1 Basic Equations of Hydromechanics

1.1 General Transport Equation

The basic equations in hydromechanics are transport equations of mass, momentum, energy, etc. A unified treatment of all equations is possible. For this purpose we use the following abstract transported quantities:

- **Extensive quantity** $m$
- **Intensive quantity** $\phi$ (intensity of the quantity $m$)
- **Flux** $\vec{j}$ of the quantity $m$
- **Sources / sinks** $s$ of the quantity $m$

Extensive quantities (e.g. volume, mass, energy) are additive. A mass $m_1$ added to a mass $m_2$ yields a total mass $m_1 + m_2$. Intensive quantities (e.g. density) are specific quantities (extensive quantities related to mass or volume) and are therefore not additive. A body with density $\rho_1$ added to a body with density $\rho_2$ (not equal $\rho_1$) does not result in a body of density $\rho_1 + \rho_2$. Through integration of the intensive quantity over the system (here: volume) we obtain the associated extensive quantity.

$$ \int m = \int \phi d\Omega $$

Consider a control volume $\Omega$ with boundary $\Gamma$. Fig. 1-A shows the notation used in the equations below:

The transport equation is obtained from the balance of the extensive quantity over the control volume. The balance states that the sum of all fluxes through the boundary and all fluxes from sources and sinks $s$ within the control volume is equal to the storage of the extensive quantity in the control volume per unit time. The minus sign in front of the boundary term is explained by the fact that the surface normal $n$ is (by convention) directed from the inside to the outside of the control volume.

$$ -\int \vec{j} \cdot \vec{n} d\Gamma + \int s d\Omega = \frac{\partial}{\partial t} \int \phi d\Omega $$

Using the divergence theorem (Green-Ostrogradski-Gauss, $-\int \vec{j} \cdot \vec{n} d\Gamma = -\int \nabla \cdot \vec{j} d\Omega$) we obtain the differential form.
\[ \frac{\partial \phi}{\partial t} + \nabla \cdot \vec{j} = s \]  \hspace{1cm} (1-3)

The relevant equations in hydromechanics all follow from Eq (1-3) through appropriate choice of the quantities \(m, \phi, s\) and \(\vec{j}\).

The general transport equation can also be derived as a balance over a volume element, shown in Fig. 1-B.

With respect to the correct units, the continuity equation follows

\[ (\text{Flux}_{\text{in}} - \text{Flux}_{\text{out}}) \cdot A \cdot \Delta t + s \cdot V \cdot \Delta t = \text{Storage} \]  \hspace{1cm} (1-4)

\[ [j(x) - j(x + \Delta x)] \cdot A \cdot \Delta t + s \cdot V \cdot \Delta t = m(t + \Delta t) - m(t) \]  \hspace{1cm} (1-5)

Divisions by \(\Delta x, \Delta A\) and \(\Delta t\) deliver

\[ -\frac{j(x + \Delta x) - j(x)}{\Delta x} + s = \frac{\phi(t + \Delta t) - \phi(t)}{\Delta t} \]  \hspace{1cm} (1-6)

and for \(\Delta t, \Delta x \to 0\)

\[ \frac{\partial \phi}{\partial t} + \frac{\partial j}{\partial x} = s \]  \hspace{1cm} (1-7)

Generalization to three dimensions results in

\[ \frac{\partial \phi}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = s \]  \hspace{1cm} (1-8)

Considering the definition of the divergence \((\nabla \cdot \vec{j})\), the equation above equals Eq (1-3).
1.2 **Transport of fluid mass (continuity equation)**

Taking \( m = M \) = mass, \( \phi = \rho \) = density and \( \bar{J} = \bar{u}\rho \) = mass flux \([\text{kg}/(\text{s} \ \text{m}^2)]\), we obtain the continuity equation for mass

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\bar{u}\rho) = 0
\]  
\( \text{(1-9)} \)

The alternative derivation by balance over a volume element can be done analogously to the general transport equation. The storage can be seen as change of the intensive quantity, and the fluxes equal \( \bar{j}_m = \rho(x) \cdot \bar{u}_x(x) \) and \( \bar{j}_{out} = \rho(x+\Delta x) \cdot \bar{u}_x(x+\Delta x) \)

\[
\left[ \rho(x) \cdot \bar{u}_x(x) - \rho(x+\Delta x) \cdot \bar{u}_x(x+\Delta x) \right] \cdot A \cdot \Delta t = M(t+\Delta t) - M(t)
\]  
\( \text{(1-10)} \)

Divisions by \( \Delta x, \Delta A \) and \( \Delta t \) yield

\[
- \frac{\rho(x+\Delta x) \cdot \bar{u}_x(x+\Delta x) - \rho(x) \cdot \bar{u}_x(x)}{\Delta x} = \frac{\Delta (\rho / \sqrt{V})}{\Delta t} = \frac{\Delta \rho}{\Delta t}
\]  
\( \text{(1-11)} \)

and for \( \Delta t, \Delta x \rightarrow 0 \)

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \bar{u}_x)}{\partial x} = 0
\]  
\( \text{(1-12)} \)

Generalized to 3D, Eq (1-9) results.

