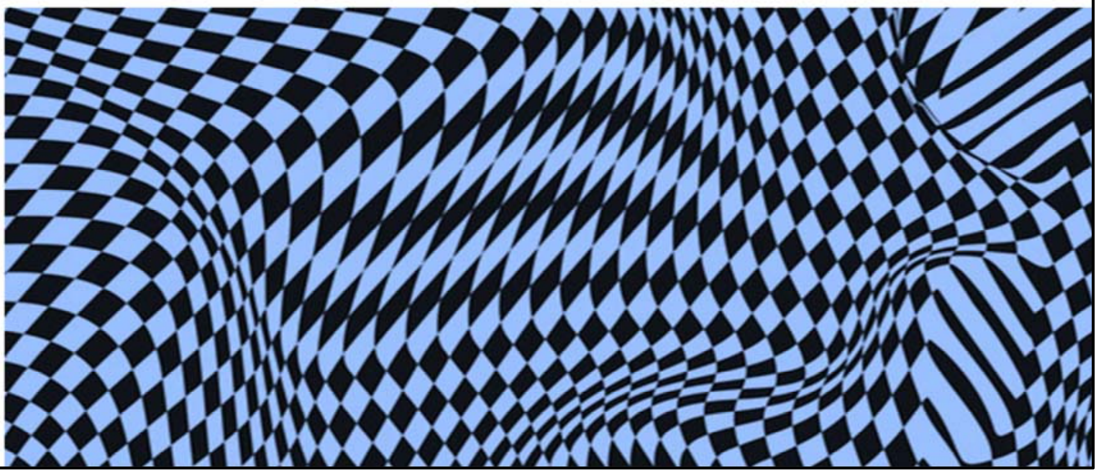
Displacement and distortion

Displacement and distortion
Compatibility conditions
Plane strain case



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By splitting the displacement gradient into a symmetric and antisymmetric part one obtains the distortion tensor and the rotation tensor for the geometrically linear theory. After showing its tensor character, all results we obtained for stress tensors can be translated to strains. To get unique relations for the strain tensor, compatibility conditions must be fulfilled. A special case is the plain strain condition.

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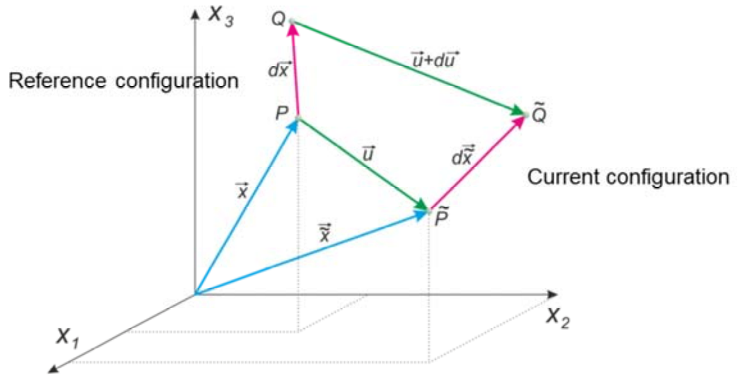
Displacement and displacement gradien



Displacement point P: $u_i = \tilde{x}_i - x_i$

Displacement infinitesimal line element dx: $d\vec{x} \rightarrow d\tilde{\vec{x}}; Q \rightarrow \tilde{Q}$

Displacement point Q: $u_i + du_i$

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j = u_{i,j} dx_j =: H_{ij} dx_j$$

Displacement gradient: $H_{ij} = \frac{\partial u_i}{\partial x_j}$

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*An arbitrary material Point P of a body is given by its position vector x. This is called reference configuration and could for example be an undeformed one.
*By external action the position of the Point i now P~ in the current configuration with the position vector x~.
*The displacement of Point P is hence vector u. Well this is nice, but we have no clue, if this is rigid body motion or deformation.
*For this we take a neighboring Point Q that can be reached by the infinitesimal line element dx. We deform and dx → dx~ and Q → Q~.

*Since we are in a continuum, u is a steady vector field and one can calculate the total differential. The displacement gradient H describes the relative motion of neighboring points and contains now the stretching as well as the rotation of the line element. It is only valid for small displacements.

Note that the description of the length change can be made in two different ways: in the Lagrangian description, where the motion of a material point is followed. Independent variables are then the material coordinates. In the Eulerian description the state in a fixed position in space is considered. This is very useful for flow problems, but in elasto mechanics, the Lagrangian description is preferred.

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Strain and rotation tensor

Decomposition of the displacement gradient into symmetric and anti-symmetric part:

$$H_{ij} = u_{i,j} = \underbrace{\frac{1}{2}\left(u_{i,j} - u_{j,i}\right)}_{\text{Infinitesimal Rotation tensor}} + \underbrace{\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)}_{\text{Infinitesimal Strain tensor}} = \omega_{ij} + \varepsilon_{ij}$$

$$\omega_{ij} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix}$$

$$\omega_{ij} = -\omega_{ji} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

$$\omega_{12} = -\omega_{21} = \frac{1}{2}(\alpha + \beta) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$$

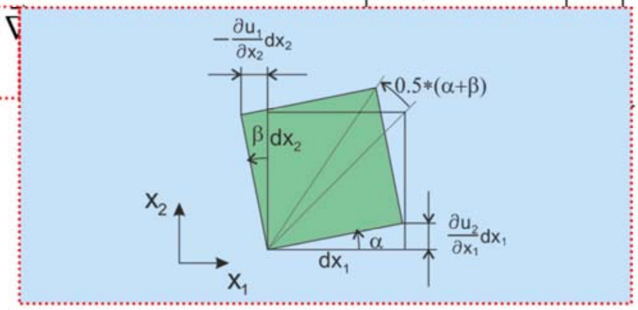
$$\omega_{23} = -\omega_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right)$$

$$\omega_{31} = -\omega_{13} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right)$$

$$\vec{\omega} = \frac{1}{2}\,\mathrm{rot}\,\vec{u} = \frac{1}{2}\nabla \ldots$$

$$\omega_{ij} = \begin{bmatrix} 0 & \ldots \\ -\omega_3 & \ldots \\ \omega_2 & \ldots \end{bmatrix} \qquad \begin{bmatrix} u_{3,2} - u_{2,3} \\ \ldots \end{bmatrix} \qquad \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Rigid body rotation about the x_3 axis



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The deformation gradient H describes the relative displacement of neighboring points and contains as such stretching of the line element as well as its rotation. If it is decomposed into a symmetric (epsilon) and an anti-symmetric (omega) component (w12=-w21), the rotation can be splitted off. Now we obtain an infinitesimal rotation and distortion tensor. The component of the rotation tensor describe the rotation of the element as one can see in the picture. For small deformations alpha=du2/dx1 and beta=-du1/dx2 for a rotation about the x3 axis. For the rotation angle of the diagonal about the x3 axis one obtains (equation). Analogous things are true for the rotation around the other 2 axes. Hence one has a rigid body rotation, that do not lead to a distortion of the element and hence do not lead to stresses. Consequently they cannot help us to reduce the under determination of the system. Hence the rotation components will not enter the elasticity law and are not further considered. We focus on the distortions.

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Strain and rotation tensor

Decomposition of the displacement gradient into symmetric and anti-symmetric part:

$$H_{ij} = u_{i,j} = \underbrace{\frac{1}{2}\left(u_{i,j} - u_{j,i}\right)}_{\text{Infinitesimal Rotation tensor}} + \underbrace{\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)}_{\text{Infinitesimal Strain tensor}} = \omega_{ij} + \varepsilon_{ij}$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Technical shear strain

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right),$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

Mathematical shear strain

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

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The symmetric part of the displacement gradient H is equal to the strain tensor of the geometric linear theory. However one has to be careful. On distinguished between technical and mathematical shear strains. Taking the displacement u,v,w in x,y,z direction the technical shear strain is defined as follows: The mathematical one is different by a factor of ½. It is important to know what shear strain measure is used, e.g. when the results of computer programs are interpreted. The strain tensor is a 2nd order tensor that can be considered the same way as the stress tensor: by transformation, principal axis systems and decomposition into hydrostatic and deviatoric parts.

