ETH Zürich
Department for Chemistry and Applied Biosciences
Institute for Chemical and Bioengineering

Lecture notes for the course of

Chemical Process Control

(Regelungstechnik)

Frühlingssemester 2024

Prof. Robert N. Grass

Organization of the Course

Lecture:	Friday	11:45 – 13:30 in HCI H 8.1
Exercise:	Wednesday	11:45 – 12:30 in HCI H 8.1
Website:	https://fml.ethz.ch/lectures/RegTech.html	

Lecturer:

Prof. Robert N. Grass	HCI E 111	Tel. 36 344	E-Mail: robert.grass@chem.ethz.ch

Assistant:

Andreas Gimpel	HCI E 106	Tel. 32909	E-Mail: andreas.gimpel@chem.ethz.ch

Recommended Reading:

G.F. Franklin, J.D. Powell and A. Emami-Naeini
Feedback Control of Dynamic Systems
Sixth Edition, 2010, Prentice Hall, Upper Saddle River
ISBN 0-13-032393-4

D.E Seborg, T.F. Edgar and D.A. Mellichamp
Process Dynamics & Control
Third Edition, 2011, Wiley, New York
ISBN 0-471-00077-9

J. Lunze
Regelungstechnik 1
10.Auflage, 2007, Springer, Berlin
ISBN 978-3-642-53908-4

These lecture notes are intended to support students in learning for the exams in Process Control. They are not understood as a complete and self-explanatory scriptum, and therefore are only useful in connection to visiting the actual course.

Students willing to learn without visiting the course are advised to follow one of the books mentioned above, or the online course by Prof. Manfred Morari published on YouTube: http://tinyurl.com/jshqba8

Table of Content

1 Introduction

1.1 Engineer's View of the World

Changes of state can be described by differential equations that transform the current steady state, that depends on certain input variables and some additional external variables into another steady state.

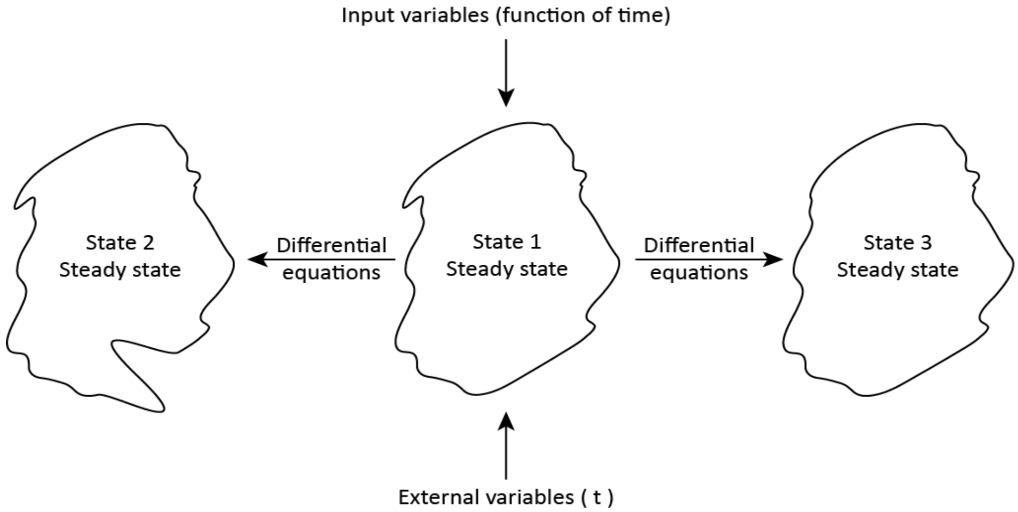


Figure 1-1. Different state transformations showing the important variables.

1.2 What is Control?

In short terms: Do as I say!
But how can you be sure that it is really done?

1) Do R!
2) Did you get R? No: Change something
   Yes: Keep everything as it is

Telling and checking make up control as you know it from teaching kids or pets.

EXAMPLE 1.2-1: Maintaining Body Temperature

Our body temperature is controlled by different factors, which you can see in Figure 1-2.

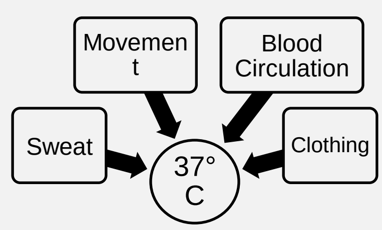


Figure 1-2. Influence factors on body temperature.

EXAMPLE 1.2-2: Oven

The simplest idea of a temperature control system for an oven can be seen in Figure 1-3.

Do T=170°C!

If T < 170°C → heat
If T > 170°C → do nothing

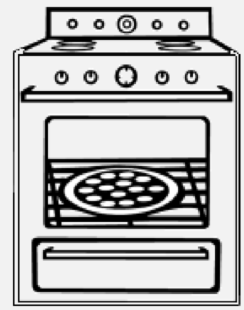


Figure 1-4. Oven.

EXAMPLE 1.2-3: Production plant

A very general flowchart of a chemical production plant plant is depicted in Figure 1-4. In order that the plant can produce the outputs properly all inputs need to be controlled.

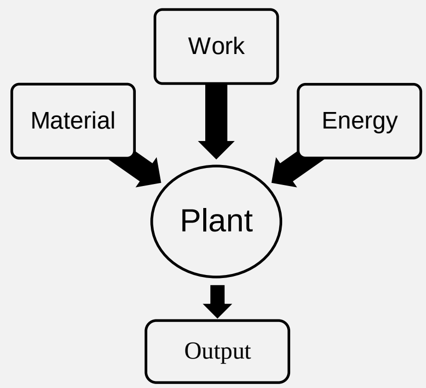


Figure 1-3. Plant flowdiagram.

1.3 Basic Elements of Process Control

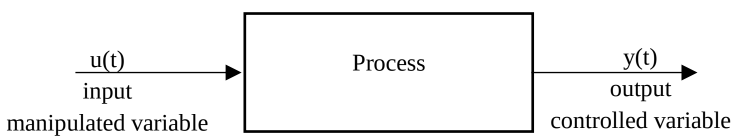


Figure 1-5. Basic elements of Process Control

Process (in general) = machine/system equipment + actuators (servomotors) + measuring device (sensors)

1.4 History of Process Control

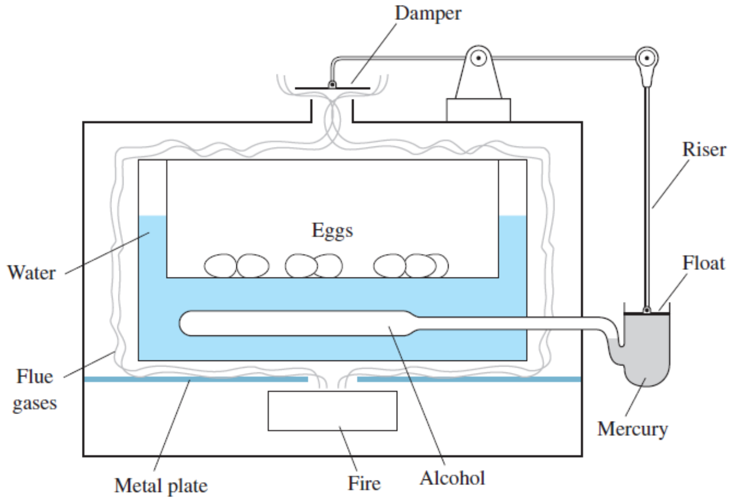


Figure 1-6. Drebbels incubator (Brutkasten)

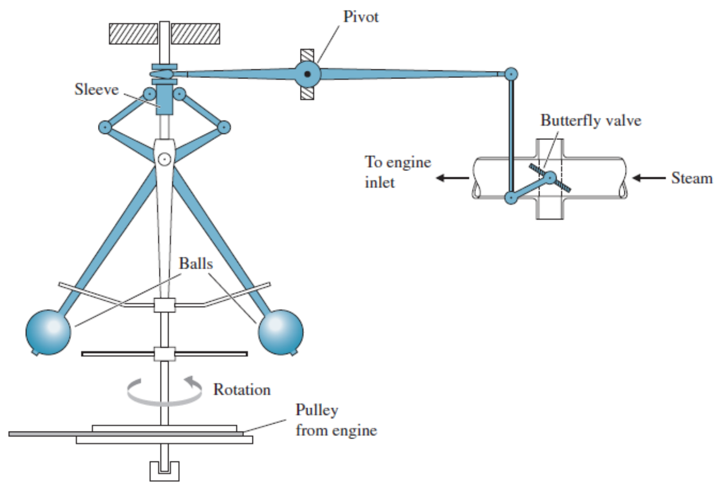


Figure 1-7. James Watts governor for steam machines.

1.5 Process Control in Chemical Industry

Goals of the Industry: Maximize: output
Minimize: energy, work, raw materials

→ **This is Chemical Engineering**

<u>The ideal case:</u>
Design plant (steady state) → Turn plant on → Walk away → DOESN'T WORK!

<u>Why not ?:</u>
Nature (external variables) not taken into account (temperature at day/night/seasons, unreliable products)

Machine		Nature
⬇		⬇
Chemical Engineering		Chaos
⬇	Process Control	⬇
Design	⟷	Dynamic

EXAMPLE 1.5-1: Coca Cola Plant

The mixing process has to run at steady state to guarantee for the same quality of the product at any time, but there are some problems:

- Sugar from sugar beets has different concentrations depending on the season etc.
- Vanilla concentration changes, too.
- Phosphoric acid from stone → natural product and therefore also unsteady.

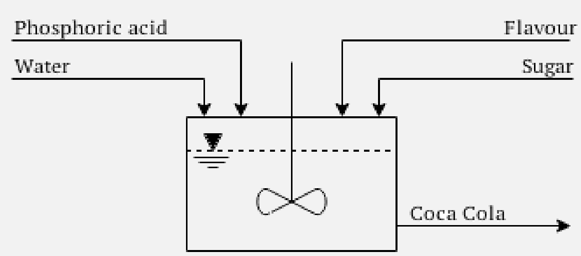


Figure 1-8. Simplified Coca Cola mixing process.