For an incompressible fluid \( (\rho = \text{const.}) \) the continuity equation simplifies to

\[
\nabla \cdot \bar{u} = 0
\]  
\( \text{(1-13)} \)

Applying the product rule to the second term of Eq (1-9) we obtain the continuity equation in the following form

\[
\frac{\partial \rho}{\partial t} + \bar{u} \cdot (\nabla \rho) + \rho (\nabla \cdot \bar{u}) = \frac{\partial \rho}{\partial t} + (\bar{u} \cdot \nabla) \rho + \rho \nabla \cdot \bar{u} = 0
\]  
\( \text{(1-14)} \)

Eq (1-14) can be rewritten as

\[
\frac{D \rho}{Dt} + \rho \nabla \cdot \bar{u} = 0
\]  
\( \text{(1-15)} \)

with the substantial derivative

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y} + \frac{dz}{dt} \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \bar{u} \cdot \nabla
\]  
\( \text{(1-16)} \)
The substantial derivative corresponds to the sum of local change of a quantity and its change along the trajectory \((x(t), y(t), z(t))\) prescribed by the velocity field \(\vec{u}\).

1.3 Transport of momentum (Equation of motion)

Exemplary, we only consider the momentum in x-direction. The momentum equation in x-direction is obtained using Eq (1-3) and the following quantities:

- \(m = M u_x\) momentum in x-direction in the volume \(V\)
- \(\phi = \rho u\) momentum density in x-direction
- \(\vec{f}_x = \rho u \vec{u}\) momentum flux in x-direction
- \(f_{p,x}, f_{g,x}, f_{f,x}\) force densities (volume and surface forces) in x-direction \([N/m^3]\)

\[ p = \text{pressure}, \; g = \text{gravity}, \; f = \text{friction} \]

\[
\frac{\partial (\rho u_x)}{\partial t} + \nabla \cdot (\rho u_x \vec{u}) = f_{p,x} + f_{g,x} + f_{f,x}
\]

(1-17)

In a rotating reference system, additionally the Coriolis force needs to be considered.

Eq (1-17) in 3D can be written as follows

\[
\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \bullet \vec{u}) = \vec{f}_p + \vec{f}_g + \vec{f}_f
\]

(1-18)

where \(\vec{u} \bullet \vec{u}\) is the dyadic product of the two vectors (while the scalar product is denoted as \(\vec{a} \cdot \vec{b}\))

\[
\vec{u} \bullet \vec{u} \equiv \begin{bmatrix} u_xu_x & u_xu_y & u_xu_z \\ u_yu_x & u_yu_y & u_yu_z \\ u_zu_x & u_zu_y & u_zu_z \end{bmatrix} = \vec{u} \cdot \vec{u}^T
\]

(1-19)

Applying the product rule yields

\[
\nabla \cdot (\rho \vec{u} \bullet \vec{u}) = \nabla \cdot \begin{bmatrix} \rho u_xu_x & \rho u_xu_y & \rho u_xu_z \\ \rho u_yu_x & \rho u_yu_y & \rho u_yu_z \\ \rho u_zu_x & \rho u_zu_y & \rho u_zu_z \end{bmatrix}
= \begin{bmatrix} \frac{\partial}{\partial x} (\rho u_xu_x) + \frac{\partial}{\partial y} (\rho u_xu_y) + \frac{\partial}{\partial z} (\rho u_xu_z) \\ \frac{\partial}{\partial x} (\rho u_yu_x) + \frac{\partial}{\partial y} (\rho u_yu_y) + \frac{\partial}{\partial z} (\rho u_yu_z) \\ \frac{\partial}{\partial x} (\rho u_zu_x) + \frac{\partial}{\partial y} (\rho u_zu_y) + \frac{\partial}{\partial z} (\rho u_zu_z) \end{bmatrix}
\]

(1-20)

\[(\nabla \rho \cdot \vec{u}) \vec{u} + \rho \vec{u} (\nabla \cdot \vec{u}) + \rho (\vec{u} \cdot \nabla) \vec{u} \]

and therefore
\[
\frac{\partial (\rho \ddot{u})}{\partial t} + \nabla \cdot (\rho \ddot{u} \cdot \ddot{u}) = \rho \frac{\partial \ddot{u}}{\partial t} + \ddot{u} \frac{\partial \rho}{\partial t} + \ddot{u} (\nabla \rho \cdot \ddot{u}) + \rho \ddot{u} (\nabla \cdot \ddot{u}) + \rho (\ddot{u} \cdot \nabla \ddot{u}) + \rho (\ddot{u} \cdot \nabla) \ddot{u}
\]