Strain tensor: Transformation, principal axes

Transformation relation: $\varepsilon_{i'j'} = \frac{1}{2}\left(u_{j',i'} + u_{i',j'}\right) = a_{i'k}a_{j'l}\varepsilon_{kl}$

Principal axis with:

- Principal strains $\varepsilon_I, \varepsilon_{II}, \varepsilon_{III}$
- Perpendicular principal axes
- Disappearance of shear strains $\varepsilon_{ij} = 0, \text{ für } i \neq j$
- Direction of maximum shear strains:

$$\gamma_I = \pm\tfrac{1}{2}(\varepsilon_{II} - \varepsilon_{III}), \gamma_{II} = \pm\tfrac{1}{2}(\varepsilon_{III} - \varepsilon_I) = \gamma_{\max}, \gamma_{III} = \pm\tfrac{1}{2}(\varepsilon_I - \varepsilon_{II})$$

- Invariants:

$$I_1 = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III} = \varepsilon_{kk}$$

$$I_2 = \varepsilon_I\varepsilon_{II} + \varepsilon_{II}\varepsilon_{III} + \varepsilon_{III}\varepsilon_I = \tfrac{1}{2}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij})$$

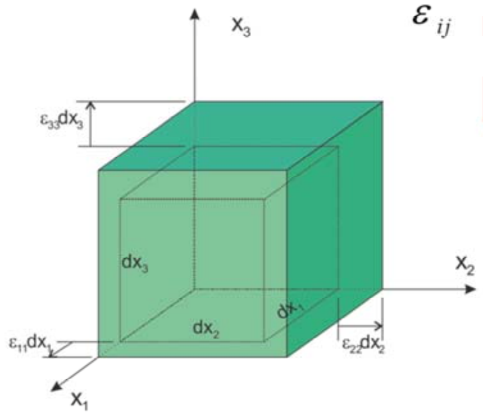
$$I_3 = \varepsilon_I\varepsilon_{II}\varepsilon_{III} = \tfrac{1}{2}\varepsilon_{ikm}\varepsilon_{jln}\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{mn} = \det \underline{\underline{\varepsilon}}$$

Decomposition of the tensor:

$$\varepsilon_{ij} = \varepsilon_{ij}^H + \varepsilon_{ij}^D = \tfrac{1}{3}\varepsilon_{kk}\delta_{ij} + \underbrace{\left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right)}_{e_{ij}}$$

As for the stress tensor transformation relations can be used. Also there exists a principal axis system.

Strain tensor: Decomposition of the strain tensor



$$\varepsilon_{ij} = \varepsilon_{ij}^{H} + \varepsilon_{ij}^{D}$$

$$= \tfrac{1}{3}\varepsilon_{kk}\delta_{ij} + \underbrace{\left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right)}_{e_{ij}}$$

$$I_1^D = \varepsilon_I^D + \varepsilon_{II}^D + \varepsilon_{III}^D = 0$$

$$I_2^D = \tfrac{1}{6}\left[\begin{matrix}(\varepsilon_I - \varepsilon_{II})^2 + \\ (\varepsilon_{II} - \varepsilon_{III})^2 + (\varepsilon_{III} - \varepsilon_I)^2\end{matrix}\right]$$

$$I_3^D = \varepsilon_I^D \varepsilon_{II}^D \varepsilon_{III}^D$$

→Shape change

$$\varepsilon_V = \frac{\Delta dV}{dV} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = I_1 = 3\varepsilon^H$$

→Volume change (dilatation)

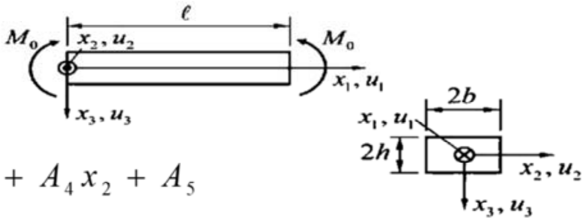
Octahedral strain:

$$\varepsilon_{okt} = \frac{1}{3}I_1 ; \gamma_{okt} = \sqrt{-\tfrac{2}{3}I_2^D}$$

The strain tensor can be analogously decomposed into volumetric and deviatoric part. The volume change can be calculated via the first invariant of the strain tensor. Since the hydrostatic part describes the volume change, the deviatory part describes the shape change of the volume element. Of cause here again three invariants exist. Finally octahedral strains can be calculated, analogous to the octahedral stresses.

Strain tensor

The displacement distribution in a beam with rectangular cross-section , loaded by end moments is given in elasticity theory by:



$$u_1 = \frac{c}{R} x_1 x_3 + A_1 x_3 + A_2 x_2 + A_3$$

$$u_2 = -\frac{c}{R} \nu x_2 x_3 - A_4 x_3 - A_2 x_1 + A_6$$

$$u_3 = -\frac{c}{2R}\left[x_1^2 + \nu (x_3^2 - x_2^2) \right] - A_1 x_1 + A_4 x_2 + A_5$$

ν Poisson's number, *R* radius of curvature of the beam, *c* a constant that relates to M_0 , and constants A_1 , ..., A_6 that can be determined by boundary conditions.

a. Calculate the strain and rotation tensor.
b. Calculate the volume change.
c. What are the constants A_1 , ..., A_6 assuming that the displacement and rotation in the origin is zero.
d. Calculate with these constants the rotation in point (l,0,0) from the rotation tensor as well as from the deflection line.
e. From the beam equation *M =-EIw"* calculate the constant *c* .

Compatibility conditions

Kinematic equations: $$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{j,i} + u_{i,j}\right)$$

$$2\varepsilon_{12,12} = u_{1,212} + u_{2,112} = \left(u_{1,1}\right)_{,22} + \left(u_{2,2}\right)_{,11} = \varepsilon_{11,22} + \varepsilon_{22,11}$$

6 compatibility conditions:

$$\eta_{11} := \varepsilon_{22,33} + \varepsilon_{33,22} - 2\varepsilon_{23,23} = 0, \qquad \eta_{12} := \left(\varepsilon_{23,1} + \varepsilon_{13,2} - \varepsilon_{12,3}\right)_{,3} - \varepsilon_{33,12} = 0,$$

$$\eta_{22} := \varepsilon_{33,11} + \varepsilon_{11,33} - 2\varepsilon_{31,31} = 0, \qquad \eta_{23} := \left(\varepsilon_{31,2} + \varepsilon_{21,3} - \varepsilon_{23,1}\right)_{,1} - \varepsilon_{11,23} = 0,$$

$$\eta_{33} := \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0, \qquad \eta_{31} := \left(\varepsilon_{12,3} + \varepsilon_{32,1} - \varepsilon_{31,2}\right)_{,2} - \varepsilon_{22,31} = 0,$$

In general: $$\eta_{ij} = \varepsilon_{ikm}\varepsilon_{jln}\varepsilon_{kl,mn} = 0$$

Example: $$\eta_{23} = \varepsilon_{2km}\varepsilon_{3ln}\varepsilon_{kl,mn} = \varepsilon_{2km}\left(\varepsilon_{312}\varepsilon_{k1,m2} + \varepsilon_{321}\varepsilon_{k2,m1}\right) = \varepsilon_{2km}\left(\varepsilon_{k1,m2} - \varepsilon_{k2,m1}\right)$$
$$= \varepsilon_{231}\left(\varepsilon_{31,12} - \varepsilon_{32,11}\right) + \varepsilon_{213}\left(\varepsilon_{11,32} - \varepsilon_{12,31}\right)$$
$$= \varepsilon_{31,21} + \varepsilon_{21,31} - \varepsilon_{23,11} - \varepsilon_{11,23} \square$$

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Starting point are the kinematic equations. If displacements u_i are given, stains can be calculated in unique way by differentiation. However if we want to calculate from the 6 given strains epsilon_ij by integration the displacements we have 6 equations for 3 unknowns u_i. In other terms we are 3 time kinematic over determined. Hence the 6 strains epsilon_ij can not be independent from each other and must fulfill compatibility conditions. One obtains these by differentiating twice. In principle 3*3*3*3=81 equations would be possible this way. However, since the strain tensor is symmetric and one can change the order of derivations and can forget about trivialities like i=j=k=l=1,2,3 it reduces to 6. In general one can write these equations in a very compact way using index notation with the LEVI-CIVITA tensor. Eta is also called incompatibility tensor and the=0 is the compatibility condition. This can be controlled for the example ij=23. The equation eta_ij=0 can also be understood as a linear coupled second order differential equation system to obtain the six unknown components of the strain tensor. Hence the strains would be calculated from the compatibility conditions in unique way. Since eta_ij,i =0 these 6 components are linearly dependent and one does not obtain the 6 strains.

