Mechanics of Building Materials

Failure criteria for concrete

F. Wittel
To model the behavior of steel reinforced concrete, it is necessary to understand three fields:

1. **Constitutive behavior of concrete**: For the description of the complicated behavior of concrete under different stress combinations, there exist a multitude of approaches, that can be grouped in this way:
   1. Representation of given stress-strain curves by curve fitting methods, interpolation or mathematical functions. Uniaxial and quasi-uniaxial models; multi axial models, multi-parameter models etc.
   2. Models based on linear or non-linear elasticity theory
   3. Models based on perfect or work-hardening plasticity theory.

2. **Plastic behavior of the steel reinforcement**: thin cross sections $\rightarrow$ only axial load $\rightarrow$ uniaxial plasticity works quite good, in its simples form ideally linear elastic, perfectly plastic ignoring Bauschinger effects.

3. **Composite behavior**, hence the pull-out behavior between steel and concrete, that can be quite complicated. In practical applications this is often considered as a perfect compound.

But for now, let’s have a look, why the behavior of concrete is such a complicated story.
As always, before you model, you need to understand the system behavior, here the mechanical behavior for one – two and three-dimensional stress states. This is not only an important step for selecting the relevant phenomena for the mathematical model, but also helps for the parameter identification of the constitutive model.

First we will look at unreinforced concrete and this is already rather complicated. In principle we are dealing with a so-called cohesive frictional material that consists of rather stiff particles or aggregates in diverse shapes and sizes, glued together by a cementitious matrix. In general the aggregate content is that high, that one obtains a random granular packing, and forces are transmitted at contact points, resulting in a complicated force network. But even if it would be less particles in a mortar matrix, the stress field is far from simple, as we see on the right side. Additionally cement undergoes enormous shrinkage upon hydration and drying resulting in excessive residual stresses. Often those are well above the strength of the cement matrix and the cement-aggregate interphase. In particular they are reduced by shrinkage, however excessive micro-cracking is observed, additionally complicating the 3D stress states that are present and for us hard to imagine. To make things worse, production induced defects, e.g. by the flow direction upon production result in anisotropic behavior.

Under load cracks behave differently under tension and compression. While in compression, cracks simply close and are capable of transmitting large compressive stresses, tension leads to crack opening. At first this is elastic, but for higher stress, micro-cracks can start to grow. Of course micro cracks can not only grow under the tensile mode, but also under the two shear modes. To conclude, concrete is so
complicated, that up to date, not sufficient representative volumes can be calculated. Hence we are left to making phenomenological considerations of larger material volumes under load.
In general concrete is brittle under tension and has limited deformability under compression. Typical values for strength under tension are 8-12 times smaller than under compression. A similar ratio hold for the ultimate strains. Under compression 0.25% are normal, then one enters the instable regime of strain softening up to a strain of about 0.35%. The brittle behavior under tension does not mean, that linear-elastic behavior rules. It can be strongly non-linear elastic and is dominated by the growth of a dominant crack. Under compression however multiple cracks form and develop instantaneously perpendicular to the maximum principle stress. As we remember the MOE is for concrete the secant modulus under compression at a loading and unloading between the lower stress of 0.5MPa and an upper one of 1/3 of the cube compressive strength.

When we look at the longitudinal and transversal strain behavior of a concrete sample under uniaxial compression, we realize, that the curves are strongly determined by the mechanisms of internal crack growth.

1. The axial compression strain curve exhibits an almost linear behavior up to about 33% of the cube compressive strength \( f_c \). From about 30% on, the energy is sufficient to drive local crack growth and thus marking the end of the elastic regime. In the regime from 30-50% \( f_c \) the interphase cracks between aggregate and mortar grow and one observed progressing aggregate debonding. This is followed by increasing softening behavior up to 80% \( f_c \) where fabric destruction sets in, best to be observed by looking at the volumetric strain (turning point in the strain curve). Fabric destruction results in stronger softening up to the point of failure with the cube strength. From about 75% \( f_c \) cracks reach critical length and in the instable
phase cracks can grow autonomously.

2. A constant Poisson's number corresponds to a line in the volumetric strain. This is observed for up to 65%\(f_c\) before it starts to deviate and finally the material dilates. The reason is from about 80%\(f_c\) the emergence of small cracks parallel to the load line that result in increased transverse strain (\(\nu_{\text{concrete}} \sim 0.18\), metal \(\sim 0.33\)). On the right side of the origin we plot volume decrease, on the left increase. The stress corresponding to the smallest value of volumetric strain is called critical stress.