The curly bracket marks the continuity equation (1-4), which is equal to zero according to mass conservation. Extraction of \( \rho \) yields

\[
\frac{\partial (\rho \ddot{u})}{\partial t} + \nabla \cdot (\rho \ddot{u} \cdot \ddot{u}) = \rho \frac{\partial \ddot{u}}{\partial t} + (\ddot{u} \cdot \nabla) \ddot{u} = \rho \frac{D \ddot{u}}{D t}
\]

Inserting the expressions for pressure and gravity force density into Eq (1-17) leads to

\[
\frac{\partial (\rho u_x)}{\partial t} + \nabla \cdot (\rho u_x \ddot{u}) = \frac{\partial p}{\partial x} + \rho g_x + f_{f,x}
\]

The equations for momentum in y- and z-direction can be derived analogously. Inserting Eq (1-22) leads to the equation of motion in three dimensions

\[
\rho \frac{D \ddot{u}}{D t} = -\nabla p + \rho \ddot{g} + f_f
\]

Eq (1-24) ist known as **Navier-Stokes equation**. The friction term \( f_f \) depends on the rate of deformation and vanishes for \( \ddot{u} = 0 \).

A material law describes the dependence between deformation rate and friction force (see section 1.3.2).

**1.3.1 Friction force on a volume element**

Fig. 1-C shows the 6 stresses acting in x-direction on a unit volume element (\( \Delta x = \Delta y = \Delta z = 1 \)).
Meaning of the indices: e.g., for $\tau_{xz}$, $z$ denotes the direction of the surface normal (bold arrow), while the stress itself is acting in $x$-direction.

These stresses form a 2\textsuperscript{nd} order tensor (stress tensor)

$$
\tau = \begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z \\
\end{bmatrix}
$$

As shown later in the material law (section 1.3.2), $\tau$ is symmetric. The 9 unknown entries of the tensor therefore reduce to 6 unknown components, which require 6 constitutive equations for the closure of the problem.

The $x$-component of the friction force according to Fig. 1-C (per unit volume) is

$$
f_{f,x} = \frac{\partial \sigma_x}{\partial x} \Delta x + \frac{\partial \tau_{xy}}{\partial y} \Delta y + \frac{\partial \tau_{xz}}{\partial z} \Delta z = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \quad (1-26)
$$

For the 3D case, it follows

$$
\bar{f}_f = \nabla \cdot \tau \quad (1-27)
$$

### 1.3.2 Material law

Considering water as a Newtonian fluid is a very good approximation. This means that the stress tensor is proportional to the rate of deformation tensor, which will be derived in the following.

Fig. 1-D illustrates the possible deformations experienced by a fluid element. $u_x$ and $u_y$ are the velocity components in $x$- and $y$-direction, respectively. We look at an infinitesimal small volume element. Therefore, small changes can be approximated by a Taylor series of $1\textsuperscript{st}$ order.
The compression in Fig. 1-D (left) denotes a volume change of the element. The relative volume change over time results after division by \( \Delta x, \Delta y, \Delta z \) and \( dt \)

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \vec{u}
\]  

(1-28)

The deformations in Fig. 1-D result from the relative motion of \( P \) with respect to \( O \). They consist of shear and rotation.

The assumption \( \sin d\alpha = d\alpha \) (for small angles) yields

\[
d\alpha = \left( u_x + \frac{\partial u_x}{\partial x} \Delta x \right) dt - u_x dt = \frac{\partial u_x}{\partial x} dt
\]

\[
d\beta = \left( u_y + \frac{\partial u_y}{\partial y} \Delta y \right) dt - u_y dt = \frac{\partial u_y}{\partial y} dt
\]

(1-29)

The shear rate is written as

\[
\frac{d\alpha + d\beta}{dt} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}
\]

(1-30)

and the rotational velocity is

\[
\frac{d\beta - d\alpha}{dt} = \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}
\]

(1-31)

For \( d\beta = -d\alpha \), no shear but only rotation appears; however, for \( d\beta = d\alpha \) rotation disappears and only the shear rate remains.