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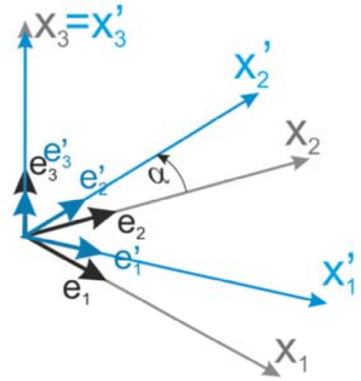
Plain strain (PE)



Long components: $\varepsilon_{3i} = \varepsilon_{i3} = 0$

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$$

Transformation matrix:

$$a_{i'j} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Principal strain: $\varepsilon_{I/II} = \frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) \pm \sqrt{\frac{1}{4}(\varepsilon_{11} - \varepsilon_{22})^2 + \varepsilon_{12}^2}$

Invariants: $I_1 = \varepsilon_{\alpha\alpha}; I_2 = \frac{1}{2}\left(\varepsilon_{\alpha\alpha}\varepsilon_{\beta\beta} - \varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta}\right)$

Compatibility condition: $\eta_{33} = \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0$

Summary: state of strain

Strain	$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$
shear	$\varepsilon_{12} = \varepsilon_{21}; \varepsilon_{23} = \varepsilon_{32}; \varepsilon_{13} = \varepsilon_{31}$
Principal strains	$\varepsilon_{I}; \varepsilon_{II}; \varepsilon_{III}$
Invariants	$I_1; I_2; I_3$
Kinematics	$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$
Boundary conditions	$u_i = u_i\big\|_S$
Compatibility	$\eta_{ij} = \varepsilon_{ikm}\varepsilon_{jln}\varepsilon_{kl,mn} = 0$

Let's summarize: By splitting the displacement gradient into a symmetric and anti-symmetric part one obtains the distortion tensor and the rotation tensor for the geometrically linear theory. After showing its tensor character, all results we obtained for stress tensors can be translated to strains. To get unique relations for the strain tensor, compatibility conditions must be fulfilled. A special case is the plain strain condition.

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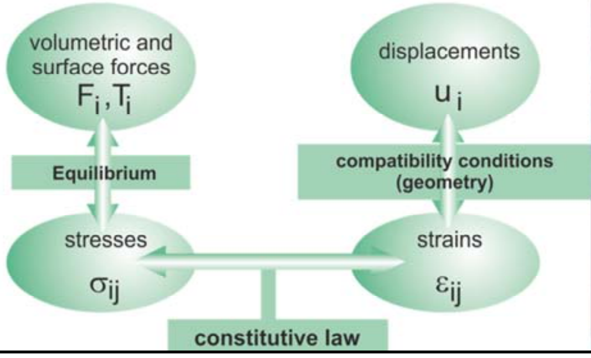
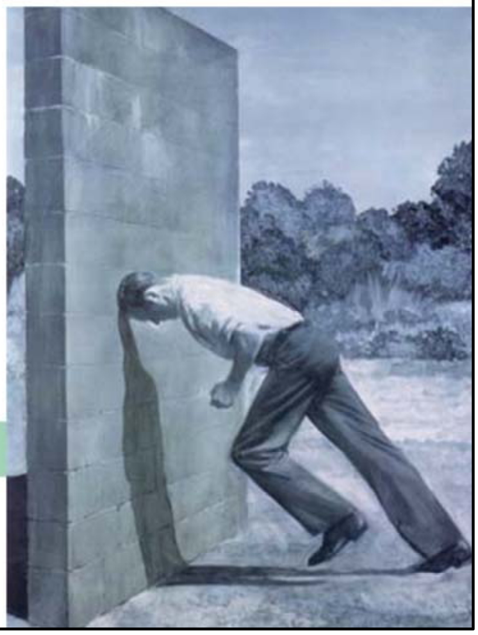
Elasticity law

Stiffness and compliance tensor
Elastic potential and strain energy
Material symmetries
Generalized Hook's law





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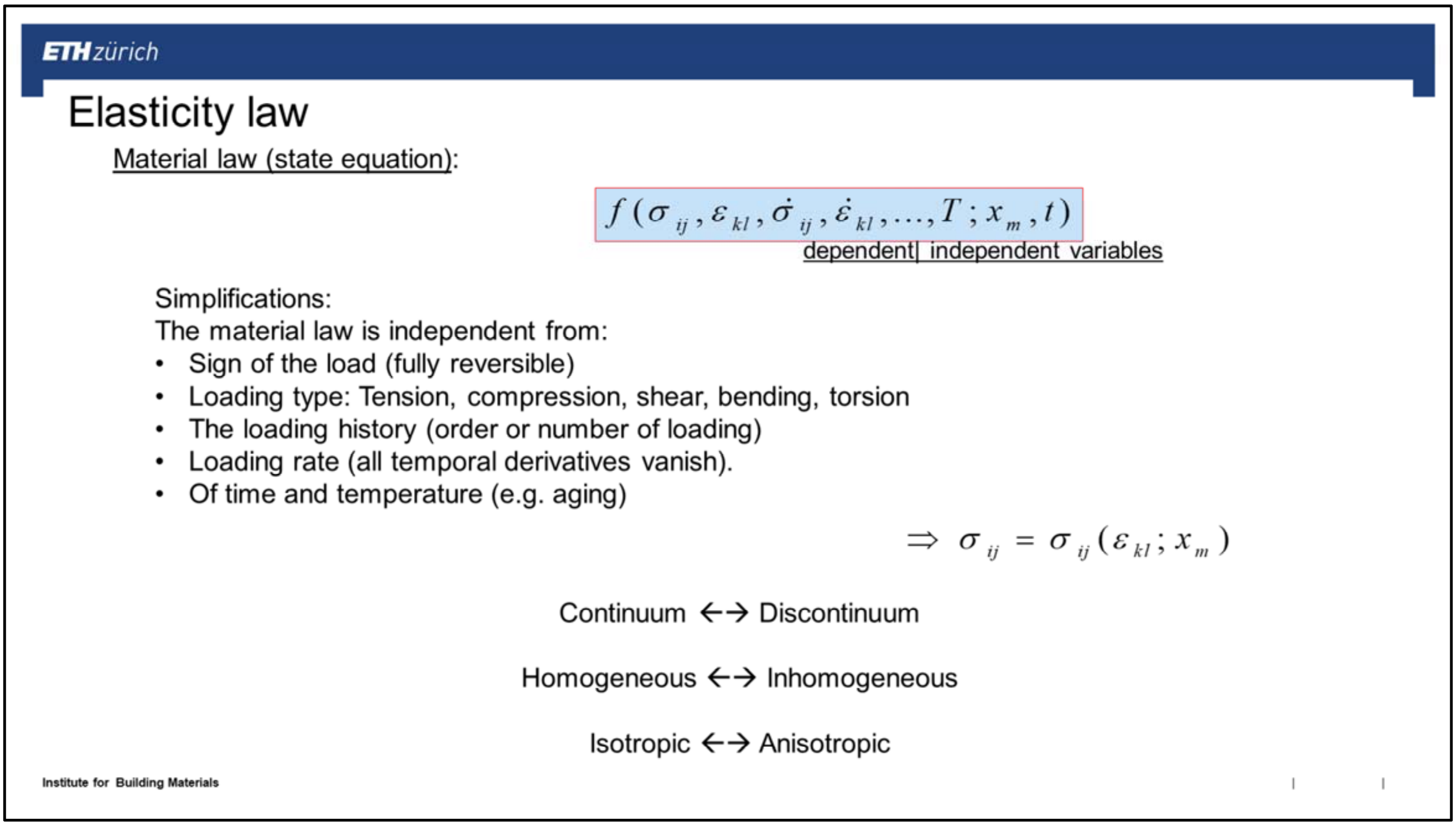
The elasticity law interconnects stresses and strains via elasticity tensors. In the general case one obtains a stiffness or compliance tensor with 21 independent components, whose number is further reduced by material symmetry. Isotropic material has only 2 parameters.