2 Representation of Systems

Systems in process control are represented by block representation:

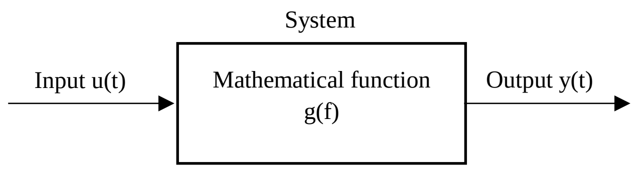
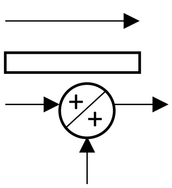


Figure 2-1. Simplified representation of a system that transfers an input into an output.

Mathematically described by:

$$y(t) = g\big(u(t)\big)$$

where g is a transfer function.



Variable with time dependence (vector of variables)
Functions transfer one variable into another
Mathematical operations

SISO   Single input single output (most of this course)
MIMO   Multiple input multiple output. In this case u(t) and y(t) are vectors (see last chapter).

2.1 Feedforward Control

Feedforward control is also called open loop control (deutsch: Steuerung). The controller produces a signal u(t), which is independent of the current value of y(t).

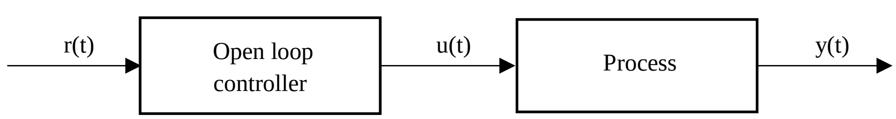


Figure 2-2. Block representation Feedforward Control.

r(t)   reference point
u(t)   input
y(t)   output
Controller   mathematical construct/function

Goal   y(t) = r(t)

Just telling, no checking
→ “stupid” system because it can’t react to changes and works only “by design”

2.2 Feedback Control

Feedback control is also called closed loop control (deutsch: Regelung).

The controller produces u(t) based on feedback of the real value of the controlled variable y(t) and comparison with the desired value r(t).

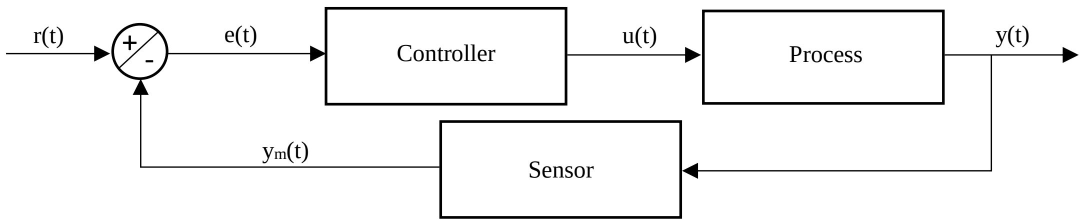


Figure 2-3. Block representation of feedback control.

e(t)	error = r(t) – y_m(t)
r(t)	reference point
u(t)	input
y(t)	output
Controller	mathematical construct/function
Sensor	measures y(t)
y_m(t)	measured output

→ “intelligent” system, which can react to changes but has a disadvantage because of the measurement that has to be done.

EXAMPLE 2.2-1: An Easy Static Model

It is given that:

$$\frac{10\ mph\ change\ of\ velocity}{1°\ throttle\ angle\ change}$$

And:

$$\frac{0.5\ mph\ change\ of\ velocity}{1\ \%\ road\ grade\ change}$$

Therefore, part of the control system will be given by:

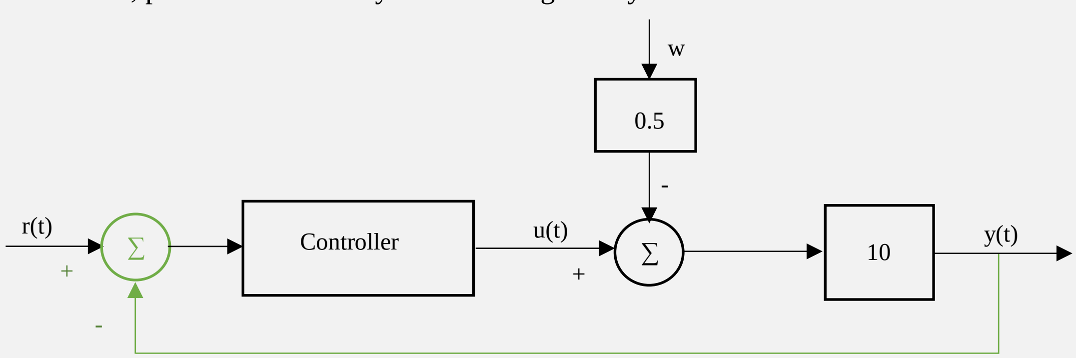


Figure 3-5. Control cycle of a car speed system**.** Green arrows stand for feedback control.

In the case of **open loop control** only the black part is relevant (**Controller = 1/10**). For the velocity you get:

$$\begin{aligned} y_{ol} &= 10(u - 0.5\,w) \\ &= 10\left(\frac{r}{10} - 0.5\,w\right) \\ &= r - 5w \end{aligned}$$

This means that on a road with no grade (w = 0): $y_{ol} = 55\ mph = r$. But on a road where w = 1: $y_{ol} = 50\ mph\ \neq r$ this are already 10 % error.

In the case of **closed loop control** also the green arrows have to be considered (assume c**ontroller gain= 100**). For the velocity you get:

$$y_{cl} = 10u - 5w$$
$$u = 100(r - y_{cl})$$

And combined:

$$\begin{aligned} y_{cl} &= 1000r - 1000y_{cl} - 5w \\ 1001\,y_{cl} &= 1000r - 5w \\ y_{cl} &= 0.999r - 0.005w \end{aligned}$$

For a grade where w = 1 and r = 55 mph $y_{ol} = 54.94\ mph$ (0.1% error). For w = 0, $y_{ol} = 54.945\ mph$ → there will be also an error for flat roads = remaining steady state error.

Goal of Feedback Control

- Compensation of ‘uncertainties’ (disturbances)
- Uncertainties can be compensated with bigger gains (factor that multiplies with the difference)

Problems

- Feedback gain cannot be randomly chosen (stability problems)
- Steady state error remains

3 System Modelling

Why should the systems be modelled? → understanding of the problem/system
→ enable manipulation and control of the system

Purpose of Modelling Dynamic Systems

- Understanding through analysis of model properties and simulation
- Control unit design and testing
- Model as explicit integral part of control units (model predictive control)

Two different types of modelling methods

- Establish fundamental physical laws and constitutive relationships = fundamental model
- Adjusting of mathematical (not based on physics) descriptions based on experimental data = black box model

→ in practice mostly a combination of both models is used.

EXAMPLE 3.0-1: Car with Cruise Control

Goal: car should drive with constant speed

Assumptions: Ignore impacts that do not contribute a lot to the problem
e.g. loss of mass due to burning fuel

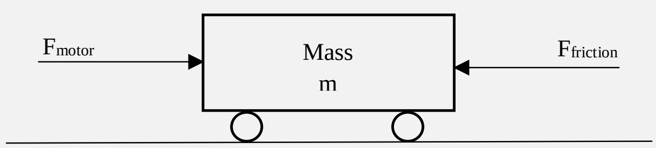


Figure 3-1. Simplified model of problem.

Mathematical description:

$$m\ddot{x} = F_{motor} - F_{friction}$$

For air resistance it holds that the friction force is proportional to the velocities and you have two different regimes:

If v small $\quad F_{friction} = b_1\dot{x}$

If v high: $\quad F_{friction} = b_2\dot{x}^2$

Therefore, we can write for small v:

$$\ddot{x} + \frac{b_1}{m}\dot{x} = \frac{F_{motor}}{m}$$

How does accelerating work:

The throttle valve is what should be controlled with the controlling system.

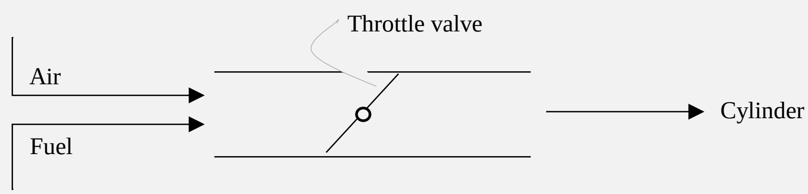


Figure 3-2. Acceleration system.

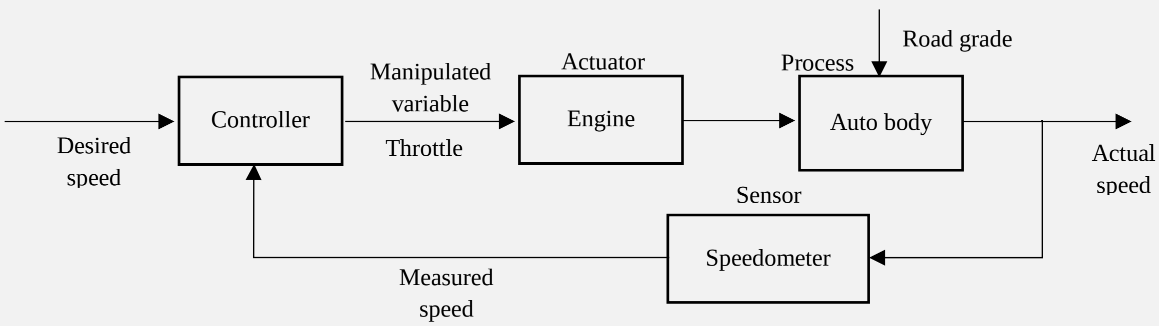


Figure 3-3. Control cycle of the acceleration system in a car with cruise control.

The control cycle is represented in Figure 3-3.

EXAMPLE 3.0-2: Car Damping

If mass is distributed uniformly then:

$$m_2 = 0.25\, M_{car}$$

Physical modelling of springs:

Hook's Law: $F = k\,\Delta l$

Damper: $F = d\, v$

Where *d* is the damping constant.

Therefore, the forces are:

$$F_{wheel} = k_1(x - r)$$
$$F_{spring} = k_2(y - x)$$
$$F_{damper} = d_2(\dot{y} - \dot{x})$$

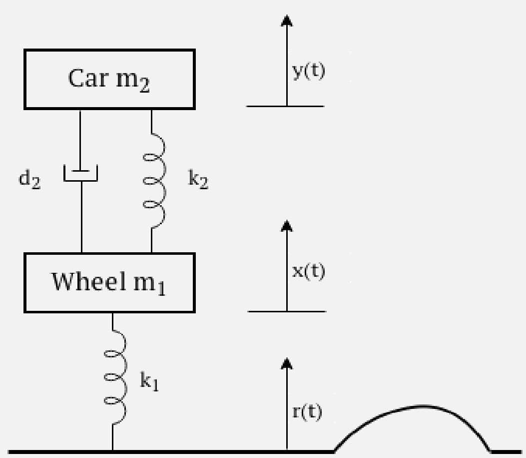


Figure 3-4. Problem description.