The general form of the rate of deformation tensor in 3D reads

\[
\frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

mit \( x, y, z \rightarrow x_i, i = 1, 2, 3 \)

\( \text{bzw. } x_j, j = 1, 2, 3 \)

(1-32)

Rotation does not cause any friction, because no change of body forces is involved. The stresses can therefore be written as a function of the symmetric part of the deformation tensor only. The matrix with components \( e_{ij} \) describes the change of shape (or form) and volume. The contribution from the change of shape must constitute a traceless matrix, which means that the diagonal elements have to add up to zero. We can thus apply the following decomposition of the rate of deformation tensor
\[ E = \begin{bmatrix} e_{11} - \frac{e_1}{3} & e_{12} & e_{13} \\ e_{21} & e_{22} - \frac{e_1}{3} & e_{23} \\ e_{31} & e_{32} & e_{33} - \frac{e_1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Change of shape (shear / rotation)

Change of volume (compression)

where \( e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \) and \( e_j = e_{11} + e_{22} + e_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \vec{u} \).

Therefore, a volume change only appears for non zero \( e_j \) (and therefore also \( \nabla \cdot \vec{u} \)). From Eq (1-28) it follows that only compressible fluids can experience a change in volume.

The constitutive relation for a Newtonian fluid in its most general form reads

\[ \tau_{ij} = \lambda e_j \delta_{ij} + 2\eta e_{ij} \]  

(1-34)

where \( \eta \) is the shear viscosity and \( \lambda \) is the so-called second viscosity. The latter is due to the fact that, when a compressible fluid is compressed or expanded evenly and without shear, it may still exhibit a form of internal friction that resists its flow. These forces are related to the rate of compression or expansion multiplied by a factor expressed by \( \lambda \).

From Eq (1-33) assuming that \( \lambda \) is small (i.e. negligible) and for \( i \neq j \) we obtain

\[ \tau_{ij} = 2\eta e_{ij} = \eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(1-35)

(see also Hydraulik I for one-dimensional flows) and for \( i = j \)

\[ \sigma_i = 2\eta \left[ \frac{\partial u_i}{\partial x_i} - \frac{e_i}{3} \right] = 2\eta \frac{\partial u_i}{\partial x_i} - \frac{2}{3} \eta e_i \]  

(1-36)

where we have used that \( \lambda \equiv -\frac{2}{3} \eta \) for water and no summation over repeated indices applies here.
The stress tensor is symmetric and the gradient of the stress tensor becomes

\[
\nabla \cdot \tau = \eta \left( \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) \right) + \frac{2}{3} \eta \nabla (\nabla \cdot \tilde{u}) \]

\[(1-37)\]

With \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) (\(\text{laplace operator}\)) follows for the friction term

\[
\nabla \cdot \tau = \vec{f}_f = \eta (\nabla (\nabla \cdot \tilde{u}) + \Delta \tilde{u}) - \frac{2}{3} \eta \nabla (\nabla \cdot \tilde{u}) = \frac{1}{3} \eta \nabla (\nabla \cdot \tilde{u}) + \frac{\eta \Delta \tilde{u}}{\text{compression force}} + \frac{\eta \Delta \tilde{u}}{\text{friction force acting on volume element}} \]

\[(1-38)\]

For incompressible fluids, the friction term simplifies considerably. Using the continuity equation \( \nabla \tilde{u} = 0 \) (Eq (1-13)), Eq (1-38) can be written as follows

\[
\nabla \cdot \tau = \vec{f}_f = \eta \Delta \tilde{u} \]

\[(1-39)\]

### 1.4 Navier-Stokes Equations

#### 1.4.1 Navier-Stokes equation for compressible fluids

The \(x\)-component of the Navier-Stokes equation is written as

\[
\rho \frac{D(u_x)}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \eta \Delta u_x + \frac{1}{3} \eta (\nabla (\nabla \tilde{u}))_x \]

\[(1-40)\]

with

\[
\Delta u_x = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \]

\[(1-41)\]

and the substantial derivative from Eq (1-16)

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \]

\[
\]
The general formulation in 3D is

\[
\rho \frac{D\vec{u}}{Dt} = \frac{\partial (\rho \vec{u})}{\partial t} + \nabla (\rho \vec{u} \cdot \vec{u}) = -\nabla p + \rho \ddot{g} + \frac{1}{3} \eta \nabla (\nabla \cdot \vec{u}) + \eta \Delta \vec{u}
\]  

(1-42)

### 1.4.2 Navier-Stokes equations for incompressible fluids

By use of \( \eta \nabla (\nabla \cdot \vec{u}) = 0 \) the formulation for incompressible fluids is obtained as

\[
\rho \frac{D\vec{u}}{Dt} = -\nabla \ddot{p} + \rho \ddot{g} + \eta \Delta \vec{u} \quad \text{or} \quad \frac{\partial (\rho \vec{u})}{\partial t} + \nabla (\rho \vec{u} \cdot \vec{u}) = -\nabla p + \rho \ddot{g} + \eta \Delta \vec{u}
\]  

(1-43)

Under isothermal conditions (\( T = \text{const.} \)) and for the 3D case, the 4 unknown functions (\( u_x, u_y, u_z, p \)) are obtained by solving 4 equations (3 Navier Stokes + 1 continuity equation). In addition, initial and boundary conditions have to be specified.