3. The behavior under tension is similar to the one under compression, only with strength values being significantly lower with \(f_t=5-10\%f_c\).
The shape of the stress-strain curve is similar for different concrete qualities (low, mean, high strength concrete). The linear regime is however more pronounced for high strength with respect to low strength concrete. It is interesting to note that all maxima are around a strain of 0.2%. The softening for high strength concrete is typically more pronounced, since it behaves more brittle.

There are numerous equations for describing the compression-strain behavior of concrete. In general those are fits to experimental data. One example is the fit following the CEB (Comitee Euro-International du Beton) model. Note that this relation does not really work good for the post-peak behavior. This regime, that is for example important for calculating concrete structures needs more complicated approaches.
When engineers realized that multi-axial stress states have a significant influence on strength, they started to develop strength envelopes for different biaxial stress states. Here you see the envelopes for 3 different concrete types. When normalized by their respective cube compression strength, they collapse. The reason is that they share common failure modes like failure under tension in crack planes perpendicular to the maximum principal stress orientation. Hence superimposed tensile stresses have an enormous significance on the failure criteria and mechanism. For different combinations of strength components one observed different damage models.

The envelope for concrete is relatively independent on the loading history, hence the principal stress criterion can work. The maximum compressive strength can be reached, when a compressive stress is applied perpendicular to the main compressive stress that closes the cracks. Increase of 25% for sig1/sig2=0.5 and 16% for sig1/sig2=1. At biaxial compressive-tensile stress, the compressive strength is observed to decrease almost linearly, while tensile-tensile stress is similar to uni-axial tensile failure.
After strength envelopes were made, researchers focused on the stress-strain curved and made all kinds of strain controlled tests at various load combinations.
A widely used function for describing the stress-strain relation for uni-axial and quasi uni-axial (biaxial) stress states is the relation following Liu, Nilson, Slate. The fundamental concept of the model is the reduction of the biaxial stress state in a quasi-uniaxial, equivalent state. The principal stress ratio is used to reduce the tangent modulus, what is supposed to consider the effect of the biaxiality. The model works quite well for constant Poisson’s numbers, hence up to the fabric failure (80%f_c) and for plane problems like beams, thin shells, plates. Large deformations and 3D stress states can however not be captured by such a simple model.
When we look at the mechanical behavior, it gets quickly evident, that the strength behavior of concrete for multi-dimensional stress states is a function of the stress state and not just a simple, uncoupled limitation of a simple tensile, compressive or shear stress. A correct prediction therefore needs the interaction of the different components of the stress state.

The shape of the failure envelope for concrete is best looked at in the principal stress state in form of intersections with the deviatoric plane and the meridian plane. Those are the planes that comprise the hydrostatic axis.
But what do we call failure? Is it creep, crack initiation, the ultimate stress or deformation? We take failure as the ultimate load that can be transmitted by a material. Additionally we assume concrete to be initially isotropic, in other words production induced anisotropy is neglected. As we already know, failure criteria have to be invariant with respect to a given coordinate system. For many isotropic materials the failure condition can be written in terms of the principal stresses. For multi-axial stress states it is difficult to make a physical (geometrical) interpretation of failure on this basis. The decomposition of stress tensors into volumetric and deviatory parts and the use of respective invariants $I_i$ and $J_i$ simplify this interpretation. Additionally the use of invariants results also in invariant constitutive relations.

In general we distinguish (for concrete) between tensile and compressive failure. Tensile failure is rather brittle and defined by the initiation and growth of one dominant crack, while for compressive failure many cracks grow simultaneously, degrading the material volume as such. So let’s look at these invariants to get to their physical interpretation.
Let's repeat: The stress tensor can be decomposed into a hydrostatic (spherical) and a deviatoric component. All properties \( \sigma_i \), \( I_i \), \( J_i \) are scalar invariant with respect to the CSYS. For failure criteria mainly three invariants \( I_1 \) (pure hydrostatic stress), \( J_2 \) and \( J_3 \) (invariant for pure shear) are important.
Let’s look at the octahedral plane. This is the cutting plane perpendicular to the principal diagonal of the principal stress CSYS. Hence there exist 8 of those. In this cutting plane we can define a normal and a shear stress. The direction of the shear stress is defined by the similarity angle that can be expressed in terms of the two invariants J2, J3 of the deviatoric stress. The stress state can thus not only be described by the invariants I1, J2, J3 but alternatively via the invariant sets I1, J2, Theta, or sigma_oct, Tau_oct, Theta. (theta=0→sig1=sig2; theta=60°→sig2=sig3)