We have seen that stresses cannot be solely calculated from equilibrium conditions. The problem was three times undetermined. Then we tried to describe the problem kinematicaly and saw that it is three times over determined. However to solve questions of elasticity theory, a coupling between stress and strain is needed, what is done by materials laws that comprise the physical behavior of the body. It is a state equation with dependent and independent variables. It contains implicit material properties that characterize the physical behavior of the body. We are here to take a deeper look into exactly this. However before it gets complicated, we can make it simple and make some assumptions such as: ….

Hence the stress state in a point of the body depends with theses assumptions only from the deformation state and NOT how it was reached (history). It is as well not dependent on its neighborhood (locality). If all material points are of identical phase (solid/liquid) it is a continuum. Porous media is not. If all points in the body have identical material properties, the material is homogeneous (independent on x_m). Composites like concrete are not. If all material properties are independent on the reference frame, it is isotropic. Composites, wood and others are not.

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Elasticity law: stiffness tensor

HOOKEs law (linear relation between strain and stress):

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$$

4th order tensor → 3^4=81 elements

Symmetry of σ_{ij} and ε_{ij} results in $E_{ijkl}=E_{jikl}=E_{ijlk}=E_{jilk}$ →6*6=36 independent constants.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & E_{1131} & E_{1112}\\ E_{2211} & E_{2222} & E_{2233} & E_{2223} & E_{2231} & E_{2212}\\ E_{3311} & E_{3322} & E_{3333} & E_{3323} & E_{3331} & E_{3312}\\ E_{2311} & E_{2322} & E_{2333} & E_{2323} & E_{2331} & E_{2312}\\ E_{3111} & E_{3122} & E_{3133} & E_{3123} & E_{3131} & E_{3112}\\ E_{1211} & E_{1222} & E_{1233} & E_{1223} & E_{1231} & E_{1212}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\\ 2\varepsilon_{12}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{1}\\ \sigma_{2}\\ \sigma_{3}\\ \sigma_{4}\\ \sigma_{5}\\ \sigma_{6}\end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16}\\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26}\\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36}\\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46}\\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56}\\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{1}\\ \varepsilon_{2}\\ \varepsilon_{3}\\ \varepsilon_{4}\\ \varepsilon_{5}\\ \varepsilon_{6}\end{bmatrix}$$

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$$

⇩

$$\sigma_{A} = E_{AB}\varepsilon_{B}, \quad A, B = 1, 2, \ldots, 6$$

Symmetry: $E_{AB} = E_{BA}$

→ 21 independent constants

→ (aelotropic body)

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For now let's stay general. Elastic material behavior can be described via the Hooks law. It is a linear relation between stress and strain. Hence a linear relation between two second order tensors needs a 4th order tensor with 3^4=81 components. It is called stiffness tensor and its components stiffness components. By symmetry of sigma and epsilon (6 independent constants each) one can reduce it to 36 independent constants that are needed to describe a material . The 9x9 equation system can be reduced to 6x6. Since the stiffness tensor is symmetric as well (most the time), it can further be reduced with E_AB=A_BA to 21 independent components. Such a body is called aelotropic.

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Elastic law: compliance tensor

Compliance tensor:

$$D_{ijkl} = E^{-1}_{ijkl}$$

$$D_{ijmn} E_{mnkl} = \delta_{ik}\delta_{jl}$$

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & 2D_{1123} & 2D_{1131} & 2D_{1112} \\ & D_{2222} & D_{2233} & 2D_{2223} & 2D_{2231} & 2D_{2212} \\ & & D_{3333} & 2D_{3323} & 2D_{3331} & 2D_{3312} \\ & & & 4D_{2323} & 4D_{2331} & 4D_{2312} \\ & sym. & & & 4D_{3131} & 4D_{3112} \\ & & & & & 4D_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = D_{ijkl}\sigma_{kl}$$

Shorter:

$$\varepsilon_A = D_{AB}\sigma_B, \quad A, B = 1, 2, \ldots, 6$$

→ D is the inverse of E: $D_{AB} E_{BC} = \delta_{AC}$

The inverse of the stiffness tensor is called compliance tensor. Of course also here the symmetries hold and one can write : … With the index notation everything gets again very compact. One can make the inverse, but the number of independent elements of course does not change. In principle on has to obtain these from material tests, what means and enormous experimental burden. However materials often have internal symmetries, what means an significant reduction of the number of independent components.

Crafting unusual mechanical properties Metamaterials, which have engineered internal structure, can have very different responses from conventional materials when compressed. COVER Illustration of a 3D chiral elastic metamaterial that is being compressed from above, causing the material to twist (along with the usual axial compression and lateral stretching or expansion). The darkest orange area denotes the highest degree of deflection. The twist motions, forbidden in ordinary elastic continua, aid the design of complex mechanical architectures. See pages 994 and 1072.

Illustration: C. Bickel/*Science*

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Elasticity law: material symmetries

Symmetry with respect to one plane:

→Monoclinic, resp. monotropic material

Examples symmetric with respect to the x_1-x_2 plane $\left(\vec{e}_{1'}, \vec{e}_{2'}, \vec{e}_{3'}\right) = \left(\vec{e}_1, \vec{e}_2, -\vec{e}_3\right)$

When transforming, material properties are not allowed to change

$$a_{i'j} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ sym. & & -1 \end{bmatrix}$$

Example: $E_{i'j'k'l'} = a_{i'i} a_{j'j} a_{k'k} a_{l'l} E_{ijkl}$

$$E_{1'1'} = E_{1'1'1'1'} = a_{1'i} a_{1'j} a_{1'k} a_{1'l} E_{ijkl} = E_{1111} = E_{11}$$

$$E_{1'4'} = E_{1'1'2'3'} = a_{1'i} a_{1'j} a_{2'k} a_{3'l} E_{ijkl} = -E_{1123} = -E_{14} = 0$$

$$E_{AB} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & E_{16} \\ & E_{22} & E_{23} & 0 & 0 & E_{26} \\ & & E_{33} & 0 & 0 & E_{36} \\ & & & E_{44} & E_{45} & 0 \\ & sym. & & & E_{55} & 0 \\ & & & & & E_{66} \end{bmatrix}$$

Symmetry: $E_{AB} = E_{BA}$

→ 13 independent constants (Monotropic body)

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The internal symmetries of technical materials mean, that the material behavior is not supposed to change under certain transformations. The transformation matrix contains all transformations. As one can see a component is positive, when the index 3 is 0,2, or 4 times in there and negative if 1 or 3 times. Since an inverted sign would be in contrary to the symmetry, these components must be zero. This way one obtains a reduced stiffness tensor for a monoclinic material with 13 independent constants. As one can see here, pure shear deformation causes also normal stresses, since these coupling terms are not zero, and pure strain can also result in shear stress. Consequently the principal orientations of the stress and strain tensor do not coincide.

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Elasticity law: material symmetries

Symmetry with respect to two perpendicular planes:

→orthotropic material

$$E_{AB} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ & E_{22} & E_{23} & 0 & 0 & 0 \\ & & E_{33} & 0 & 0 & 0 \\ & & & E_{44} & 0 & 0 \\ & sym. & & & E_{55} & 0 \\ & & & & & E_{66} \end{bmatrix}$$

→ 9 independent constants (orthotropic body)

With engineering constants E_i, G_{ij}, ν_{ij}:

$$D_{AB} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix}$$

Symmetry condition $D_{AB}=D_{BA}$:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad \text{für } i \neq j$$

Positive definite D_{AB}:

1. $E_1, E_2, E_3, G_{23}, E_{31}, E_{12} > 0$
2. $|\nu_{ij}| < \sqrt{\frac{E_i}{E_j}}$
3. $\nu_{12}\nu_{23}\nu_{31} < \frac{1}{2}\left(1 - \nu_{12}^2 \frac{E_2}{E_1} - \nu_{23}^2 \frac{E_3}{E_2} - \nu_{31}^2 \frac{E_1}{E_3}\right) < \frac{1}{2}$

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If additional symmetry to a perpendicular plane is enforced, 4 further components vanish, and one can see that it is the same as having 3 symmetry planes. Such a material is called orthotropic. Crystals, wood, composites, rolled steel and others are examples for orthotropy, even though for rolled steel orthotropy is more in the strength tensor.

9 independent parameters are needed to define an orthotropic body. As one can see, shear strains only lead to shear stresses. If the symmetry planes are not identical to the coordinate axes, one obtains a fully occupied matrix, however composed still of 9 independent parameters.