For varying temperature, in addition to the momentum equation also the energy equation has to be used (Section 1.5).

By rearranging Eq (1-43), it is possible to obtain the transport equation for the vorticity vector \( \vec{\omega} \)

\[
\vec{\omega} = \nabla \times \vec{u} = \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
\begin{pmatrix}
u_x \\
u_y \\
u_z
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial z} \\
\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial x} \\
\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial y}
\end{pmatrix}
\]  

(1-44)

Applying the curl (\( \nabla \times \)) on the Navier Stokes equation divided by \( \rho \) (for simplicity we consider an incompressible, homogeneous fluid, Eq (1-43)), one obtains (for \( \nu = \eta / \rho \))

\[
\nabla \times \left( \frac{\partial \vec{u}}{\partial t} + \nabla (\vec{u} \cdot \vec{u}) + \frac{1}{\rho} \nabla p - \ddot{g} - \nu \Delta \vec{u} \right) = 0
\]  

(1-45)

Further, we invoke the following vector identities:

a) The divergence of a rotational field vanishes identically

\( \nabla \cdot \vec{\omega} = \nabla \cdot (\nabla \times \vec{u}) = 0 \)  

(1-46)

b) The curl of a gradient vanishes identically

\( \nabla \times \nabla a = 0 \)  

(1-47)
The transport of energy is described considering the following quantities:

- \(\varphi \rightarrow \rho (e + \frac{\vec{u}^2}{2})\): internal energy + kinetic energy per unit volume

- \(j \rightarrow \varphi \cdot \vec{u} = \rho (e + \frac{\vec{u}^2}{2}) \cdot \vec{u}\)

- \(s \rightarrow \) Work done by volume forces and surface forces acting on the fluid volume element. Dissipation through heat transfer.

\[
\frac{\partial}{\partial t} \left[ \rho \left( e + \frac{\vec{u}^2}{2} \right) \right] + \nabla \cdot \left( \rho \left( e + \frac{\vec{u}^2}{2} \right) \vec{u} \right) = \begin{align*}
\nabla \cdot (k \nabla T) - \nabla \cdot (\vec{u} p) + \nabla \cdot (\vec{u} \tau) + \rho g \vec{u} \end{align*}
\]  

The new variable \(e\) requires a new constitutive relation. This additional relation is obtained from the equation of state.
Absorption of radiation may be taken into account and added to the energy equation via additional source terms. The stress tensor \( \tau \) remains the same as the one used in Eq (1-25).

### 1.6 Problems in the solution of the Navier-Stokes Equations

The Navier-Stokes Equations are very difficult to solve. Analytical solutions only exist for simple limiting cases with geometrically simple boundary conditions. The main reason for the problematic nature of the equations lies in the non-linearity of the advective acceleration. This term is the main cause of turbulences. A Direct Numerical Simulation (DNS) is currently only possible for laminar and weakly turbulent flows. For real turbulent flows, at least \( 10^9 \) nodes would be required for an adequate spatial resolution (the size of the smallest eddies is about \( 10^{-3} \times \) domain size). This huge number of nodes is currently not feasible to treat numerically. Further, it is not clear how useful such a solution would be, as it would cover the stochastic-chaotic behaviour of the turbulence and would only include information in a statistical sense.

Ways to circumvent these problems are found by the following approaches:

- Averaged equations: Reynolds equations plus closure model (turbulence model)
- Large Eddy Simulation (LES)
- Approximations of the averaged equations (similar to Hydraulik I for stationary flows), e.g., boundary layer equations, equations averaged over the cross-section, etc.

### 1.7 Reynolds Equations

For engineering applications, the main focus is often on the mean flow, whereas the turbulent fluctuations are typically of secondary importance. When modelling turbulent flows this is accounted for using statistical approaches. The physical quantities \( \phi \) are thereby decomposed over the time interval into a mean \( \bar{\phi} \) and a fluctuating part \( \phi' \)

\[
\begin{align*}
\bar{u} &= \bar{u} + u' \\
p &= \bar{p} + p'
\end{align*}
\]  

Each term in the Navier Stokes equation is averaged over the time interval

\[
\frac{\overline{\partial (\bar{u} + u')}}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} = \frac{\partial \bar{u}}{\partial t}
\]  

\[
\nabla (\bar{p} + p') = \nabla \bar{p} + \nabla p' = \nabla \bar{p} + \nabla p' = \nabla \bar{p}
\]  

\[
\Delta (\bar{u} + u') = \Delta \bar{u} + \Delta u' = \Delta \bar{u}
\]
Inserting these terms into the Navier Stokes equations (here for incompressible fluids) leads to the Reynolds equations

\[
\frac{\partial \rho \tilde{u}}{\partial t} + \nabla (\rho \tilde{u} \cdot \tilde{u}) + \nabla (\rho \cdot \tilde{u} \cdot \tilde{u}) = -\nabla p + \rho g + \eta \Delta \tilde{u} \tag{1-57}
\]

with \( \tilde{u} \cdot \tilde{u} \) as the dyadic product from Eq (1-19).