The main aim is a better physical interpretation of the invariants. However there are also practical reasons, since the calculation of the principal stresses sigma_i do not need the solution of the characteristic cubic equation sigma^3-I1sigma^2+I2sigma-I3=0 but can be solved by:

\[
\sigma_{oct} = \frac{1}{3} I_1 = \sigma_m; \quad \tau_{oct} = \sqrt{\frac{2}{3} J_2}
\]

Similarity angle θ (Lode angle):

\[
\cos 3\theta = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{1/2}}, \quad \text{mit } 0 \leq \theta \leq \frac{\pi}{3}
\]

\[
\sigma_{oct} - \tau_{oct} - \theta \quad \text{space:}
\]

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix} = \begin{bmatrix}
\sigma_{oct} \\
\sigma_{oct} \\
\sigma_{oct}
\end{bmatrix} + \frac{2}{\sqrt{3}} \sqrt{J_2} \begin{bmatrix}
\cos \theta \\
\cos \left(\theta - \frac{\pi}{3}\right) \\
\cos \left(\theta + \frac{\pi}{3}\right)
\end{bmatrix}
\]

- Clearer physical interpretation
- Simplified calculation of principal stresses
Imagine an infinitesimal spherical volume element. On each surface point there acts a stress vector with a normal and a shear component. By integrating over the surface, we obtain the mean normal stress $\sigma_m$. For the shear stress this is a little bit more complicated, since at each point on the surfaces shear stresses can have different orientations. Since the orientation of the stress has no influence on the physical failure mechanism, the mean value of the orientations can be considered by the expression...

Hence the mean stress can be used to interpret the invariants. $I_1$ is obvious, and $J_2$ is directly connected to the mean shear stress. We already got to know the hydrostatic axis in the principal stress space, know that the deviatoric plane is perpendicular to it and that the deviatoric plane that contains the origin is called pi-plane. Each stress vector in the pi-plane is a pure shear stress state without hydrostatic stress component. Each stress vector OP can thus be decomposed in a hydrostatic part CHI (ON) and a deviatoric part rho (NP). Those are the first two Haigh-Westergaard coordinates. The angle theta is defined by the projection of the sigma1 axis onto the vector NP. This is the Lode angle that gives the ratio of the mean principal stress with respect to the largest and smallest principal stress. If $\sigma_2=\sigma_3$, the Lode angle is 60°, if $\sigma_1=\sigma_2$, the Lode angle is 0°. Hence it is an indicator of the magnitude of the middle principal stress with respect the min and max principle stress. The plane that is defined by the Chi-rho vectors and that contains the hydrostatic axis is called meridial plane.
Many constitutive models do not use stress but strain invariant, that we just summarize for completeness. Since the stress tensor is like the strain tensor a second order tensor, everything is similar with principal strains, invariants, octahedral ones.
Let’s get back to the constitutive relation for concrete. If a constitutive law is formulated in terms of invariants, it is independent on the used coordinate system. Additionally the curve has to be convex and continuously differentiable. In the deviatoric plane, the failure curve has three-fold symmetry, as long as concrete behaves isotropic. This is logical, since the indices 1,2,3 can be arbitrarily exchanged. This is very practical, since we only have to examine the regime 0<\theta<60 experimentally and the rest comes from the symmetry condition. For low hydrostatic values, the curve is almost triangular, while for larger hydrostatic stress, the failure curve is almost circular. This means that the dependence of the failure curve from the middle principal stress vanishes with increasing hydrostatic stress.

The plane that is given by the chi-rohv vectors and contains the hydrostatic axis is called meridian plane. The shape of the failure curve in this plane describes how the deviatoric stress \rho, that can be taken by the material changes as result of the hydrostatic stress. The extremal lode angles are called tensile (0°) (for pure tensile load) and compressive (60°) (for pure compressive load) meridians. They are essential in the experimental determination of the failure envelope. From experiments one can see that meridians are curvilinear and convex. Since concrete in principle does not fail under hydrostatic compression, the hydrostatic axis is not intersected. Note that sometimes a shear meridian is used, what corresponds to a Lode angle of 30°. This is pure shear stress \[\sigma_1, (\sigma_1+\sigma_3)/2, \sigma_3\], superimposed by the hydrostatic stress \[\sigma_m=1/2(\sigma_1+\sigma_3)\].
Failure criteria for concrete – experimental evaluation

Non-linear function in meridian plane
Non-affine intersection curve in the deviatory plane as function of $\xi$
Change from triangular to circular shape with increasing volumetric part
In principle the failure bodies for concrete look all alike, but the real behavior of concrete is relatively complicated. Additional to many other factors, the mechanical properties of aggregate, mortar, production and load-induced damage are significant, resulting in a large variability of the failure details. For us this means, that drastic idealization are needed for the mathematical model of the strength characteristic of concrete. Consequently not only one model exists for describing the behavior for concrete under all possible conditions. Even if such a model would exist, it would be so complicated that it could not be used for the stress analysis of practical problems. Hence simplifying models and criteria have to be used to describe relevant characteristics.