When using engineering constants for MOE, MOS and Poisson rations in the material orientations, one obtains a compact shape of the compliance matrix. Due to the symmetry conditions one can write: The matrix has to be positive definite, all main diagonal elements have to be positive, all 2nd order sub determinants have to be positive and the 3x3 sub determinant has to be positive. Additionally the following inequalities have to be fulfilled, not to violate energy balance (first law of thermodynamics).

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Elasticity law: material symmetries

Symmetry with respect to two perpendicular planes:

→ 9 independent constants
(Orthotropic body)

With engineering constants E_i, G_{ij}, ν_{ij} follows

$$E_{AB} = \begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{a}E_1 & \frac{\nu_{12}-\nu_{13}\nu_{32}}{a}E_2 & \frac{\nu_{13}-\nu_{12}\nu_{23}}{a}E_3 & 0 & 0 & 0 \\ & \frac{1-\nu_{31}\nu_{13}}{a}E_2 & \frac{\nu_{23}-\nu_{21}\nu_{13}}{a}E_3 & 0 & 0 & 0 \\ & & \frac{1-\nu_{12}\nu_{21}}{a}E_3 & 0 & 0 & 0 \\ & & & G_{23} & 0 & 0 \\ & sym. & & & G_{31} & 0 \\ & & & & & G_{12} \end{bmatrix}$$

with $a = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31}$

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By re-inversion of the compliance matrix one obtains the stiffness matrix in engineering constants here given for the sake of completeness.

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Elasticity law: material symmetries

Rotation symmetry with respect to one axis:

→transversal isotropic material

Example rotational symmetry with the x_3 axis.

$$E_{i'j'k'l'} = a_{i'i} a_{j'j} a_{k'k} a_{l'l} E_{ijkl}$$

$$a_{i'j} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Symmetry: $E_{AB} = E_{BA}$

→ 5 independent constants

$$E_{AB} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ & E_{11} & E_{13} & 0 & 0 & 0 \\ & & E_{33} & 0 & 0 & 0 \\ & & & E_{44} & 0 & 0 \\ & sym. & & & E_{44} & 0 \\ & & & & & \frac{1}{2}(E_{11} - E_{12}) \end{bmatrix}$$

$$E_{11} = E_{22}$$
$$E_{13} = E_{23}$$
$$E_{44} = E_{55}$$
$$E_{66} = \frac{1}{2}(E_{11} - E_{12})$$

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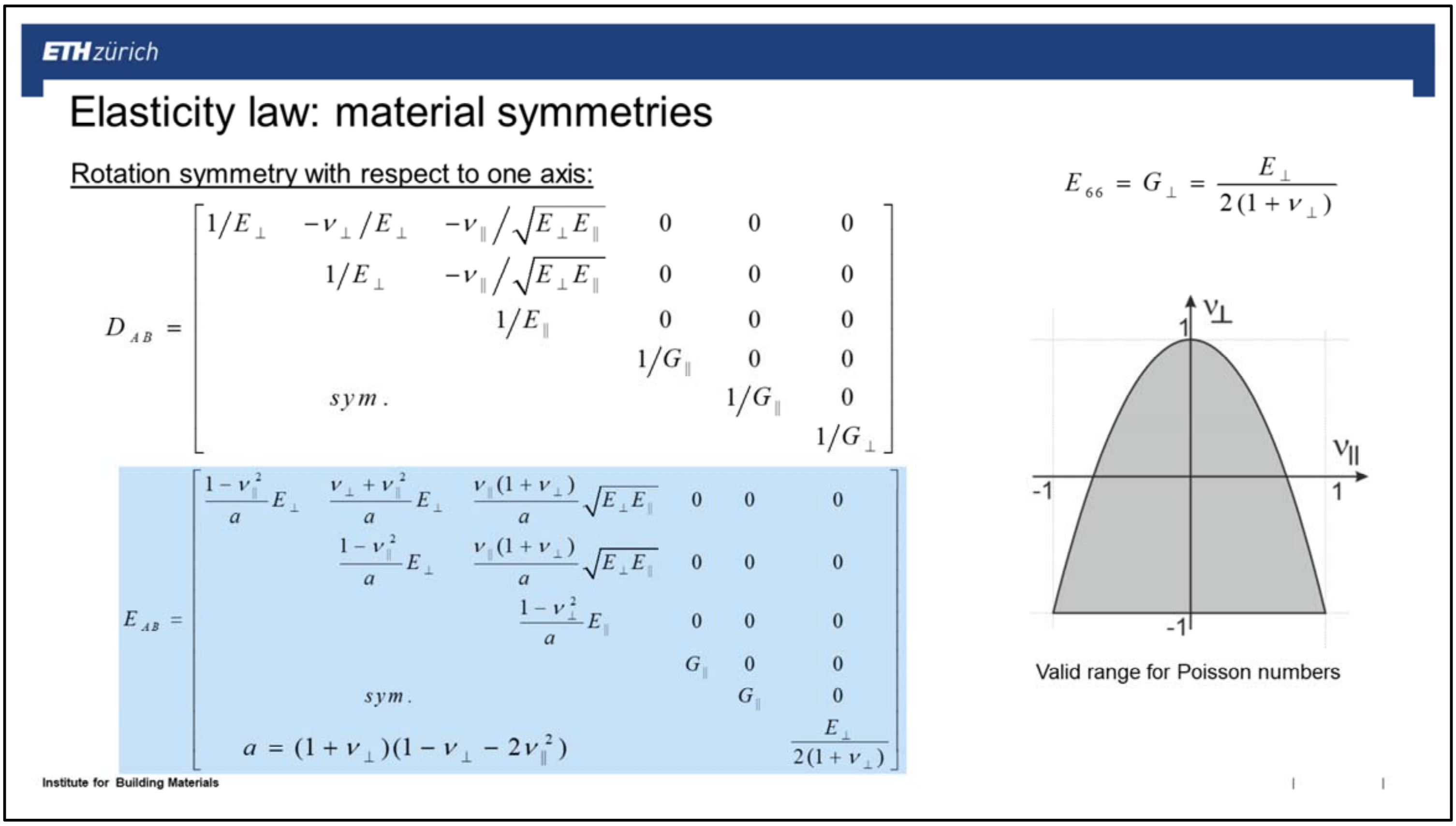
Rotational symmetric behavior with respect to one axis is given e.g. for a fiber bundle model. We do not make a derivation of the individual constants here. If you are interested, take a good mechanics book. Transverse materials are described by 5 independent constants:



If we call all elements parallel to the isotropy axis (3) as II and the perpendicular ones as _I_, then the compliance matrix can be written as:

Its inversion again leads to the stiffness tensor. One can see that the matrix population is identical to the one for orthotropic material. Again the request for positive definite elements limits the range of material parameters. The valid range is plotted here.

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Elasticity law: material symmetries

Rotation symmetry with respect to two axes:

→isotropic material

$$E_{11} = E_{33}$$
$$E_{12} = E_{13}$$
$$E_{44} = E_{66}$$

$$E_{AB} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0 \\ & E_{11} & E_{12} & 0 & 0 & 0 \\ & & E_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(E_{11}-E_{12}) & 0 & 0 \\ & sym. & & & \frac{1}{2}(E_{11}-E_{12}) & 0 \\ & & & & & \frac{1}{2}(E_{11}-E_{12}) \end{bmatrix}$$

Invariance with respect to rotation about an additional coordinate axis already results in isotropy. This additional constraint can be enforced by coordinate exchange from the simple rotational symmetry. …. The stiffness matrix gets a very simple shape: For isotropic material only 2 material constants are important, or have to be experimentally obtained, what significantly simplifies the experimental campaign, since they can be even obtained from 1 material test.