For the non-linear convective acceleration term, the averaging produces new terms, containing \( \tilde{u}' \).

Due to the additional term (Reynolds stress) in the equation, the solution of the Reynolds equations requires an additional closure relation, a turbulence model. The main challenge in turbulence modelling is the closure of this system by using suitable approaches.

### 1.7.1 Eddy viscosity principle

The term \( \nabla (\rho \cdot \tilde{u} \cdot \tilde{u}) \) describes the friction, which originates from the “clashing” of one eddy into the other. This friction is analogous to the molecular friction, but viewed on a larger scale (the size of a turbulent eddy). Turbulence models develop expressions for the velocity fluctuations in order to solve the system.

Most turbulence models are based on the Boussinesq hypothesis, assuming that the Reynolds stresses are proportional to the velocity gradients, as

\[
-\nabla (\rho \cdot \tilde{u} \cdot \tilde{u}) = \eta_{\text{eddy}} \Delta \tilde{u} \tag{1-58}
\]

\( \eta_{\text{eddy}} \) is the eddy viscosity. Different from the kinematic viscosity, the eddy viscosity is not a material constant, but only depends on the flow.

The Reynolds stresses can be written in a more general form

\[
\rho \tilde{u}_i \tilde{u}_j = -\eta_{\text{eddy}} \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} + \frac{2}{3} \sum_k \frac{\partial \tilde{u}_k}{\partial x_i} \delta_{ij} \right) + \frac{2}{3} \rho k \delta_{ij} \tag{1-59}
\]

where the term in the brackets is analogous to Eq (1-36), the last term is a turbulent pressure term that is due to “collisions” of vortices and the turbulent kinetic energy is expressed as

\[
k = \sum_i \frac{u_i u_i}{2} \tag{1-60}
\]

Taking the gradient of Eq (1-59) yields Eq (1-58) for the case when the flow is incompressible and turbulent pressure is not important. The application of a constant eddy viscosity is very limited. Therefore, more complicated turbulence models based on the eddy viscosity principle typically employ \( \eta_{\text{eddy}} \) as a function of the flow field. Well known models of this type are listed in the following.
Mixing length model (L. Prandtl)

\( \eta_{\text{eddy}} \) is calculated using characteristic scales (for size and velocity) of turbulent eddies

\[
\nu_{\text{turb}} = L_{\text{turb}} \cdot U_{\text{turb}} \tag{1-61}
\]

\[
\eta_{\text{eddy}} = \rho \cdot \nu_{\text{turb}} \tag{1-62}
\]

and empirical relationships express the dependence of these scales on inhomogeneous coordinates (for example, a linear relationship between \( L_{\text{turb}} \) and the wall normal coordinate in wall turbulence).

**k-\( \varepsilon \) Model**

The eddy viscosity is determined by additional transport equations (not reported here; the interested reader is referred to monographs such as “Turbulent Flows” by S.P. Pope) for turbulent kinetic energy

\[
k = \sum \frac{u_i u_j}{2}
\]

and its dissipation rate

\[
\varepsilon = \nu \sum_{i,j,k} \left( \frac{\partial u_i}{\partial x_j} \right) \left( \frac{\partial u_j}{\partial x_k} \right)
\]

that are coupled to the Reynolds Equations. In this model, the eddy viscosity is given by

\[
\nu_{\text{turb}} = C_\mu \frac{k^2}{\varepsilon} \tag{1-64}
\]

with the empirical constant \( C_\mu = 0.09 \).

**1.7.2 Spatially integrated Reynolds equations**

**Pipe flow**

To describe the one-dimensional pipe flow problem, the Navier Stokes equations will be written as a function of the average velocity and pressure, \( v_m \) and \( p_m \). For the derivation we also employ the continuity equation for elastic pipes and an expression for the wall friction \( \tau_0 \) from the turbulence theory.