Already in the first hour we saw different failure hypotheses. The most known are .... It is not like one can only use one criterion, one can combine them as well. An example is concrete where under tension one can use the Rankine criterion, while compression is described by a v. Mises or Coulomb criterion. Due to the complicated situation in concrete and its importance as building material, many criteria were proposed in the stress space. They can have from 1 to 5-7 independent material parameters.
For small tensile stresses and small hydrostatic pressure, concrete fails by cleavage cracking. A simple model for brittle failure is the Rankine criterion, hence the failure under maximum tensile stress, as soon as the principal stress somewhere in the material reaches the tensile strength. This is independent from the magnitude of other stress components. The tensile strength can be measured in experiments. In the principal stress space the criterion is a simple cube. Hence in the deviatoric plane it is a triangle and in the meridional one a line with different slope for tension (theta=0°) or compression (theta=60°). The failure plane is simply called tension cutoff.

For higher hydrostatic stress, shear stress criteria like the von Mises or Tresca criterion become interesting to predict the ductile failure. For the von Mises criterion one uses the octahedral shear stress, hence the radius of the circle in the deviatoric plane, while the Tresca criterion uses the maximum shear stress. The parameter k is the yield stress under pure shear and is obtained from compressive tests and the resulting compressive strength. The criteria are independent from the hydrostatic stress, what has limited validity for concrete. Nevertheless early implementations used the von Mises criterion for steel reinforced concrete with a tension cutoff to limit the material under tension. Since concrete is also not infinitely resistant towards compression, a compression cutoff in form of a maximum bearable compressive load is used, defined by the uniaxial compressive strength.
When we looked at the material behavior of concrete, we realized a strong sensitivity of the strength with respect to the hydrostatic stress. One-parameter models cannot capture this, since the shape of the intersection lines with deviatoric plane are not identical any more with respect to size and shape. For concrete we saw how it merges from almost triangular to circular shape for high hydrostatic pressures \( \chi \). With two parameters however this is also out of scope. But if the shape remains constant, 2-parameter models can change the size. The two simplest models are the Mohr-Coulomb and the Drucker-Prager criterion, that we already know.

The Mohr-Coulomb criterion has two material constants \( c \) and \( \phi \), one for cohesion and one as internal angle of friction. If the shear stress \( \tau \) reaches anywhere a value that depends linearly on the normal stress in that plane, the material fails. In principle stress space we obtain a hexagonal pyramid, that intersected with the deviatoric plane gives a hexagon. If \( \phi = 0 \) we obtain the Tresca criterion and \( \phi = 90 \) gives the Rankine criterion. Of course it can be again combined with other criteria like the Rankine criterion what gives a 3 parameter criterion. We will have a look at this later.

The fundamental problem of the MC criterion is that the surface is not continuously differentiable since edges exist that cause numerical problems. A smooth version is better, what leads us to the Drucker-Prager criterion. In principle it is the modification of the von Mises criterion with a hydrostatic stress. \( K \) and \( \alpha \) are positive material constants. In principle stress space the surface is a cone with the hydrostatic axis. The criterion is often used for soils.
The description of concrete failure via the Mohr-Coulomb with combined tension cutoff was used quite some time and even today it is used for the sake of simplicity in first approximations. The MC criterion with the irregular, hexagonal pyramid has the 2 parameters cohesion $c$ and internal angle of friction $\phi$. The intersection lines with meridians are lines and the one with the $\pi$ plane ($\sigma_1 + \sigma_2 + \sigma_3 = 0$) is a hexagon. The characteristic length $\rho_{t0}$ and $\rho_{c0}$ in the $\pi$-plane correspond to a Lode angle of $0^\circ$ and $60^\circ$. Consequently the length can be calculated directly.