Newton:

$$m_1\ddot{x} = -(x - r)k_1 + (\dot{y} - \dot{x})d_2 + (y - x)k_2$$
$$m_2\ddot{y} = -(\dot{y} - \dot{x})d_2 - (y - x)k_2$$

The process follows:

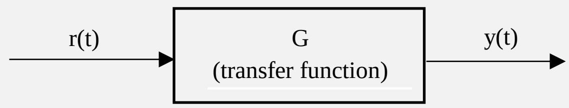


Figure 3-5. Block description for car damping**.**

EXAMPLE 3.0-3: Pendulum

If an external torque T_c is exerted on the pendulum the following equations hold:

Newton:

angular movement	$I\,\ddot{\theta} = \sum T \;=\; T_c - \;mgl\;\sin\theta$
inertia	$I = ml^2$
harmonic oscillator (if T_c =0)	$\frac{g}{l} = \;\omega^2$
	$m\ddot{x} \;= \sum F$

And from force equilibrium:

$$m\,l^2\,\ddot{\theta} = T_c - mgl\;\sin\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}$$

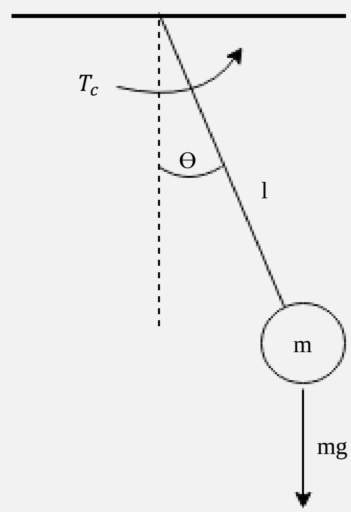


Figure 3-6. Problem description.

For $\theta \ll 1$ it holds that $\sin\theta = \;\theta$, therefore the solution is:

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{T_c}{ml^2}$$

The corresponding block diagram is as follows:

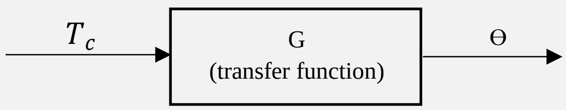


Figure 3-7. Block description of a pendulum with external torque.

EXAMPLE 3.0-4: Water Tank

Mass balances:

$$\frac{dM}{dt} = w_{in} - w_{out}$$

$$\frac{d\,(h\,A_g\,\rho)}{dt} = w_{in} - w_{out}$$

Where ρ and A_g are independent of t, therefore:

$$\frac{A_g\,\rho\,dh}{dt} = w_{in} - w_{out}$$

w_{out} can be found using the Bernoulli equation:

$$g\,\rho\,h + \frac{1}{2}\rho v^2 + p = const.$$

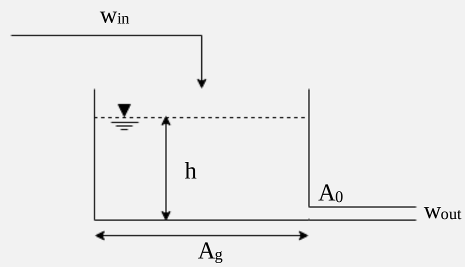


Figure 3-8. Problem description.

By writing the Bernoulli equation for the top ($v = 0$) and bottom ($h = 0$) of the tank we can get the following expression for the bottom velocity:

$$v_{bottom} = \sqrt{2\,g\,h}$$

And finally:

$$w_{out} = A_0\sqrt{2\,g\,h}\;\rho$$

By plugging this into the mass balance and rearranging for dh/dt:

$$\dot{h} = \frac{1}{\rho\,A_g}\,(w_{in} - w_{out}) = \frac{1}{\rho\,A_g}\left(w_{in} - \rho\,A_0\sqrt{2gh}\right)$$

Where w_{in} has units: kg s^{-1}

EXAMPLE 3.0-5: Thermocouple

Problem parameters:

A	Surface area thermocouple
C_p	Heat capacity thermocouple
h	Heat transfer coefficient
γ	Thermo element constant
y	Measured voltage $y = \gamma T'$
T	Temperature of fluid
T'	Temperature of thermocouple
M	Mass of thermometer

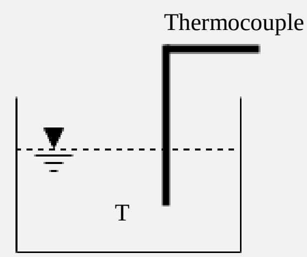


Figure 3-9. Problem description (side view).

There will be a temperature gradient between the thermocouple and the surrounding water, that can be described by Fig 3.10:

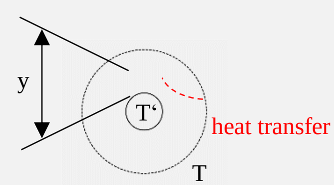


Figure 3-10. Problem description (top view).

Energy balance:

$$\frac{dE}{dt} = E_{in} - E_{out}$$
$$\frac{d\left(MC_p\,T'\right)}{dt} = hA(T - T')$$
$$\frac{d\left(MC_p\frac{y}{\gamma}\right)}{dt} = hA\left(T - \frac{y}{\gamma}\right) \qquad y = \gamma T'$$
$$\frac{C_pM}{hA}\frac{dy}{dt} + y = \gamma T$$

Where $\frac{C_pM}{hA}$ is the time constant τ of the Thermocouple, i.e. high A and low M desired.
The measured temperature is time dependent no matter how good the sensor. It will approach the true temperature only asymptotically but with a speed proportional to 1/τ.

EXAMPLE 3.0-6: Heating

Water is heated in a radiator with hot air. We assume $T_L = T_L^0$ and $T_W = T_W^0$ (instantaneous heat transfer and ideal mixing) like in a CSTR. The temperatures inside the radiator are independent of space but will change with time. The energy balance reads as follows:

$$\frac{dE}{dt} = m_L c_L \frac{dT_L}{dt} = w_L c_L T_L' - w_L c_L T_L - \frac{T_L - T_W}{R}$$
$$\frac{dE}{dt} = m_W c_W \frac{dT_W}{dt} = w_W c_W T_W' - w_W c_W T_W + \frac{T_L - T_W}{R}$$

Where c is the heat capacity and T the temperature of the liquid and water, w is the mass flux and R the resistance to heat transfer. This is a MIMO system and already quite complicated. It can be simplified when we look only at one in- and outlet and keep the other two constant so we can treat it like a SISO.

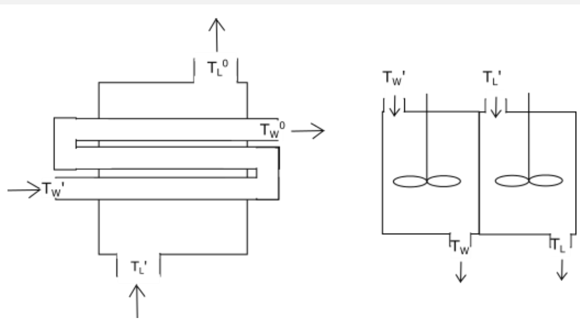
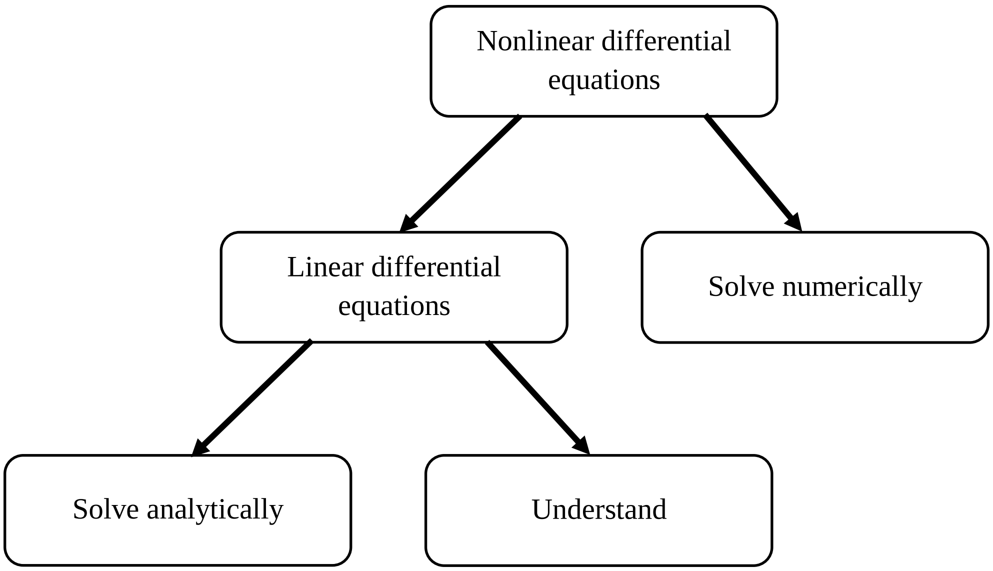


Figure 3-11. Problem description Radiator.