We assume that the pipe axis is parallel to the x-axis (Fig. 1-E). With this assumption, the y- and z-components of the Navier-Stokes equations are not considered, they become equations for the pressure.
Averaging the horizontal velocity component over the cross section (y-z plane), one obtains

\[ u_m = \frac{1}{A} \int_A u_x dA = \frac{Q}{A} \quad (1-65) \]

The x-component of the Navier Stokes equation (Eq (1-23)), averaged over the cross section (y-z plane), yields

\[ \frac{1}{A} \int_A \left[ \rho \frac{\partial u_x}{\partial t} + \rho u_x \frac{\partial u_x}{\partial x} - \rho g \sin \alpha + \frac{\partial p}{\partial x} - f_{f,x} \right] \cdot dA = 0 \quad (1-66) \]

Mixed components and internal friction drop out with the integration. Only the friction at the wall \( \tau_0 \) remains (see Fig. 1-F)

\[ \frac{Force}{Volume} = \frac{\tau_0 \cdot \Delta x \cdot U}{\Delta x \cdot A} = \frac{\tau_0}{R_{hy}} \quad (1-67) \]

with \( R_{hy} \) equal the hydraulic radius. From the relation \( \Delta h = \lambda \frac{\Delta x u_m^2}{2r \frac{2g}{2}} \) (Darcy-Weisbach, see Hydraulics I, \( \lambda = f(Re, k/d) \)) and \( \Delta p = \rho g \Delta h \) follows

\[ \Delta p \cdot A = \frac{\rho \cdot \lambda \cdot \Delta x \cdot u_m^2 \pi r^2}{4r} = \tau_0 \cdot 2r \pi \cdot \Delta x \quad (1-68) \]
and therefore
\[ \tau_0 = \frac{\lambda}{8} \rho u_m^2 \]  \hspace{1cm} (1-69)

Finally, Eq (1-66) can be written as
\[ \rho \frac{\partial u_m}{\partial t} + \rho \alpha' u_m \frac{\partial u_m}{\partial x} - \rho g \sin \alpha + \frac{\partial p_m}{\partial x} + \frac{\tau_0}{R_{hy}} = 0 \]  \hspace{1cm} (1-70)

with the velocity correction factor
\[ \alpha' = \frac{1}{u_m} \int_A u_x^2 \, dA \]  \hspace{1cm} (1-71)

The introduction of this correction factor is necessary because \( \langle v^2 \rangle \neq \langle v \rangle^2 \).

The velocity correction factor describes the non-uniformity of the velocity profile (Hydraulics I). For uniform flow we have \( \alpha' = 1 \). In the following we also approximate \( \alpha' = 1 \) for turbulent pipe flow.

The friction slope is defined as
\[ I_R = \frac{\tau_0}{\rho g R_{hy}} \]  \hspace{1cm} (1-72)

The momentum equation with the assumption \( \alpha' = 1 \) then reads
\[ \frac{\partial u_m}{\partial t} + u_m \frac{\partial u_m}{\partial x} - g \sin \alpha + \frac{1}{\rho} \frac{\partial p_m}{\partial x} + g \cdot I_R = 0 \]  \hspace{1cm} (1-73)

Next to the momentum equation, the continuity equation is the second equation for the unknowns \( v_m \) and \( p_m \). When integrating the continuity equation over the coordinates \( y \) and \( z \), one has to consider that the pipe is an elastic boundary, which changes in time due to pressure fluctuations. Thus, a continuity equation for elastic pipes is required, for which the cross sectional area \( A \) is not constant.

Mass balance per unit pipe length [kg/m] results according to Eq (1-12) and the averaged velocity
\[ u_m = \frac{1}{A} \int_A u_x \, dA \] from Eq (1-65)
\[ \frac{\partial (\rho A)}{\partial t} + \frac{\partial (\rho A u_m)}{\partial x} = 0 \]  \hspace{1cm} (1-74)

Applying the product rule and dividing by \( A \) yields
\[ \frac{\partial \rho}{\partial t} + \frac{\rho}{A} \frac{\partial A}{\partial t} + u_m \frac{\partial \rho}{\partial x} + \rho u_m \cdot \frac{\partial A}{\partial x} + \rho \frac{\partial u_m}{\partial x} = 0 \]  

(1-75)

With

\[ \frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} \cdot \frac{\partial}{\partial t} + \frac{\partial A}{\partial x} \cdot u_m \]  

(1-76)

the continuity equation for elastic pipes is obtained as

\[ \frac{\partial \rho}{\partial t} + u_m \frac{\partial \rho}{\partial x} + \rho \frac{\partial u_m}{\partial x} + \frac{\rho}{A} \frac{dA}{dt} = 0 \]  

(1-77)

with \( \rho = \rho(p) \) and \( A = A(p) \).