By the Rankine criterions the additional parameter tensile strength is introduced expanding the parameter space to 3. Also here $\rho_{t0}$ and $\rho_{c0}$ can be calculated quite simple.
The combined Mohr-Coulomb criterion with tension cutoff for maximum tensile stress has three parameters $\phi$, $c$, $f_t$, that can be nicely visualized in the Mohr’s circle. The one on the right with center in O separates the Mohr-Coulomb from the Rankine part. Let’s have deeper look how failure is predicted by the criterion.

In principle we have 2 different failure types: tensile failure (crack growth) and compressive failure (shear failure). The angle of the shear plane is nicely visualized in the Mohr’s circle as intersection of the perpendicular through the mean principal stress $1/2(\sigma_1-\sigma_3)$. If all principal stresses are different, one obtains always two different orientations for the shear plane (angle between shear plane and smallest principal stress). This is visualized in the small picture on the right. For biaxial stress of equal magnitude, this is not so simple any more and one observes crack orientation perpendicular to the largest principal strain. Hence one should use a combined criterion that uses stresses as well as strains. Tensile failure however is oriented perpendicular to the largest principal stress or strain. In reality the maximum tensile strain is dependent on the compressive stress, however in most applications this is neglected and only a scalar parameter is used.

The advantages and disadvantages of the approach are...
Compared to the von Mises, the DP criterion is much better, since it considers the hydrostatic stress. However the linear relation between I1 and sqrt(J2) (rho and chi) is not correct. Also the intersection line with the deviatoric plane is not really a circle but eccentric depending on the Lode angle. To consider these details, an additional parameter is needed.

Let’s first look at a model that considers a parabolic dependence of rho and chi, respective the octahedral shear stress from the octahedral normal stress. Bresler and Pister proposed one in 1958 and it reads....

The octahedral normal stress is positive for tension and f_c is always positive (cube strength). The constants a, b, and c are three failure parameters, that can be obtained by curve fitting to experimental data. To better approximate the regime of smaller pressure (the deviatoric intersections are almost triangular) the criterion can be combined with a tension cutoff.

William and Warnke took in 1974 a different path. They focused on the tensile regime and the region of small pressure. In this regime meridians are straight, but the deviatoric intersections are far from circular. The mean normal and shear stress can be used to express the failure curve as function rho(teta), what is an elliptical curve in the regime 0<\theta<60°. The elliptical failure curve fulfills the condition of symmetry, differentialbility and convexity.

An alternative function was proposed by Argyris. The problem however is, that the convexity is only given for rho_t/rho_c>0.78 what is a narrow regime rarely met in
practice.
If one is looking for curved meridians and non-circular deviatoric intersections, more than 3 parameters are required. There are a number of 4 and 5 parameter models that share both features.

The 4-parameter model of Ottosen gets the 4 parameters $a, b, k_1, k_2$ by adjusting to different tests for uniaxial compression, tension, biaxial compression and one tri-axial test. The criterion is valid for all stress states, meridians are parabolic and deviatoric intersections non-circular. For different values of the parameters, it reduces to different types of criteria like $a = b = 0$, $\lambda = \text{const}$. Is the von Mises or for $a = 0$ and $\lambda = \text{const}$. The Drucker-Prager criterion is obtained.

The 4-parameter Hsieh-Ting-Chen model (HTC) uses combinations of invariants and the largest principal stress. The material parameters $a, b, c, d$ have to be obtained experimentally as well. As one can see along the compressive meridian ($60^\circ$) there is an edge, what can cause instability in implementations. The criterion again contains the von Mises, Drucker-Prager and the Rankine criterion for different sets of parameters.

The 5-parameter model by Willam & Warnke (1974) has no such edges and is continuously differentiable everywhere. We will focus on this criterion in the following.
After completing his undergraduate Dipl.-Ing. degree at the Technical University of Vienna, Austria, in 1964, Professor Willam continued his studies with graduate work at California State University San Jose and at the University of California Berkeley. In 1969 he received his Ph.D. degree with a dissertation on ‘Finite Element Analysis of Cellular Structures’. Upon returning to Europe in 1970, he directed a large-scale R&D project at the University of Stuttgart to develop the 3-dim finite element code SMART for the analysis and design of pre-stressed concrete reactor vessels. This provided an opportunity to focus on computational aspects of thermo mechanical analysis of materials and structures. In 1980 he was promoted to lecturer for Structural Mechanics in Aeronautical and Aerospace Engineering with a habilitation thesis on ‘Finite Element Discretization of Quasi static Problems in Space and Time’. In 91 he was appointed Professor of Civil Engineering at the University of Colorado Boulder, where he was and is still teaching a variety of undergraduate and graduate courses in structural mechanics and materials. In 1988 he accepted the responsibility to chair the Institute of Mechanics at the University of Karlsruhe, Germany, from which he resigned in 1990 to return to CU-Boulder.