4 Differential Equations

General scheme of how you can deal with differential equations:



4.1 Nomenclature

PDE	Partial differential equations More than 2 variables differentiated	$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$	
ODE	Ordinary differential equations Order: highest differential	$\frac{du}{dt} = u$	first order
		$\frac{d^2u}{dt^2} + 4\frac{du}{dt} = 2$	second order
	Linear: only linear terms in u	$u^{(n)}(x) = \sum_{k=0}^{n-1} a_k(x)u^{(k)}(x) + g(x)$	

Example:	$\frac{d^2u}{dt} + \omega_0 u = 0$	linear second order
	$\frac{du}{dt} = u^2$	non-linear system

4.2 Numerical Solution

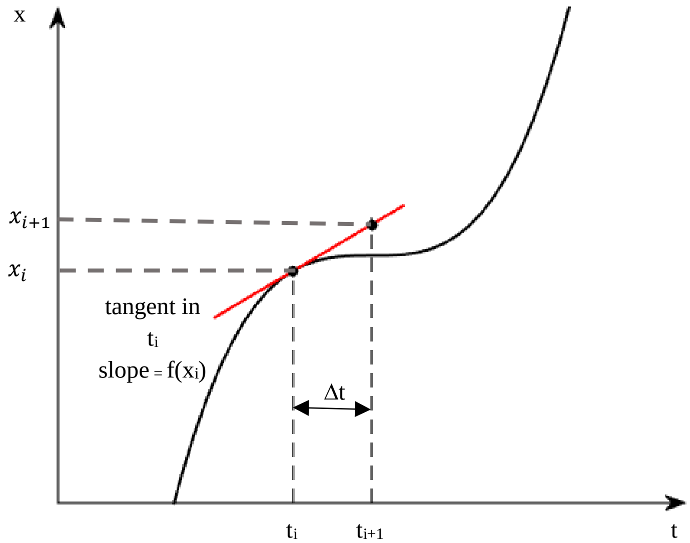


Figure 4-1. Euler method.

$$\dot{x} = f(x) = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} \approx f(x)$$

$$\frac{x_{i+1} - x_i}{\Delta t} \approx f(x)$$

Forward Euler Equation:
$x_{i+1} = x_i + \Delta t\, f(x_i)$

If $\Delta t \to dt \quad \to$ No error

4.3 Analytical Solution

General form of an ODE $\quad \dot{x} = Ax + Bu$

Homogeneous $\quad \dot{x} = Ax$

$x(t) = Ce^{At} \qquad C = const.$

Inhomogeneous $\quad \dot{x} = Ax + Bu$

Guess solution (variation of parameters)

$$x(t) = e^{At} \cdot C(t)$$

Differential of guess

$$\dot{x}(t) = AC(t)e^{At} + e^{At} \cdot \dot{C}(t)$$

Compose with original equation

$$\dot{x}(t) = Ax + Bu$$
$$Bu = e^{At} \cdot \dot{C}(t)$$
$$\dot{C}(t) = Bu\, e^{-At}$$

$$\mathrm{x(t)} = \mathrm{e}^{\mathrm{At}} \cdot \left(C(t_0) + \int_{t_0}^{t} Bu(\tau)e^{-A\tau}d\tau\right) \text{ with } \mathrm{C(t_0) = x_0}$$

$$\to x(t) = e^{At}\left(x_0 + \int_0^t e^{-A\tau}Bu(\tau)d\tau\right)$$

4.4 Linearization

Assume for small Δx: f(x) is linear, this is only valid in the near neighborhood of x.

$$f(x) \approx f(\bar{x}) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{\bar{x}} (x-\bar{x})$$

In vector format:

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

$$f(x) \approx f(\bar{x}) + \mathrm{J}|_{\bar{x}} (x-\bar{x})$$

$$\mathrm{J} = \frac{\mathrm{d}f}{\mathrm{d}x^T} = \begin{pmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}x_1} & \frac{\mathrm{d}f_1}{\mathrm{d}x_2} \dots \\ \frac{\mathrm{d}f_2}{\mathrm{d}x_1} & \dots \end{pmatrix}$$

For DGL-Systems:

u	input
y	output
x	state
$\dot{x}, y$	transfer functions

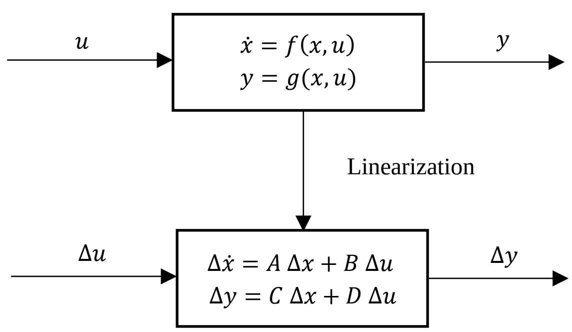


Figure 4-2. Linearization of DGL Systems.

Linearization of $\dot{x}$ gives:

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{x} &\approx f(\bar{x},\bar{u}) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{\bar{x}} (x-\bar{x}) + \left.\frac{\mathrm{d}f}{\mathrm{d}u}\right|_{\bar{u}} (u-\bar{u}) \\ &\approx 0 + A(x-\bar{x}) + B(u-\bar{u}) \\ \rightarrow \dot{x} &= A\,\Delta x + B\,\Delta u = \Delta\dot{x} \end{aligned}$$

Same with y gives:

$$\begin{aligned} y &= g(x,u) \\ y &\approx g(\bar{x},\bar{u}) + \left.\frac{\mathrm{d}g}{\mathrm{d}x}\right|_{\bar{x}} (x-\bar{x}) + \left.\frac{\mathrm{d}g}{\mathrm{d}u}\right|_{\bar{u}} (u-\bar{u}) \\ &\approx \bar{y} + C(x-\bar{x}) + D(u-\bar{u}) \\ \rightarrow \Delta\mathrm{y} &= C\,\Delta x + D\,\Delta u \end{aligned}$$

EXAMPLE 4.4-1: Water Tank

Remember from Example 3.0-4:

$$\frac{dh}{dt} = \frac{1}{A\rho}\left(w_{in} - A_0\rho\sqrt{2gh}\right)$$

Constant	A_0, ρ, A, g
Variables	h, w_{in}

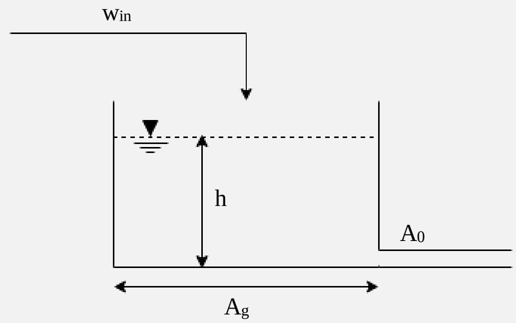


Figure 4-3. Problem description.

Nonlinear in h

$$\frac{dh}{dt} = f$$

$$f \approx f_{ss} + \left.\frac{\partial f}{\partial h}\right|_{ss}\Delta h + \left.\frac{\partial f}{\partial w_{in}}\right|_{ss}\Delta w_{in}$$

Steady state:

$$0 = \frac{dh}{dt} = \frac{1}{A\rho}\left(w_{in}^{ss} - A_0\rho\sqrt{2gh^{ss}}\right)$$

$$h^{ss} = \frac{1}{2g}\left(\frac{w_{in}^{ss}}{\rho A_0}\right)^2$$

Linearization:

$$f \approx 0 - \frac{A_0}{A}\frac{\sqrt{2gh^{ss}}}{2h^{ss}}\Delta h + \frac{1}{A\rho}\Delta w_{in}$$

$$\frac{d\Delta h}{dt} = \frac{d(h - h^{ss})}{dt} = \frac{dh}{dt} - \frac{dh^{ss}}{dt} = \frac{dh}{dt}$$

$$f = \frac{dh}{dt} = A'\Delta h + B'\Delta w_{in} = \frac{d\Delta h}{dt}$$

Therefore, we get the following linear equation:

$$A' = -\frac{\rho A_0^2 g}{A w_{in}^{ss}};\quad B' = \frac{1}{A\rho}$$

Valid only close to steady state.

4.5 Superposition

If an input signal is complex, you could split the complex input into multiple additive simpler inputs. If the system is linear, the output of the complex input can be calculated by summing up the outputs of the simpler inputs.

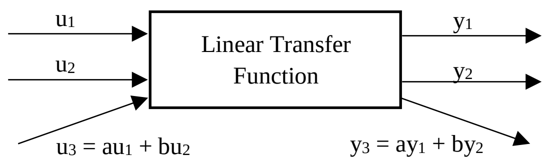


Figure 4.4. Superposition of multiple input signals.

EXAMPLE 4.5-1: Superposition

$$\dot{y} + ky = u$$

1) $\dot{y}_1 + ky_1 = u_1$

2) $\dot{y}_2 + ky_2 = u_2$

3) $u_3 = au_1 + bu_2$

$$\dot{y}_3 + ky_3 = u_3$$

Demonstrate that $y = y_1 + y_2$

Which means: $\dot{y} = \dot{y}_1 + \dot{y}_2$

and $u = u_1 + u_2$

$$(a\dot{y}_1 + b\dot{y}_2) + k(ay_1 + by_2) = au_1 + bu_2$$

$$a(\dot{y}_1 + ky_1 - u_1) + b(\dot{y}_2 + ky_2 - u_2) = 0$$

Because $\dot{y} + ky - u = 0$ ▪

4.6 Convolution

Simple inputs → g(t) → outputs

	Input $u(t)$	Output $y(t)$
Pulse	$\delta(t)$	$g(t)$ pulse response
Pulse with time delay	$\delta\,(t-\tau)$	$g(t-\ \tau)$
Weighted pulse	$u(\tau)\ \delta\ (t-\tau)$	$u(\tau)\ g\ (t-\tau)$
Generic Input $u(t)$	$u(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t-\tau)d\tau$	$y(t) = \int_{-\infty}^{+\infty} u(\tau)g(t-\tau)d\tau$

$$y(t) = \int_{-\infty}^{+\infty} u(\tau)g(t-\tau)d\tau$$
$$y = u * g$$

If $u(t) = 0$ for $t < 0$

then
$$y(t) = \int_{0}^{+\infty} u(\tau)g(t-\tau)d\tau$$

And if additionally $g(t) = 0$ for $t < 0$

then
$$y(t) = \int_{0}^{t} u(\tau)g(t-\tau)d\tau$$

5 Laplace Transformation

5.1 Laplace Transformation

The Laplace transformation of a function $f(t)$ $(f(t) = 0,\ t < 0)$ is defined as

$$\int_0^\infty f(t)e^{-st}\, dt \ = F(s) = \ \mathcal{L}\{f(t)\}$$

(Which is equivalent to the "continuous power series" $\int_0^\infty f(t)x^t dt$ with $s = -\ln(x)$.
It exists for all functions, which fulfill the condition of Dirichlet:

$$\int_0^\infty |f(t)|e^{-\sigma t}\, dt \ < \infty$$

For a large enough σ

EXAMPLE 5.1-1: for f(t) = 1

$$F(s) = \int_0^\infty 1\, e^{-st}\, dt = \lim_{A\to\infty} -\frac{e^{-st}}{s}\bigg|_{t=0}^{t=A} = \lim_{A\to\infty}\left(\frac{e^{-sA}}{s} + \frac{1}{s}\right) = \frac{1}{s}$$

5.2 Inverse Laplace Transformation

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\sigma_j - j\infty}^{\sigma_j + j\infty} e^{st}F(s)\, ds = \begin{cases} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

If $F(s)$ is a fraction, the easiest way to find $f(t)$ is the partial fraction expansion of $F(s)$ and the use of table (5.3) of simple functions.