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Elasticity law: material symmetries

Rotation symmetry with respect to two axes:

→isotropic material

$$D_{AB} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ & 1/E & -\nu/E & 0 & 0 & 0 \\ & & 1/E & 0 & 0 & 0 \\ & & & 1/G & 0 & 0 \\ & sym. & & & 1/G & 0 \\ & & & & & 1/G \end{bmatrix} \quad \text{with} \quad G = \frac{E}{2(1+\nu)}$$

with

$$E > 0, \quad -1 < \nu < \frac{1}{2}$$

$$E_{AB} = \begin{bmatrix} (1-\nu)E/a & \nu E/a & \nu E/a & 0 & 0 & 0 \\ & (1-\nu)E/a & \nu E/a & 0 & 0 & 0 \\ & & (1-\nu)E/a & 0 & 0 & 0 \\ & & & E/a(1+\nu) & 0 & 0 \\ & & sym. & & E/a(1+\nu) & 0 \\ & \text{with} \quad a = (1+\nu)(1-2\nu) & & & & E/a(1+\nu) \end{bmatrix}$$

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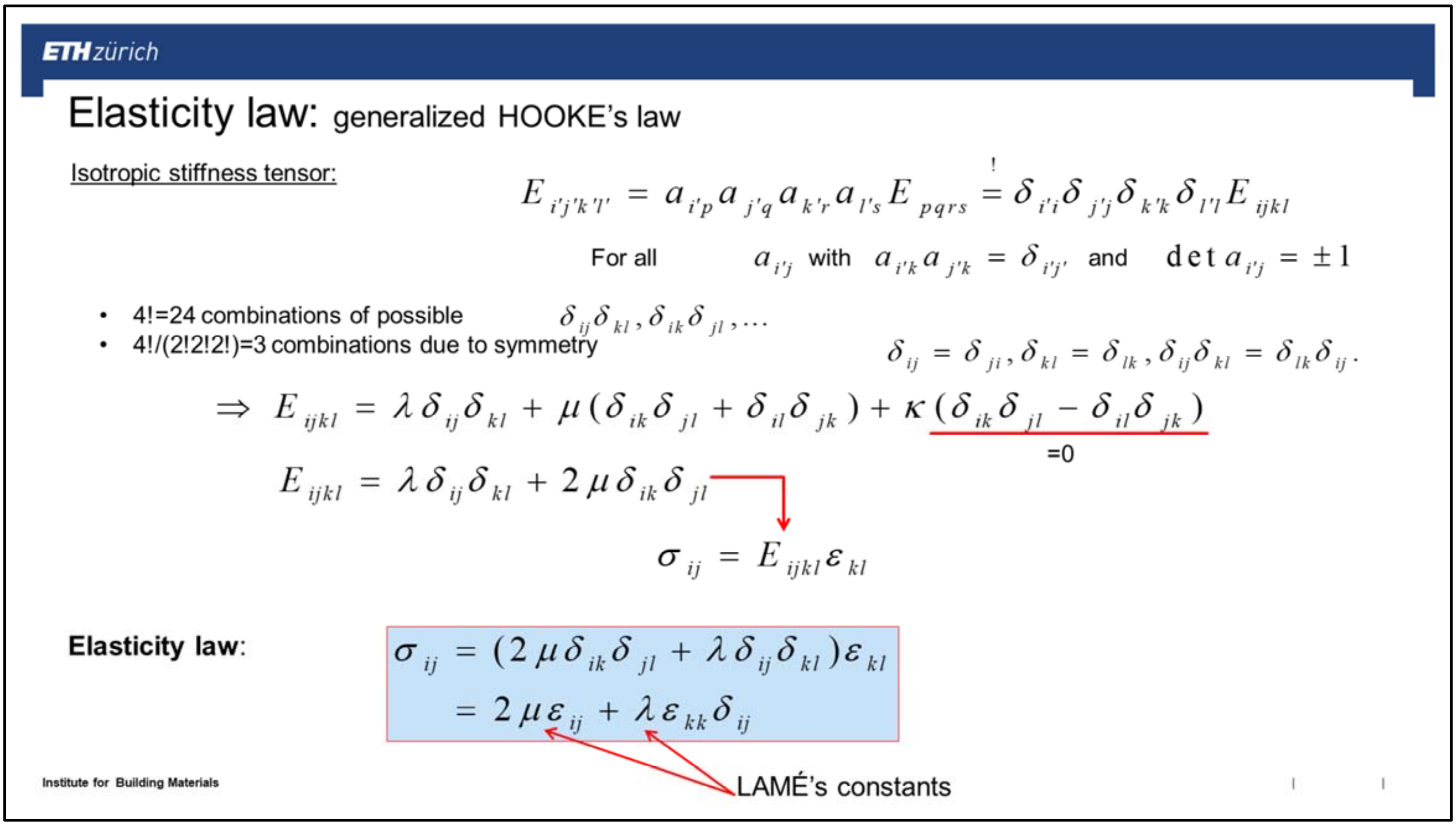
The compliance matrix in engineering constants E,G, nu has again to be positive definite, what leads to the conditions…. Again the stiffness matrix in engineering constants can be obtained by the inversion of the compliance matrix and reads… Of course there are crystals with other crazy symmetry conditions like tetragonal, trigonal, cubic and so on. However the chance that you will meet them in an engineering environment is rather low.



Let's stay with an isotropic material for some time. We require for isotropic bodies that components of the stiffness tensor are identical in all orthonormal coordinate systems. For an isotropic 4th order tensor this means: …..
All components in the rotated CSYS have to be identical to the ones in the initial CSYS. Our E_ijkl can hence only be from different combinations of delta_ij, delta_kl (with index variation there are 4!=24 possibilities). By symmetries this reduces to 3 and hence the form of the tensor. This is the most general form of an isotropic 4th order tensor. Since E_ijkl has to be symmetric with respect to the indices i,j, and k,l, the bracket with kappa is cancelled out and one obtains the stiffness tensor that comprises 2 free constants, that are called Lamé constants. If we relate this to the kinematic and kinetic properties, one obtains the elasticity law. If the strain tensor is known, we can now calculate the stress tensor.

Elasticity law: generalized HOOKE's law

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

Trace: $\sigma_{ii} = 2\mu\varepsilon_{ii} + \lambda\varepsilon_{kk}\delta_{ii} = 2\mu\varepsilon_{ii} + 3\lambda\varepsilon_{kk} = (2\mu + 3\lambda)\varepsilon_{ii}$

$$\Rightarrow \varepsilon_{kk} = \frac{\sigma_{kk}}{2\mu + 3\lambda}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda\delta_{ij}}{2\mu(3\lambda + 2\mu)}\sigma_{kk}$$

Uniaxial stress state:

$$\sigma_{11} = \sigma_0; \quad \text{all other } \sigma_{ij} = 0$$

$$\varepsilon_{11} = \frac{1}{E}\sigma_0; \quad \varepsilon_{22} = -\frac{\nu}{E}\sigma_0;$$

$$\varepsilon_{11} = \frac{1}{E}\sigma_0 = \frac{1}{2\mu}\left(1 - \frac{\lambda}{3\lambda + 2\mu}\right)\sigma_0 = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma_0$$

$$\varepsilon_{22} = \frac{\nu}{E}\sigma_0 = \frac{1}{2\mu}\left(0 - \frac{\lambda}{3\lambda + 2\mu}\right)\sigma_0 = \frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_0$$

$$\Rightarrow \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}; \quad \mu = \frac{E}{2(1+\nu)}$$

Plain shear stress:

$$\sigma_{12} = \sigma_{21} = \tau_0; \quad \text{all other } \sigma_{ij} = 0$$

$$\gamma_{xy} = 2\varepsilon_{12} = 2\frac{1}{2\mu}\sigma_{12} = \frac{\tau}{\mu} = \frac{\tau}{G}$$

$$\Rightarrow \mu = G$$

$$G = \frac{E}{2(1+\nu)}$$

If we want to calculate the strains for a given stress state, we have to invert the elasticity law. This means we have to solve the 6x6 equation system. Let's first look at the trace of the stress tensor. After renaming i to k we obtain a relation for epsilon_kk, that we can insert into the elasticity law to obtain epsilon_ij.

However since the LAME's constants are a little bit abstract, we want to express them in engineering constants like MOE, MOS, Poisson's number nu and bulk modulus K. To do this, we only look at two well defined stress states: uniaxial tension and a plain shear stress state. By equating coefficients (Koeffizientenvergleich) we obtain the Lame's constant. Don't forget about the technical shear strain gamma, that is twice the mathematical one. As we can see, E, G, nu are not independent from each other.