The failure criterion proposed by Willam and Warnke has several features: It has non-linear meridians and elliptical intersection curves in the deviatoric plane that are not self-affine. Let’s first look at the elliptic meridians. Ellipses are very suitable, since the requirement of symmetry, convexity and continuity are automatically fulfilled. Additionally an ellipse can deform towards a circle, what implies that for rho_t=rho_c, von Mises or even Drucker-Prager are included.
• The standard shape of the ellipse is: .... As visible, the failure curve P1-P-P2 is approximated with rho, theta in polar coordinates by a quarter of the ellipse P1-P-P2-P3 with the half axes a,b of the ellipse. The symmetry condition requires that the position vectors rho_c, rho_t intersects the ellipse in the points P1, P2 at a 90° angle. The smaller half axis, that corresponds to the tensile vector rho_t has to be chose such that the normality condition in point P2 is fulfilled.
• For this purpose the normal vector is calculated by partial derivation: ..... In point P2 this is (as one can obtain from simple geometrical considerations ....
• For each point on the ellipse, given by the position vector rho_c, rho_t the condition must be fulfilled: ... with the normality condition one obtains ...
• The point P2 can then be written with rho_c, rho_t and b:
• By inserting everything in the formulation for the half axes a and b:
Now we have a and b, but also x and y that need to be eliminated by a transformation into polar coordinates rho-theta. With some rearrangements und insertion of the half axis we finally obtain the failure curve in the rho-theta coordinates. \( Rho_t/rho_c = 1 \) gives the DP criterion while \( = 0.5 \) gives almost triangular intersection curves similar to the Rankine criterion.
The shape of the meridians is given by variation of the mean shear stress in the compressive and tensile regime as function of the mean normal stress. Willam & Warnke chose a parabolic function (2nd order) that comes with 2x3 parameters. The condition that the two curves must intersect on the hydrostatic axis (point simga_m0/f_c=chi_0) reduces the independent parameters to 5. Hence those have to be experimentally determined first to construct the failure envelope. The respective experimental points are inserted in the plot. In the next step the meridians are connected by ellipsoidal surfaces we obtained before.

\[ \rho(s_{m},\theta) = \text{equation of the ellipse with the relation of } \sigma_{m} \text{ and } \rho_{c}, \rho_{t} \text{ from the parabolic ansatz.} \]

As we can see the failure envelope is the mean shear stress as function of the mean normal stress and the Lode angle. Hence it is for sure not affine and not circular. Let’s summarize:
Failure criteria for concrete -5P models

WILLAM-WARNKE criterion (1974):

\[ f(w_m, i_m, \theta) = \sqrt[3]{\frac{i_m}{\rho(w_m, \theta)}} - 1 = 0 \]

- Has 5 parameters \( \rightarrow \) 5 experimental data points are needed.
- Includes all stress invariants in the form \( f_1(J_1, J_2, \epsilon), \) resp. \( f_1(w_m, i_m, \theta). \)
- Is for \( \rho > 0.5 \) a continuous differentiable surface.
- Is periodic in the deviatoric plane with three-fold symmetry.
- Has non circular shape in the deviatoric plane ranging from almost triangular to circular with increasing hydrostatic stress \( \rightarrow \) not self-affine.
- Has parabolic meridians.
- Has intersection in the deviatoric plane consisting of elliptical sections.
- Convex surface in the deviatoric plan as well as meridian if:

\[
\begin{align*}
    a_1 > 0, & \quad a_2 \leq 0, \quad a_3 \leq 0 \\
    b_1 > 0, & \quad b_2 \leq 0, \quad b_3 \leq 0 \\
    \rho(w_m) & \quad \frac{\rho(w_m)}{\rho_x(w_m)} > \frac{1}{2}
\end{align*}
\]

- Failure surface opens up towards negative hydrostatic axis, \( a_3 < 0 \) however lead to intersection with hydrostatic axis \( \rightarrow \) opposed to observations for high \( \xi. \)
- Is valid for all stress combinations and gives in all practically relevant domains, including tension good agreement with reality.
- Contains for different parameter choices different models (von Mises, Drucker-Prager, 3P-William-Warnke)
To obtain the 5 parameters, 5 different experiments have to be made: Those are...