$\mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\}$; p(s), q(s) are polynomials; order of p(s) ≤ order of q(s) (true for all physical systems).

$$\frac{p(s)}{q(s)} = \sum_{i=1}^{n}\frac{A_i}{s - s_i} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_n}{s - s_n}$$

s_i = roots of q(s) = 0, $A1, A2,$ ... partial fractions

5.3 Important Laplace Transformations

The following LPT have to be remembered.

$f(t)$	$F(s)$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{b}e^{-at}\sin bt$	$\frac{1}{(s+a)^2+b^2}$
$1-e^{at}$	$\frac{a}{s(a-s)}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

5.4 Properties of Laplace Transformations

1) Time shift
   $f(t-\tau)$ $\leftrightarrow F(s)\, e^{-\tau s}$

2) Superposition
   $af_1(t)+bf_2(t)$ $\leftrightarrow\ aF_1(s)+bF_2(s)$

3) Convolution
   $f_1(t)*f_2(t)$ $\leftrightarrow F_1(s)\cdot F_2(s)$
   $\int_0^t f_1\,(t)\,f_2(t-\tau)d\tau$

4) Final value theorem
   $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\cdot F(s)$ (not applicable if f diverges e.g. maximum one node in the origin and the real part of all other nodes is negative)

5) Initial value theorem
   $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$

6) Differentiation

   $f'(t); f(0)$ $\leftrightarrow\ sF(s)-f(0)$

   $f''(t); f(0); f'(0)$ $\leftrightarrow s^2F(s)-s\,f(0)-f'(0)$

   $f^n(t); f(0)=0;\ f^{n-1}\,(0)=0$ $\leftrightarrow\ s^nF(s)$ All initial conditions are 0.

7) Integral:
   $\int_0^t f(\tau)d\tau$ $\leftrightarrow \frac{F(s)}{s}$

EXAMPLE 5.4-1: Laplace Transformation

$$\frac{\mathrm{d}y}{\mathrm{d}t}+ay=0$$
$$\underbrace{s\,Y(s)}_{s(Y(0)-f(0))}\ -1+aY(s)=\ 0$$
$$Y(s)=\frac{1}{s+a}\quad\rightarrow\quad y(t)=\ e^{-at}$$

EXAMPLE 5.4-2: Linear Dynamic System

$$\tau \frac{\mathrm{d}\, y(t)}{\mathrm{d}\, t} + y(t) = \gamma\, T(t)$$
$$\tau(s\, Y(s) - y(0)) + \ Y(s) = \gamma t(s)$$
$$Y(s) = \frac{\gamma}{\tau\, s + 1} T(s) + \frac{\tau(y(0))}{\tau s + 1}$$

EXAMPLE 5.4-3: Final Value Theorem

→ **REMEMBER:** the final value theorem can only be used if the finite final value exists.

Easy example:

$$Y(s) = \frac{3(s+2)}{s(s^2 + 2s \ + \ 10)}$$
$$y(t \to \infty) = \lim_{s \to 0} s \frac{3(s+2)}{s(s^2 + \ 2s \ + \ 10)} = \frac{3 \cdot 2}{10} = 0.6$$

But if there is no finite final value, the final value theorem does not work, even though you might get a solution:

$$Y(s) = \frac{3}{s(s-2)}$$

→ with the final value theorem you would get:

$$y(t \to \infty) = -\frac{3}{2}$$

But actually:

$$y(t) = \frac{3}{2}(-1 + e^{2t}) \to \lim_{t \to \infty} y(t) = \infty$$

In this case you cannot use the final value theorem because the function diverges.

In General

$$Y(s) = \frac{U(s)}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} + \frac{I_{n-1}}{a_0 s^n + a_1 s^{n-1} + \cdots a_n}$$

Output = Input + Initial condition

Characteristic polynomial $\quad a_0 s^n + a_1 s^{n-1} + \cdots + a_n$

Characteristic equation $\quad a_0 s^n + a_1 s^{n-1} + \cdots + a_n \overset{!}{=} 0$

→ Solutions of characteristic equations → roots of system s_i, s_j

5.5 Initial Conditions

In process control we are often looking at the system behavior relative to a reference state. We start in the steady state ($x_{ss}. y_{ss}, u_{ss}$) and look at the deviation variables ($\Delta x, \Delta y, \Delta u$), which represent the deviation from it. This is useful as we obtain a simple transfer function that expresses changes from the steady state.

Looking at the example of the thermocouple (**EXAMPLE 3.0-5**), where we had

$$\tau \frac{\mathrm{dy}}{\mathrm{dt}} + y = \gamma T$$

In the steady state $$T = T_{ss} \quad \frac{\mathrm{dy}_{ss}}{\mathrm{dt}} = 0$$

Deviation variables $$\Delta \mathrm{T} = \mathrm{T} - T_{ss} \quad \Delta \mathrm{y} = \mathrm{y} - y_{ss}$$

$$\tau \frac{\mathrm{d}\Delta \mathrm{y}}{\mathrm{dt}} + \Delta y = \gamma \Delta T \qquad \Delta y(0) = 0, \quad \Delta T(0) = 0$$

Laplace transformation $$\tau\, s\, \overline{Y(s)} + \overline{Y(s)} = \gamma\, \overline{T(s)}$$

$$\overline{Y(s)} = \frac{\gamma}{\tau s + 1} \overline{T(s)}$$

In words: „output = transfer function × input“

6 Block Diagram Algebra

Block diagram algebra can be used for the modeling of systems. When the initial conditions are zero (see 5.5) the transfer function $G(s)$ relates the inlet $U(s)$ to the outlet $Y(s)$.

$$Y(s) = G(s) \cdot U(s)$$

6.1 Series of Blocks

Two blocks in series behave multiplicative:

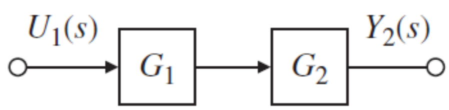


Figure 6-1. Blocks in series.

The transfer function is given by:

$$Y_1 = G_1 \cdot U_1$$
$$Y_2 = G_1 \cdot G_2 \cdot U_1$$
$$\frac{Y_2(s)}{U_1(s)} = G_1 \cdot G_2$$

6.2 Parallel Blocks

Parallel blocks behave additive as stated by the superposition principle:

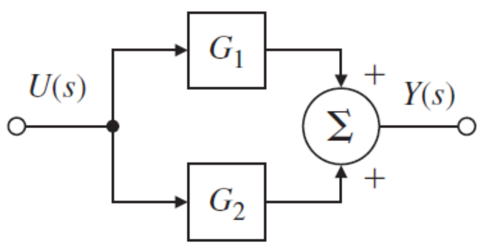


Figure 6-2. Parallel blocks.

The transfer function is given by:

$$Y = Y_1 + Y_2$$
$$Y = (G_1 + G_2) \cdot U$$
$$\frac{Y(s)}{U(s)} = G_1 + G_2$$

6.3 Feedback Control Loop

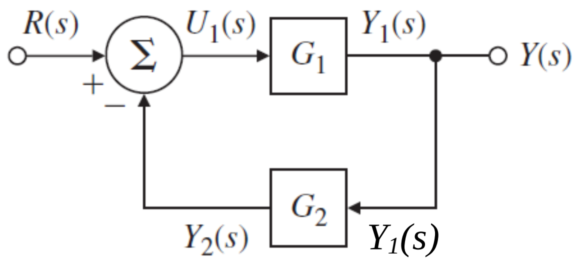


Figure 6-3. Control cycle of Feedback control.

Transfer function is given by:

$$Y_1 = G_1 \cdot U_1$$
$$U_1 = R - Y_2$$
$$Y_2 = G_2 \cdot Y_1$$
$$Y_1 = G_1(R - G_2 Y)$$
$$Y = \frac{G_1}{1 + G_1 \cdot G_2} \cdot R$$

Or written in a different way:

$$\frac{\mathrm{Y(s)}}{R(s)} = \boldsymbol{TF}_{\text{closed loop}} = \frac{\boldsymbol{TF}_{\text{input to output}}}{\mathbf{1} - \boldsymbol{TF}_{\text{loop}}}$$

In the example above:

$$\boldsymbol{TF}_{\text{input to output}} = G_1$$
$$\boldsymbol{TF}_{\text{loop}} = -\mathrm{G}_1 \cdot \mathrm{G}_2$$

6.4 Equivalent Diagrams

For the representation of more complex systems, rearranging and the use of the formulas above can be used to simplify and obtain the transfer function. Additional rules are listed below.

- Nodes can be moved by introducing blocks with the inverse of the block:

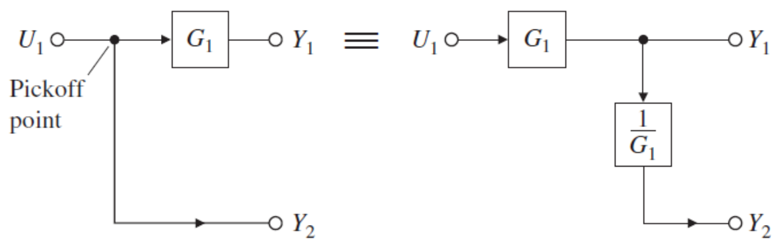


Figure 6-4. Moving nodes by introducing the inverse of the block.

- Blocks can be moved in front of a sum by introducing this block to all contributing inputs:

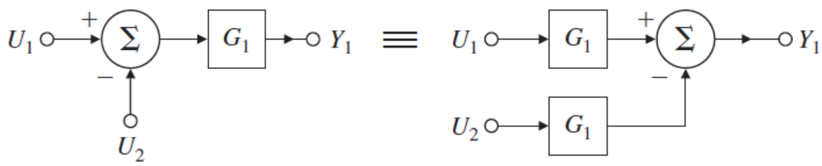


Figure 6-5. Moving a block in front of a sum by introducing it to all inputs.