**Channel flow**

The derivation of the equations is analogous to the pipe flow. For channel flow we additionally assume:

- hydrostatic pressure distribution \( h_p = z + \frac{p}{\rho g} \Rightarrow z = h_p - \frac{p}{\rho g} \)
- \( \cos \alpha \approx 1 \)
- \( \sin \alpha \approx -\frac{dz}{dx} \)
- \( \rho = \text{const.} \)

The momentum equation (1-70) with the above assumptions is recast as

\[ \frac{1}{g} \frac{\partial u_m}{\partial t} + \frac{1}{g} \alpha u_m \frac{\partial h_p}{\partial x} + \frac{\partial h_p}{\partial x} + \frac{\tau_0}{g \rho R_{xy}} = 0 \]  

(1-78)

As for the continuity equation we again have to consider that the cross-sectional area \( A \) is not constant. While the area change due to elasticity was small for the case of pipe flow, in channel flow it can become considerable because of free water surface fluctuations. Similar to pipe flow, also in channel flow the derivative \( \partial A / \partial t \) is different from 0.

The continuity equation integrated over the cross section results analogously to Eq (1-74) after division by \( \rho \)

\[ \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \]  

(1-79)
1.8 Classifications of equations in hydromechanics

The systems of equations 1-73/1-74 or 1-78/1-79 have the general form

\[
\begin{align*}
  a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial t} + c_1 \frac{\partial h}{\partial x} + d_1 \frac{\partial h}{\partial t} &= f_1 \\
  a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial t} + c_2 \frac{\partial h}{\partial x} + d_2 \frac{\partial h}{\partial t} &= f_2
\end{align*}
\]

(1-80)

For simplicity, the coefficients are assumed to be constant. Now we want to classify these functions according to different regions of the coordinate plane \(x-t\). More specifically, we look for characteristic lines along which the derivatives of \(u\) and \(h\) are undetermined, i.e. they do not exist. These lines split the coordinate space into regions which are kinematically unconnected, i.e. no solution exists which crosses these lines. With the total derivative

\[
Du = \frac{\partial u}{\partial x} \ dx + \frac{\partial u}{\partial t} \ dt
\]

\[
Dh = \frac{\partial h}{\partial x} \ dx + \frac{\partial h}{\partial t} \ dt
\]

we get the following system of equations after insertion in Eq (1-80)

\[
\begin{bmatrix}
  a_1 & b_1 & c_1 & d_1 \\
  a_2 & b_2 & c_2 & d_2 \\
  dx & dt & 0 & 0 \\
  0 & 0 & dx & dt
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial u}{\partial x} \\
  \frac{\partial u}{\partial t} \\
  \frac{\partial h}{\partial x} \\
  \frac{\partial h}{\partial t}
\end{bmatrix}
= \begin{bmatrix}
  f_1 \\
  f_2 \\
  Du \\
  Dh
\end{bmatrix}
\]

(1-82)

The derivatives are undetermined when the determinant of the matrix is zero. This means

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 & d_1 \\
  a_2 & b_2 & c_2 & d_2 \\
  dx & dt & 0 & 0 \\
  0 & 0 & dx & dt
\end{vmatrix} = 0
\]

(1-83)

and therefore

\[
(a_2 c_1 - a_1 c_2) \left( \frac{dt}{dx} \right)^2 - (a_2 d_1 - a_1 d_2) \frac{dt}{dx} + (b_2 c_1 - b_1 c_2) \frac{dt}{dx} + (b_2 d_1 - b_1 d_2) = 0
\]

(1-84)

Formation of new coefficients \(a, b\) and \(c\) yields

\[
a \left( \frac{dt}{dx} \right)^2 + b \frac{dt}{dx} + c = 0
\]

(1-85)
and from there it follows that
\[
\frac{dt}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\] (1-86)

Three possible cases for the determinant \( D \equiv b^2 - 4ac \) exist (see Abb. 1-G):

- **D > 0**: 2 real characteristics through every point, the equations are **hyperbolic**
- **D = 0**: 1 real characteristic, the equations are **parabolic** (Note: the line through P is not necessarily vertical, its slope is \(-b/2a\))
- **D < 0**: imaginary characteristics, the equations are **elliptic**

![Abb. 1-G: Classification of equations in the x–t plane](image)

In terms of flow events, a **hyperbolic** behaviour means that a system can develop from the state P only towards states in between the characteristics, as e.g. for the case of a shock wave with a given wave velocity.

**Parabolic** means that the system can only develop towards one side, nevertheless not as confined as for the hyperbolic case, where only specific velocities are possible. Finally, **elliptic** means that a system can develop from this state towards any other point on the plane, like e.g. for the case of backflows.

The classification of flow problems into these 3 types is of great interest, because the numerical solution will be different depending on the type.

The terminology originates from the geometry of conic sections, which satisfy the equation
\[
a x^2 + b xy + c y^2 + d x + e y = f = 0
\] (1-87)

It thereby follows

- \( b^2 - 4ac > 0 \) a **hyperbole**
- \( b^2 - 4ac = 0 \) a **parabola**
- \( b^2 - 4ac < 0 \) a **ellipse**.