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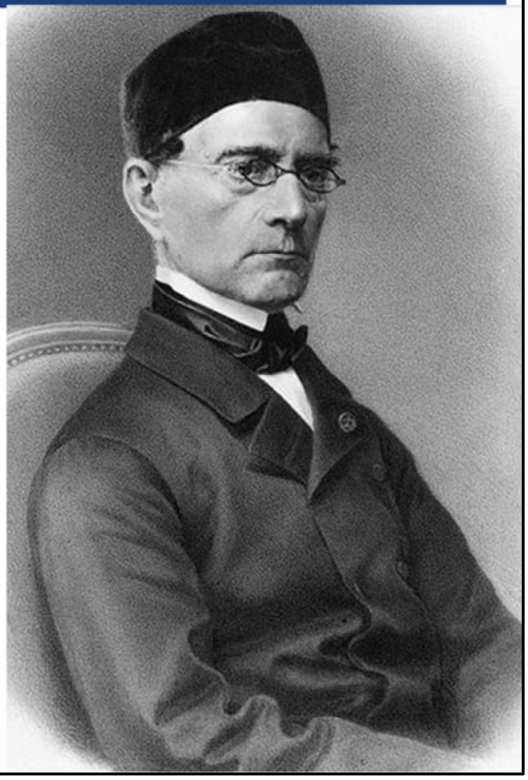

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Gabriel Lamé (1795-1870)

Gabriel Lamé (1795-1870) graduated in 1817 from the Ecole Polytechnique in Paris. He continued his studies at the famous Ecole des Mines, where he graduates in 1820 with a second degree. Also at that time technological development work was common, and the went the same year to Russia, to become director in St. Petersburg of the school for Road and transportation. He teaches in civil engineering but also does practical work in road and bridge construction in Russia. In 1832 he returns to Paris and first founds an engineering bureau. He gets the chair for Physics at the Ecole Polytechnique but remains active outside of academia as counselor and chief engineer in mine questions and railway constructions. His scientific work shows his deep love for applied math. However he also looks at abstract topics like number theory. According to Gauss, Lame is the most influential mathematician of his time. However his colleagues think different. The mathematicians think he is too practical and for natural scientists he is too theoretical. Hence we can consider him to be one of the really great minds that could bridge the worlds.

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Elasticity law: generalized HOOKE's law

	$\lambda =$	$\mu =$	$E =$	$\nu =$	$K =$	$G =$
(λ, μ)			$\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$	$\frac{\lambda}{2(\lambda + \mu)}$	$\lambda + \frac{2}{3\mu}$	
(G, K)	$K - \frac{2}{3}G$		$\frac{9KG}{3K + G}$	$\frac{3K - 2G}{6K + 2G}$		
(E, ν)	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$

Bulk modulus:

$$e = \varepsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu} = \frac{3\sigma^H}{3\lambda + 2\mu} \overset{!}{=} \frac{\sigma^H}{K}$$

$$\Rightarrow K = \frac{1}{3}(3\lambda + 2\mu) = \frac{E}{3(1-2\nu)}$$

Material law:

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

$$\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right)$$

with: $E > 0, \quad -1 < \nu < \frac{1}{2}$

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The table summarizes the relations of elasticity constants and Lame constants. The bulk modulus is the proportionality factor between the hydrostatic stress and the volumetric dilatation e. With these relations, we can write the material law in engineering constants:

The material law is valid and exactly the same for arbitrary orthonormal CSYS (e.g. cylinder coordinates (R-phi-z) or spherical coordinates (r-phi-theta)), only the indices have to be exchanged (1-2-3 by r-phi-z, resp. r-phi-theta).

Elasticity law:

Reformulate the elasticity law with decomposed hydrostatic and deviatoric part.

$$\sigma_{ij}^{D} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$\boxed{\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}} \qquad \boxed{\sigma_{kk} = \varepsilon_{kk}(2\mu + 3\lambda)}$$

$$= 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \frac{1}{3}(3\lambda + 2\mu)\varepsilon_{kk}\delta_{ij}$$

$$\boxed{\varepsilon_{ij} = \tfrac{1}{3}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}^{D}}$$

$$= 2\mu(\varepsilon_{ij}^{D} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}) + \lambda\varepsilon_{kk}\delta_{ij} - \frac{2}{3}\mu\varepsilon_{kk}\delta_{ij} - \lambda\varepsilon_{kk}\delta_{ij}$$

$$= 2\mu\varepsilon_{ij}^{D}$$

$$\boxed{\sigma_{ij}^{D} = 2\mu\varepsilon_{ij}^{D} = 2G\varepsilon_{ij}^{D}}$$

$$\varepsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu} = \frac{\sigma^{H}}{K} = \frac{\sigma_{kk}}{3K}$$

$$\boxed{\Rightarrow \sigma_{kk} = 3K\varepsilon_{kk}}$$

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Hence follows a particularly simple and easy to invert form of the elasticity law.

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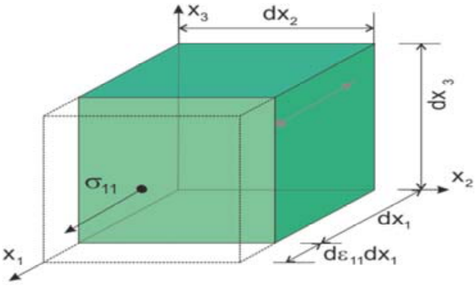
Elastic potential, strain energy

Force · displacement = work

$$\underbrace{\sigma_{11} dx_2 dx_3 \cdot d\varepsilon_{11} dx_1}_{\text{volume}} = W$$



General work increase per volume:

$$dW = \sigma_{11} d\varepsilon_{11} + \sigma_{12} d\varepsilon_{12} + \sigma_{13} d\varepsilon_{13} + \sigma_{21} d\varepsilon_{21} + \ldots$$
$$= \sigma_{ij} d\varepsilon_{ij}$$

Specific work of deformation:
(path independent for elastic body)

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\bar{\varepsilon}_{ij} = U(\varepsilon_{ij})$$

$$W = \int_0^{\varepsilon_{ij}} dU = U(\varepsilon_{ij}) \quad \text{with} \quad \sigma_{ij} d\varepsilon_{ij} = dU = \frac{\partial U}{\partial \varepsilon_{ij}} d\varepsilon_{ij}$$

Constitutive equation:

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$$

Linear elastic:

$$U = E_{ijkl} \int_0^{\varepsilon_{ij}} \bar{\varepsilon}_{kl} d\bar{\varepsilon}_{ij} \Rightarrow U = \frac{1}{2} E_{ijkl} \varepsilon_{kl} \varepsilon_{ij} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

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The deformation of an elastic body results in internal forces. Hence work is done. Let’s look at this using the differential element. The stress sigma_11 corresponds to the force sigma_11*dx2*dx3, the path is d epsilon_11*dx1, while the volume is simply dx1*dx2*dx3 and can be used to obtain volume specific work increase. We can do the same to all other components and obtain the work increase per volume. If we integrate the work from the undeformed to the deformed final state epsilon_ij, one obtains the specific work of deformation or density of the work of deformation, since it is related to the volume. If the body is elastic, it does not matter, how this state was reached and the values of the work integral is only dependent on initial and final state. Hence it does have the character of a potential. This is only possible if the integrand dW (dW) is a complete differential, and one writes…..

By equating coefficients one obtains the constitutive equation. Analogous to a conservative force, stresses in the elastic case can be derived from the potential. This is why U is also called specific strain energy or specific elastic potential or elastic potential density. In the linear elastic case the integration along the line with t*epsilon_ij results in a triangular area.

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Elastic potential, strain energy

Decomposition into volumetric and deviatoric part:

$$\sigma_{ij} = \frac{\sigma_{kk}}{3}\delta_{ij} + s_{ij}$$

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\frac{\sigma_{kk}}{3}\delta_{ij} + s_{ij}\right)\left(\frac{\varepsilon_{ll}}{3}\delta_{ij} + e_{ij}\right)$$

$$= \frac{1}{6}\sigma_{kk}\varepsilon_{ll} + \frac{1}{2}s_{ij}e_{ij} = U_v + U_g$$

U_v: Volume change energy; U_g: Distortion energy

Isotropic case:

$$U_v = \frac{K}{2}\varepsilon_{kk}\varepsilon_{ll} = \frac{K}{2}\varepsilon_v^2, \qquad U_g = G e_{ij}e_{ij}$$

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The decomposition of the stress and strain into volumetric and deviatoric part leads to the volume change energy (Volumenänderungsenergie) and distortion energy (Gestaltänderungsenergie). As one can see, both are always positive for arbitrary deformations.