- The feedback loop block diagram can be rearranged in the following way (more suitable for certain applications):

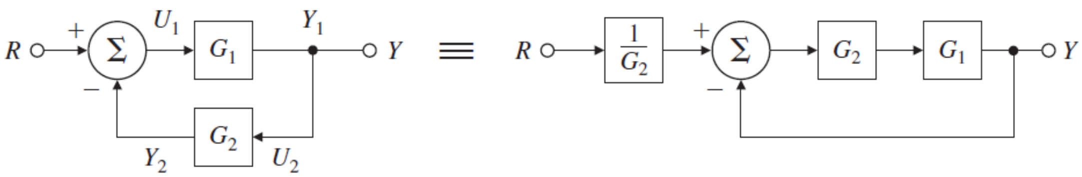


Figure 6-6. Rearranging of the feedback control loop.

7 System Response

The following system is considered

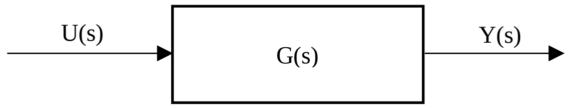


Figure 7-1. First order system.

7.1 1st Order Systems

Systems described by a first order ODE have a first order response, e.g. a CSTR without reaction.
The transfer function $G(s)$ reads:

$$G(s) = \frac{k}{\tau s + 1}$$

where k is the gain (Verstärkungsfaktor) and τ is the time constant (Zeitkonstante).

As was introduced in the previous chapter, the transfer function can be used to express the dependence of the output signal $Y(s)$ on the input signal U(s):

$$Y(s) = \frac{k}{\tau s + 1} \cdot U(s)$$

Pulse Input

In this case the input signal U(s) is:

$$U(s) = 1$$

Therefore, the Laplace transform of the output signal Y(s) becomes:

$$Y(s) = \frac{k}{\tau s + 1} \cdot 1$$

The back transformation (using the table on p.20) of Y(s) gives the output signal as a function of time y(t):

$$y(t) = \frac{k}{\tau} \cdot e^{-\frac{t}{\tau}}$$

And the slope at t = 0:

$$\left.\frac{dy}{dt}\right|_{t=0} = -\frac{k}{\tau^2}$$

Initial value theorem
$$y(t = 0) = \lim_{s \to \infty} s \cdot Y(s) = \lim_{s \to \infty} s \cdot \frac{k}{\tau s + 1} = \frac{k}{\tau}$$

Final value theorem
$$y(t \to \infty) = \lim_{s \to 0} s \cdot Y(s) = \lim_{s \to 0} s \cdot \frac{k}{\tau s + 1} = 0$$

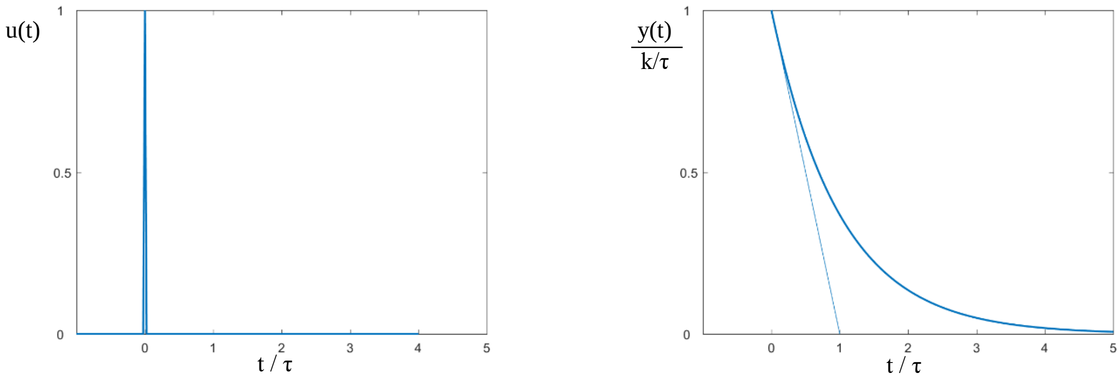


Figure 7-2. Input (left) and output (right) signal of a pulse input.

Step Input

The input signal U(s) of a step function can be written as:

$$U(s) = \frac{1}{s}$$

Therefore, the output signal Y(s) becomes:

$$Y(s) = G(s) \cdot U(s) = \frac{k}{\tau s + 1} \cdot \frac{1}{s}$$

Back transformation gives the output as a function of time y(t):

$$y(t) = k(1 - e^{-\frac{t}{\tau}})$$

Initial value theorem $\quad y(t=0) = \lim_{s\to\infty} s \cdot \frac{k}{\tau s + 1} \cdot \frac{1}{s} = 0$

Final Value Theorem $\quad y(t \to \infty) = \lim_{s\to 0} s \cdot \frac{k}{\tau s + 1} \cdot \frac{1}{s} = k$

Initial slope $\quad \dot{y}(t=0) = \lim_{s\to\infty} s \cdot s \cdot Y(s) = \lim_{s\to\infty} s \cdot s \frac{k}{\tau s + 1} \cdot \frac{1}{s} = \frac{k}{\tau}$

Remark: This approach also works for any other derivative.

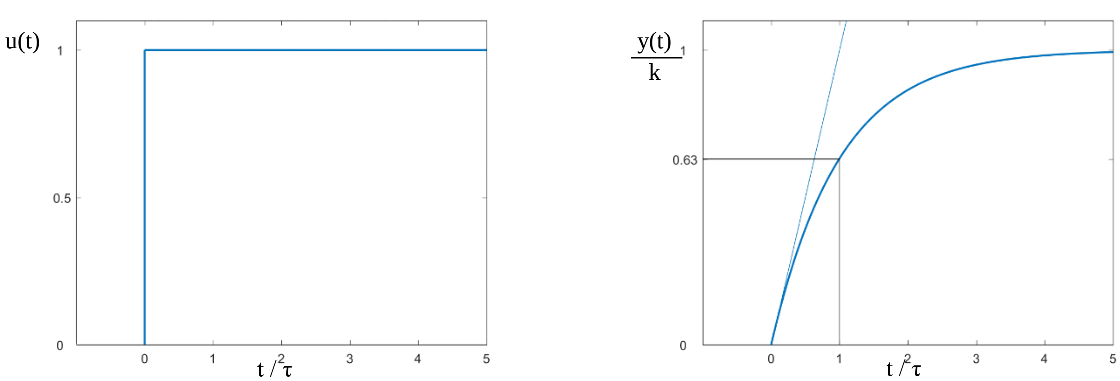


Figure 7-3. Input and output signal of a step input.

The process signal does not respond immediately to the change in input. The response is delayed and approaches the new steady state value only asymptotically. After a time equal to the time constant τ it has reached 63.2% of the input value.

How does the signal change by varying the parameters k and τ? Note that the axes are no more normalized.

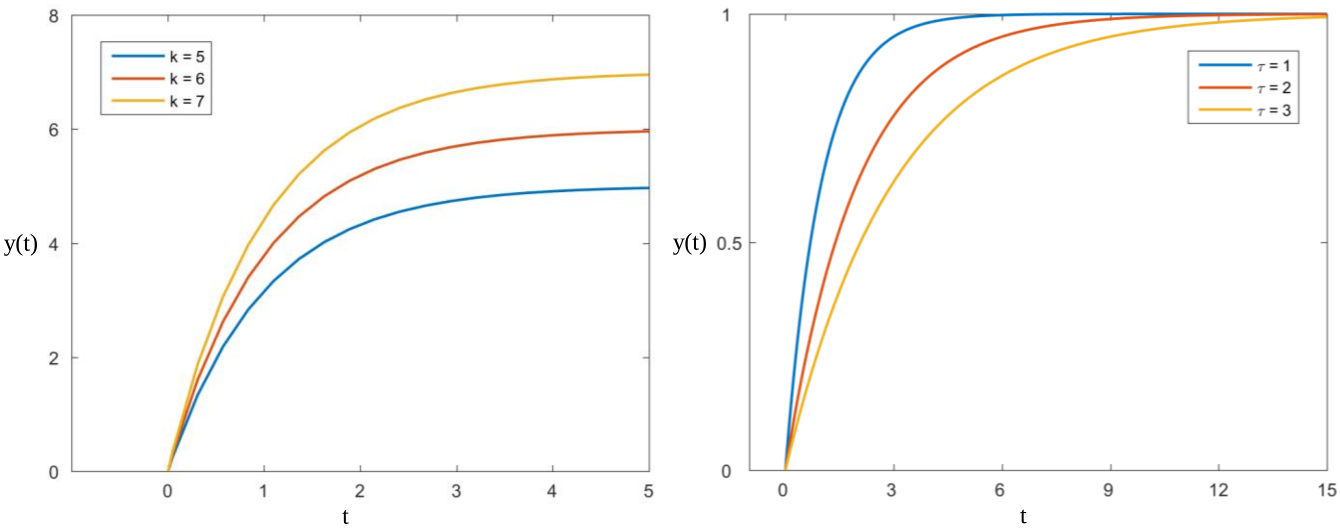


Figure 7-4. Left: Change of output signals with varying parameter k (gain).
Right: Change of output signals with varying parameter τ (time constant).

If k increases:
$\rightarrow y(t \rightarrow \infty)$ multiplied

If τ increases:
→ y(t) increases slower to asymptote

By changing the gain, the new steady state which the system approaches can be altered. It increases with increasing gain. When we vary the time constant, the speed with which the system approaches the new steady state changes. The higher the time constant, the slower the system evolves.

Ramp Input

The Laplace transformed ramp input signal $U(s)$ is:

$$U(s) = \frac{1}{s^2}$$

Therefore, the output signal $Y(s)$ is given by:

$$Y(s) = \frac{k}{\tau s + 1} \cdot \frac{1}{s^2}$$

Back transformation gives:

$$y(t) = k(t - \tau + \tau e^{-\frac{t}{\tau}})$$

Initial value theorem	$y(t = 0) = 0$
Final slope	$\dot{y}(t \to \infty) = k$
Initial slope	$\dot{y}(t = 0) = 0$

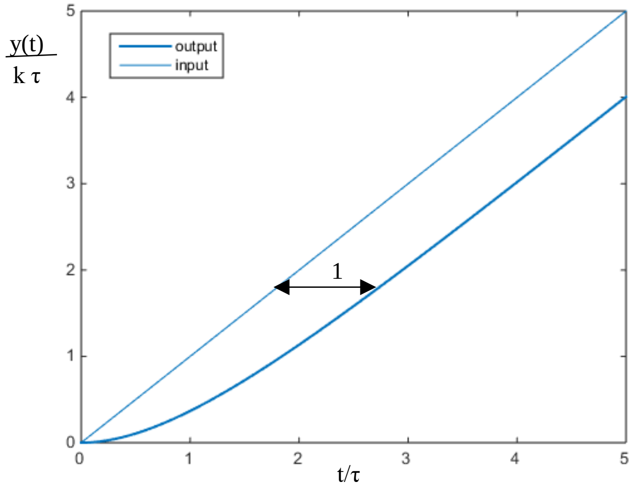


Figure 7-5. Input and output signal in case of a ramp input.