<table>
<thead>
<tr>
<th>Experimental parameter determination:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Uniaxial cylinder compressive strength: $f_c(\theta = 60^\circ, f_c &gt; 0)$</td>
<td></td>
</tr>
<tr>
<td>2. Uniaxial tensile strength $f_r(\theta = 0^\circ)$ for strength ratio $\overline{f_r} = f_r/f_c$</td>
<td></td>
</tr>
<tr>
<td>3. Uniform biaxial compressive strength $f_{bc}(\theta = 0^\circ, f_{bc} &gt; 0)$ for strength ratio $\overline{f_{bc}} = f_{bc}/f_c$</td>
<td></td>
</tr>
<tr>
<td>4. High compressive stress $(\sigma_u/f_c, \tau_u/f_c) = (-\overline{\sigma_1}, \overline{\tau_1})$ on the tensile meridian $(\theta = 0^\circ, \xi_1 &gt; 0)$</td>
<td></td>
</tr>
<tr>
<td>5. High compressive stress $(\sigma_u/f_c, \tau_u/f_c) = (-\overline{\sigma_2}, \overline{\tau_2})$ on the compressive meridian $(\theta = 60^\circ, \xi_2 &gt; 0)$</td>
<td></td>
</tr>
<tr>
<td>6. Additionally the parabolas have to intersect in the apex on the hydrostatic axis: $\rho_r(\overline{\sigma_0}) = \rho_r(\overline{\tau_0}) = 0$ b.c. $\overline{\sigma_0} = \frac{\sigma_u}{f_c} &gt; 0$</td>
<td></td>
</tr>
</tbody>
</table>

Solution of two sets of 3 linear equations to obtain $a_0, a_1, a_2, b_0, b_1, b_2$.
If we insert the relation from the table for the 5 experiments into the equation for the parabolas of the tensile and compressive meridians, we can solve for the parameters \(a0,1,2\) and \(b0,1,2\). The intersection condition of the meridians (6) gives the value for \(\zeta_0\).

If one compares the values predicted by the Willam & Warnke criterion with test data, one can see an excellent agreement. Hence a 5P WW criterion can approximate the 3D failure envelope for concrete rather good. Having the scatter of experiments in mind, the model is more than sufficient.
The high sensitivity of concrete under tension compared to its compressive performance is by far the largest problem of concrete. While under compression the material is crushed uniformly in a larger material volume, under tension the growth of a single crack perpendicular to the principle stress direction is dominant. The material experiences an extreme degradation in loading direction, while its load carrying capability perpendicular to it are almost entirely remained. Via the rough crack surfaces, even up to 60° of the maximum shear stresses can be transmitted by what is called «Aggregate interlocking». In steel reinforced concrete structures this is of enormous importance. In other words, the story does not end with the appearance of the first crack. If this would be the case, concrete would not be used as a construction material, since it contains cracks in most cases. Hence what is important is to predict their development and if we look at the image, we realize that neither maximum stress nor maximum strain criteria capture the typical concrete behavior. This is to be expected, since the failure is caused by the growth of single cracks that merge to reach extensions, where the material volume fails.

In principle there are three different approaches to describe a material with cracks:
- With smeared crack approaches
- With discrete crack
- With embedded cracks and fracture mechanical modelling (XFEM)

The choice of the right perspective strongly depends on the problem. If a general load-displacement behavior is in the focus, the smeared crack approach is a good choice, while problems that depend on the detailed, local behavior profit from a discrete model.
For those, cracks can be introduced at Finite Element boundaries or inside of elements formulation. For special fracture mechanical problems this approach has to be chosen. However in our context, and for many engineering applications, the smeared perspective is the one to look at.
The smeared crack model approach has the following characteristics: ...

For a closer look we first address the constitutive relations for undamaged concrete, then the failure criterion and the constitutive relations for the damaged concrete
So let’s start with constitutive relations for undamaged concrete. As we can see this is nothing new, since we deal with a linear elastic, isotropic material model. For completeness it is given here also for PS and PE and rotational symmetry. We always have 2 parameters that need to be determined experimentally. For moderate tensile and compressive load, the linear model is well applicable for concrete, however high compressive stress results in non-linear elastic behavior, that needs to be considered one way or the other, over this is subject to future lectures. For now we continue with the failure criterion.
We just looked at different failure criteria, MC, WR, WW and others. A very convenient way is to use a stress zoning given in the plot here. Depending on the zone, different criteria have to be chosen. The brittle behavior of the T-T and T-C zone, where micro cracks merge to form discrete macro cracks with strong localization, or the C-C zone with ductile behavior due to uniform material degradation. In the C-T zone both damage mechanisms can exits.
After fulfilling the damage criterion, cracks exist that are oriented perpendicular to the maximum principals stress direction. This automatically induces transvers isotropic material behavior with a material CSYS rotated with respect to the global one at an angle of Psi. In the first versions of smeared crack models an instantaneous, complete softening perpendicular to the crack orientation was proposed along with a full reduction of shear stiffness. In an Finite Element an integration point would be fully softened and the load would be elastically redistributed. However first one would have to transform the new stiffness matrix of the element, that is defined in the crack CSYS into the global CSYS.