Elasticity law: Generalized HOOKEs law

PE:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \varepsilon_{\alpha\beta} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}$$

PS:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \sigma_{\alpha\beta} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$G = \frac{E}{2(1+\nu)}, \quad \kappa \begin{cases} 3 - 4\nu & \text{EVZ} \\ \dfrac{3-\nu}{1+\nu} & \text{ESZ} \end{cases}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2G}\left(\sigma_{\alpha\beta} + \frac{\kappa - 3}{4}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

$$\varepsilon_{11} = \frac{1}{8G}\left((\kappa+1)\sigma_{11} + (\kappa-3)\sigma_{22}\right),$$

$$\varepsilon_{22} = \frac{1}{8G}\left((\kappa-3)\sigma_{11} + (\kappa+1)\sigma_{22}\right), \quad \varepsilon_{12} = \frac{1}{2G}\sigma_{12}$$

$$\sigma_{\alpha\beta} = 2G\left(\varepsilon_{\alpha\beta} - \frac{1}{2}\frac{\kappa-3}{\kappa-1}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

$$\sigma_{11} = \frac{G}{\kappa-1}\left((\kappa+1)\varepsilon_{11} + (3-\kappa)\varepsilon_{22}\right),$$

$$\sigma_{22} = \frac{G}{\kappa-1}\left((3-\kappa)\varepsilon_{11} + (\kappa+1)\varepsilon_{22}\right), \quad \sigma_{12} = 2G\varepsilon_{12}$$

$$\overline{U} = G\left(\varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta} - \frac{1}{2}\frac{\kappa-3}{\kappa-1}\varepsilon_{\alpha\alpha}\varepsilon_{\beta\beta}\right)$$

$$\overline{U} = \frac{1}{2}\frac{G}{\kappa-1}\left((\kappa+1)\left(\varepsilon_{11}^2 + \varepsilon_{22}^2\right) + 2(3-\kappa)\varepsilon_{11}\varepsilon_{22} + 4(\kappa-1)\varepsilon_{12}^2\right)$$

$$\sigma_{33} = \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11} + \varepsilon_{22}\right)$$

$$\varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right)$$

The isotropic elasticity law for plane problems can be given by using instead of E and nu the shear modulus and the constant kappa. For the sake of completeness, the equations are given here without derivation. Be aware that the stress state results in a three dimensional strain state and vice versa.

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Problem description

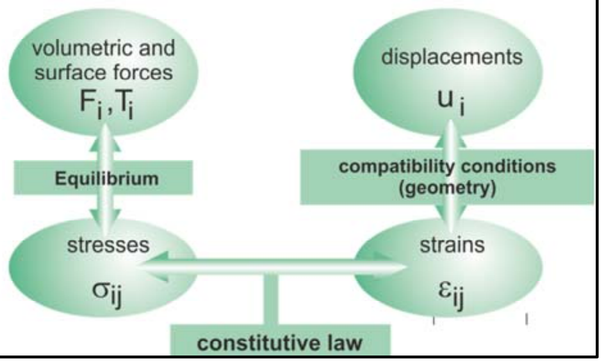
Equilibrium:	$\sigma_{ij,i} + f_i = 0$	3 equations
Kinematics:	$\varepsilon_{ij} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right)$	6 equations
Material law:	$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$, $\varepsilon_{ij} = D_{ijkl}\sigma_{kl}$	6 equations

Field properties (15):

$$u_i, \varepsilon_{ii} \text{ und } \sigma_{ij} \quad \left(\sigma_{ij} = \sigma_{ji}, \varepsilon_{ij} = \varepsilon_{ji}\right)$$

Compatibility conditions (6):

$$\eta_{ij} = \varepsilon_{ikm}\varepsilon_{j\ln}\varepsilon_{kl,mn} = 0$$



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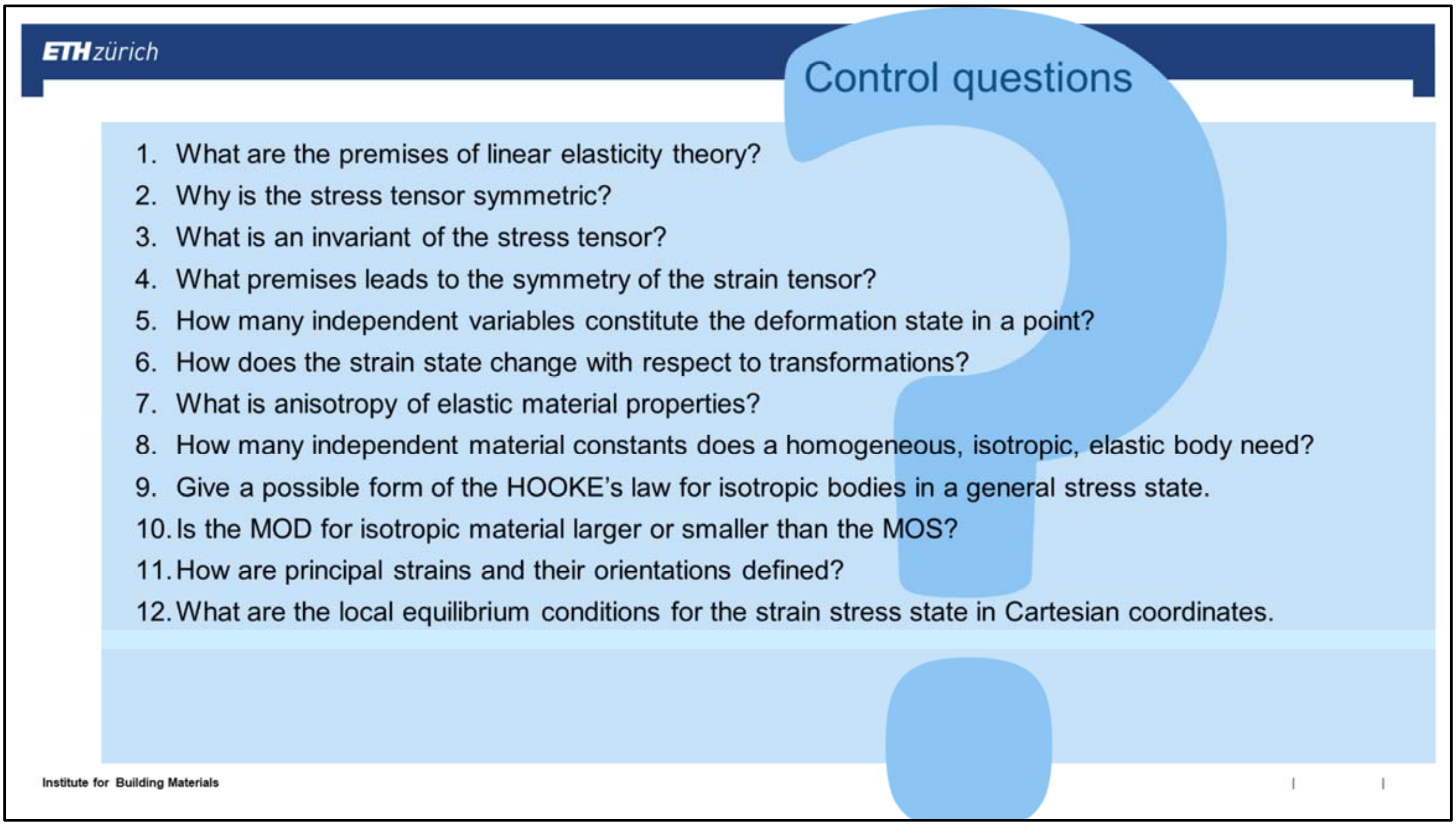
Let's summarize what we talked about so far on linear elasticity. The sum of equilibrium conditions, the kinematic equations and the 6 material laws give 15 field equations that are facing 15 unknown field equations.

All equations are linear. The material law are the projection of a stress onto a strain. The static and kinematic relations are first order partial differential equations. When integrating partial differential equations integration functions appear, that have to be determined by the boundary conditions. The solution of this boundary value problem can be made via FEM, what is not part of this lecture.

Of course we can more deeply dig into this problem, however our real interest would be lost, since we want to focus on material laws.



1. Lin el. Material behavior→ DGL 1. order
2. Moment equilibrium
3. Property for changes of CSYS
4. Potential character (order of derivations does not matter)
5. 3 displacements
6. Not a t all
7. Directional dependency
8. 2
9.
10. Larger
11. Like principal stresses

Thank you for your attention.