Note: If you plot the input and output signal in non-normalized axes, then the difference between the two signals would be the time constant τ.

7.2 2nd Order Systems

Most mechanical systems or two coupled systems of 1st order are 2nd order systems. All systems with 2nd order derivatives have a second order response, e.g. pendulum and many other mechanical systems ($F = m\ddot{x}$).

The transfer function G(s) reads:

$$G(s) = \frac{k}{\tau^2 s^2 + 2\tau\xi s + 1} = \frac{k\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where k is the gain, τ the time constant, ω_n the natural frequency and ξ the damping factor.

The roots of the characteristic polynomial

$$\tau^2 s^2 + 2\tau\xi s + 1$$

i.e. the solutions of the characteristic equation

$$\tau^2 s^2 + 2\tau\xi s + 1 = 0$$

$$s_{1,2} = -\frac{1}{\tau}\left(\xi \pm \sqrt{\xi^2 - 1}\right)$$

We can describe the shape of the solution of these systems by looking at the damping factor ξ:

$0 < \xi < 1$	roots are imaginary	$\rightarrow$	underdamped system, oscillation
$1 < \xi$	roots are real	$\rightarrow$	overdamped system, no oscillation
$1 = \xi$	only 1 discrete root	$\rightarrow$	critically damped

Step input

The output signal Y(s) of a step function of a second order system looks as follows:

$$Y(s) = \frac{k}{\tau^2 s^2 + 2\tau\xi s + 1} \cdot \frac{1}{s}$$

For a graphical representation of the different cases of damping for a second order system see Figure 7-6.

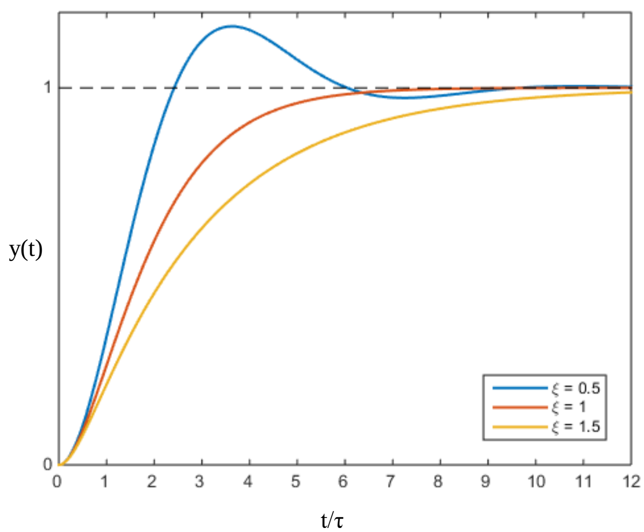


Figure 7-6. Step response of a second order system with different damping values and $\boldsymbol{k = 1}$.

Higher Order Systems

Step inputs for higher order systems can be formulated analogously (see Figure 7-7). This also models the transition of a CSTR to a PFR as a cascade of infinite CSTRs.

$$Y(s) = \left(\frac{k}{\tau s + 1}\right)^n \cdot \frac{1}{s}$$

$$\lim_{n\to\infty}\left(\frac{1}{\frac{\tau}{n}s+1}\right)^n = e^{-\tau s} \quad \xrightarrow{\mathcal{L}^{-1}} \text{time shift } f(t-\tau)$$

Remember that exponential functions in the Laplace space are just time shifts in the time space.
In this case the time constant of the total system would be equal to the sum of the time constants of the individual systems. For $n = \infty$ we would again have a plot in the form of the initial pulse but with a delay, like for a PFR.

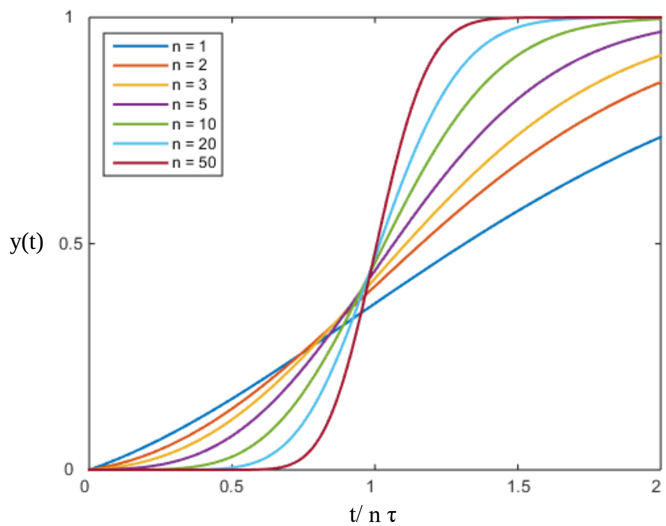


Figure 7-7. Step response of nth order systems.

7.3 Specifications within the Time Domain

For a damped second order system without roots the following holds:

Rise time: $t_r \cong \frac{1.8}{\omega_n}$

Overshoot: $M_p = \exp(\frac{\pi\xi}{\sqrt{1-\xi^2}}) \cong \begin{cases} 5\ \% & \xi = 0.7 \\ 16\ \% & \xi = 0.5 \\ 35\ \% & \xi = 0.3 \end{cases}$

Settling time: $t_s \cong \frac{4.6}{\omega_n \xi}$

Peak time: $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

- We can find the peak time and the overshoot if we set the derivative of the function zero and look for the maximum
- An additional root in the left half plane (LHP) enforces the overshoot if the root is less then factor 4 distant from the real part of the poles.
- An additional root in the right half plane (RHP) inhibits overshooting and can even lead to undershooting.
- An additional pole extends the rise time, if the additional pole is less then factor 4 distant from the complex part of the complex poles.

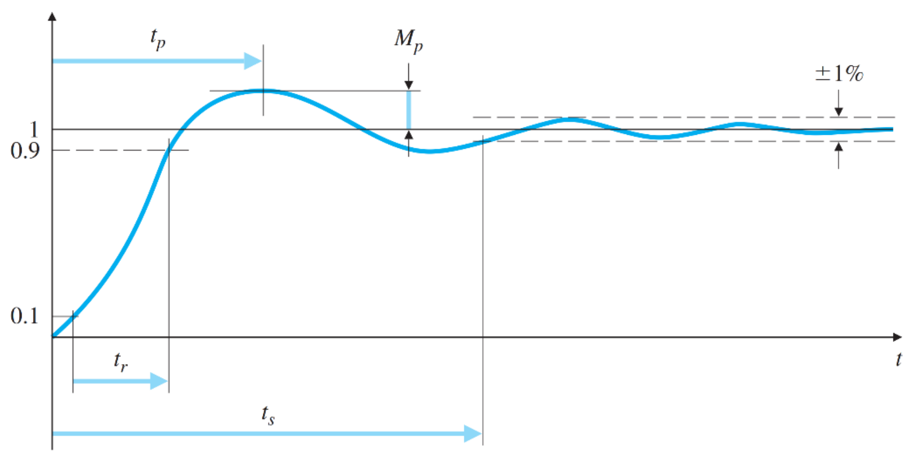


Figure 7-8. Specifications within the time domain. Figure from [FrPE06]

8 System Stability

A system is asymptotically stable, if there are no 'internal' state variables that go towards infinity but all go to zero when t → ∞. When perturbed slightly the system will go back into the steady state.

Consider the following transfer function of a system:

$$G(s) = \frac{p(s)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{p(s)}{\prod_{i=1}^{n}(s - p_i)}$$

Assume that all roots $p_1, p_2, \ldots p_n$ of the characteristic equation

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

are distinct (einfach). A differential equation with this characteristic equation has a solution of the form:

$$y(t) = \sum_{i=1}^{n} K_i e^{p_i t}$$

This system is stable, if all terms in the series go to zero for t → ∞.

In other words:

If all $p_i < 0$	for $t \to \infty$	$y(t) = 0$	converging poles stable system
If one $p_i > 0$	for $t \to \infty$	$y(t) \to \infty$	diverging poles unstable system
If one $p_i = 0$	for $t \to \infty$		neither con- nor diverging poles asymptotically stable system

If all poles are zero, these simple rules can't be applied to predict system behavior. This is due to the fact that the linearization introduces an error into the calculation. It is therefore highly unlikely that all roots are truly zero and a different approach would be needed or more terms included into the linearization.

If there are complex conjugate poles p_i, the solution would be given by an oscillatory response:

$$y(t) = A_1 e^{p_i t} = A(\cos(t) + i \sin(t))$$

.

By looking at the location of the poles one can get a feeling of the system behavior and analyze the stability. A pole to the right of the imaginary axis, with positive real part is called RHP (right-half plane pole). Such a pole will lead to an unstable system as it results in a system response which grows without bound. The term of a LHP (left-half plane pole) in contrast will go to zero and the system will approach the stable steady state.

Roots on the right hand side of the figure lead to unstable systems which are not desirable for process control applications. The more on the left, the faster the system evolves towards its steady state. When we compare the evolution speed of two roots (with roots with purely negative real parts), we must look at the rightmost root (the one closest to zero) as this one governs the speed. The system with the more negative right root will approach its steady state faster.