Now there are two different variants: One can fix the orientation angle for all consecutive increments, what is called fixed smeared crack model, or one can freely rotate them in consecutive increments what is called rotating smeared crack model. Not that even after fixing crack orientations, cracks perpendicular to the initial orientation can form.

If mü and beta=0, the system is ill-conditioned and convergence problems emerge, producing unrealistic, skewed crack pattern. By introducing a shear retention factor 0<beta<1 this problem is solved and additionally the effect of aggregate interlocking or frictional effects of shear deformation are considered. A further enhancement of model convergence is obtained by smoothly softening via a softening factor nü. This goes along with considering the reinforcement in concrete that not only bridge cracks, but also allows concrete in between cracks to contribute to the stiffness called tension stiffening effect. Also if one does tensile tests in concrete very careful, it does not fail ideally brittle
but with a tension softening effect. The stiffness reduction is only made under tension, while under compression, the material appears to be entirely healed since stresses can be fully transmitted via the crack surfaces.
We said multiple times, that concrete can fail in different ways, e.g. brittle in the T-T and T-C zone by crack growth, but also ductile. Smeared crack model however work best in the brittle failure domain. As we saw cracks are not directly but fictitiously represented by their influence on the stiffness matrix at each material point. This is why one calls it also fictitious crack models.

There are diverse ways to deal with crack orientations like fixed, adaptive fixed, or rotational free. In Abaqus for the sake of stability the fixed orthogonal crack model is used what can lead to too stiff behavior due to shear retention. By manipulating manually the shear retention factor beta as function of the deformation, this effect can be cured. Additionally the number of crack orientations at material point is limited to the dimension of the stress space (3D=3). Cracks can entirely close to transfer compressive stresses and reopen without leaving remaining deformation. A main feature of the model is the use of strain rates. For this strain rates are decomposed into elastic parts from concrete and the elastic strain rate by existing cracks. This is in particular useful with respect to the usage of plasticity material models. As fracture criterion the Rankine criterion is used.

In principle the model is defined, one has however the possibility of introducing different ways for tension-softening and shear retention, as well as for the failure envelope.
The softening behavior can be defined in different ways:
1. By giving the crack opening behavior where the reduced strength vanishes (linear reduction)
2. By a direct definition of a stress-strain curve for the softening behavior.

In general these fracture mechanical approaches are superior. The fracture energy is in principle the area underneath the tension-separation curve, here given for mode I. As for strains also the crack opening can be decomposed into the displacement part by the deformed concrete in between cracks and the real crack opening. Since we worked with strains up to this point and suddenly need a displacement, the stain has to be multiplied with a characteristic length. This however depends on the element size and order of test function. Internally, Abaqus uses (for volume elements and linear test functions) half of the space diagonal of the element. This regularization with the characteristic size of the mesh is needed, since if a localized crack is inside one element, its crack opening does not increase simply by having it inside of a larger element. Be aware that for strongly distorted elements programs hence make systematic errors.
To consider the shear retention a power law is used. The 2 parameters e_crack_max and rho define the way, the retention factor is calculated. If one has simultaneously crack opening and shear, one can nicely observe how the shear retention part at the crack surfaces is reduced to 0.
In principle the Rankine criterion is used. One can however give the relative position of failure surfaces, hence the size and position of the box in the principal stress space.
*Concrete: Yield stress, $\sigma_y$
  strength, plastic failure strain

*Tension stiffening: Linear reduction up to a crack opening of $5\times 10^{-4}$
Absolute value of the direct strain minus the direct strain at cracking

If one compares the solution with and without tension softening, one can see no
difference in the absolute value of the softening and collapse, but clearly in the
prediction of the present response. One can say, that the model works quite good, if
damage is in the T-T domain, hence dominated by crack growth. For ductile failure
however models that are based on plasticity theory are simply better. We get to this
point later on, after we learned how non-linear elastic material behavior is addressed.
Thank you for your attention.