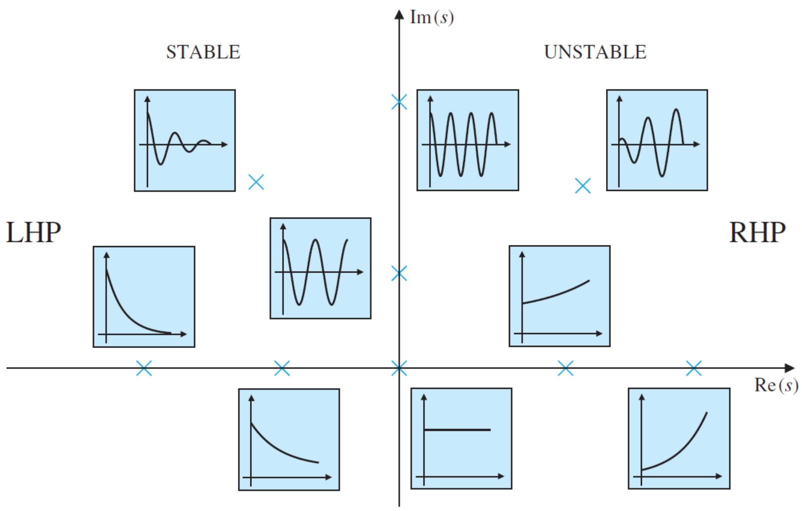


Figure 8-1. Root Locus Impulse responses as a function of the position of the roots.

9 Sensitivity

With a sensitivity analysis we can determine the effect of changes in A or due to uncertainties (if A is not known precisely).

Assumption: Engine gain goes from A to $A + \delta A$.

Open loop control:

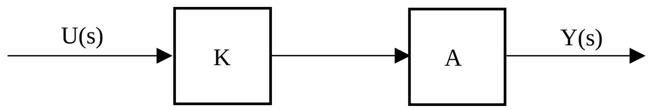


Figure 9-1. Open loop control system

$$KA = T_{ol}$$
$$K(A + \delta A) = T_{ol} + \delta T_{ol}$$

Where T_{ol} is the open loop transfer function.
Rearrange

$$K(A + \delta A) = \frac{T_{ol}}{A}(A + \delta A) = T_{ol}\left(1 + \frac{\delta A}{A}\right) = T_{ol} + \delta T_{ol} = T_{ol}\left(1 + \frac{\delta T_{ol}}{T_{ol}}\right)$$
$$1 + \frac{\delta A}{A} = 1 + \frac{\delta T_{ol}}{T_{ol}}$$

Relative change:

$$\frac{\delta T_{ol}}{T_{ol}} = \frac{\delta A}{A}$$

Hendrik W. Bode named this ratio the *Sensitivity of the transfer behavior* on changes of A.

$$S_A^T = \frac{\delta T/T}{\delta A/A} \approx \frac{A}{T}\frac{dT}{dA}$$

For open loops

$$S_A^{T_{ol}} = 1$$

When a relative change in A occurs, for an open loop, it will translate into a change in T of equal size. Thus, if the engine gains 5%, the velocity will also rise by 5%.

Closed loop control:
When we add a feedback loop to the system above

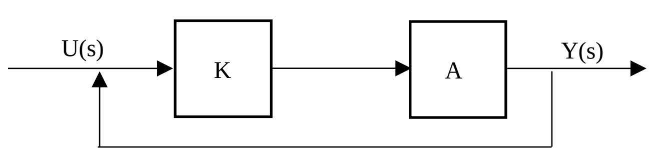


Figure 9-2. Closed loop control system

The transfer function changes and

$$\frac{AK}{1 + AK} = T_{cl}$$

$$\frac{(A + \delta A)K}{1 + (A + \delta A)K} = T_{cl} + \delta T_{cl}$$

Using the above formula for the sensitivity S_A^T:

$$S_A^{T_{cl}} = \frac{A}{\frac{Ak}{1 + Ak}} \frac{(1 + Ak)k - k(Ak)}{(1 + Ak)^2} = \frac{1}{1 + Ak}$$

Therefore, for closed loops:

$$S_A^{T_{cl}} = \frac{1}{1 + Ak}$$

This illustrates the advantage of feedback control. Sensitivity to changes is usually smaller than 1 and closed loop control reduces the effect of changes in the gain factor A by a factor of $1 + Ak$ when compared with open loop control. By choosing k large, sensitivity and the effect of disturbances on the system can be reduced.

EXAMPLE 9-1: Advantage of Feedback

Consider a closed loop with the gain k is such that

$$1 + Ak = 100$$

we can calculate the sensitivity

$$S_A^{T_{cl}} = \frac{1}{1 + Ak} = 0.01$$

a 10% change in A

$$\frac{\delta A}{A} = 0.1$$

would then lead to a change of

$$\delta T = \frac{\delta A}{A} \cdot S_A^{T_{cl}} = 0.001$$

The change in steady state gain in a closed loop would thus be 0.1%. When compared with the change for an open loop under the same conditions, which would be 10% as the sensitivity is one, the closed loop is much less sensitive.

One of the main reasons for the use of feedback control is not that it can respond easier to changes but the fact that the system is not precisely known. In order to compensate for such uncertainties a low sensitivity is desirable as it will reduce the effects of changes in the transfer function.

10 Dynamic Behavior

10.1 P-Controller

- A static k does not change the dynamics of an open loop control system
- A static k changes the dynamics of closed loop control systems

Consider the following closed loop control system with a P controller:

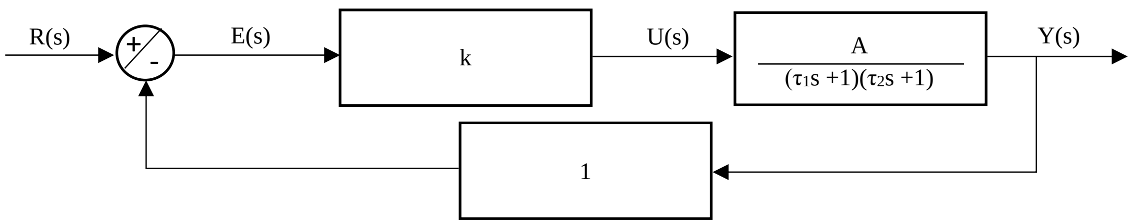


Figure 3-3. Control cycle with a second order system.

This system for example could represent two stirred tanks in series with residence times τ_1 and τ_2.

The transfer function of this system (only with a P-controller) is:

$$\frac{Y(s)}{R(s)} = \frac{kA}{(\tau_1 s + 1)(\tau_2 s + 1) + kA} \qquad \text{with} \qquad A > 0$$

The dynamics of the system can be understood by looking at the roots of the transfer function.

→ the roots of the characteristic equation as a function of the gain k are:

$$(\tau_1 s + 1)(\tau_2 s + 1) + Ak = 0$$

$$\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1 + Ak = 0$$

$$s_{1,2} = \frac{-(\tau_1 + \tau_2) \pm \sqrt{(\tau_1 + \tau_2)^2 - 4\tau_1\tau_2(1 + Ak)}}{2\tau_1\tau_2} = \frac{-(\tau_1 + \tau_2) \pm \sqrt{(\tau_1 - \tau_2)^2 - 4Ak\tau_1\tau_2}}{2\tau_1\tau_2}$$

For $k = 0$ (no P-controller) we get $s_{1,2} = \frac{-1}{\tau_{1,2}}$

For 2nd order systems:

- k affects the roots => affects the dynamics of the response
- first, increasing k will shift the roots towards the left half plane and thus will result in faster response (desired)

- at some point the discriminant becomes zero and there will be a multiple root
- further increasing k results in a negative discriminant and complex conjugate poles and thus in an oscillating response
- if k is big there is a “trade-off” between low stationary error (low sensitivity, see previous chapter) and bad dynamic behavior.

→ complex controllers with more degrees of freedom are required (see next chapter)

Plotting the behavior of the roots as a function of the parameter k gives the so called root locus:

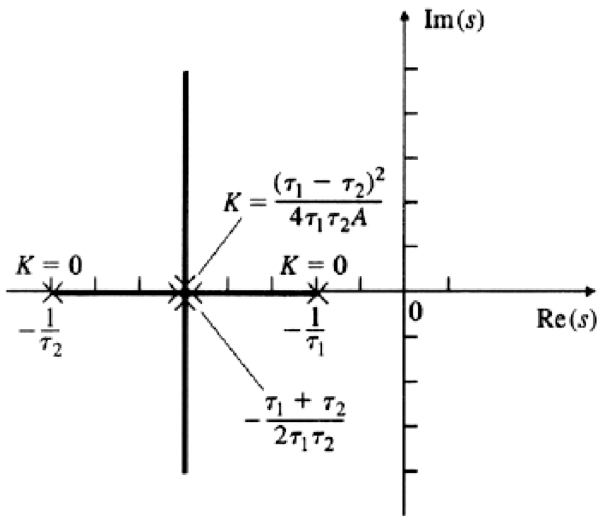


Figure 10-2. Root locus
With increasing k the roots approach the multiple root from both sides (on the horizontal) and then split when they reach it and continue on the vertical line drawn.

In the range of $0 < k < \frac{(\tau_1-\tau_2)^2}{4A\tau_1\tau_2}$ the roots are both real and negative, the system is thus stable and non-oscillating.

For $k > \frac{(\tau_1-\tau_2)^2}{4A\tau_1\tau_2}$ the roots are a complex conjugate pair and the system is oscillating.

From this analysis, k is chosen as large as possible without creating imaginary roots:

→ the more left the rightmost root, the faster the response (see Chapter 8)

EXAMPLE 10.0-1: Pendulum

Remember the variables from last time:

T_c	external force
m, g, l, I	constants
θ, T_c	variables

And the describing function:

$$f = I\ddot{\theta} = T_c - mgl\ \sin\theta$$

In steady state:

$$0 = T_c - mgl\ \sin\theta$$

If there is no external force $T_c = 0$

$$0 = -mgl\ \sin\theta$$
$$\theta^{ss} = 0 \text{ and } \quad \theta^{ss} = \pi$$

→ two steady states exist

Linearization

$$f \approx f^{ss} + \left.\frac{\partial f}{\partial T_c}\right|_{ss} \Delta T_c + \left.\frac{\partial f}{\partial \theta}\right|_{ss} \Delta\theta$$
$$I\Delta\ddot{\theta} = \Delta T_c - mgl\cos\theta^{ss}\,\Delta\theta$$

Laplace Transformation

$$Is^2\overline{\Theta} = \overline{T_c} - mgl\cos\theta^{ss}\,\overline{\Theta}$$
$$\frac{\overline{\Theta}}{\overline{T_c}} = \frac{1}{Is^2 + mgl\cdot\cos\theta^{ss}}$$

Stability of the roots

$$Is^2 + mgl\cdot\cos(\theta^{ss}) = 0$$
$$s^2 = -\frac{mgl}{I}\cdot\cos\theta^{ss}$$

For $\theta^{ss} = 0$ (pendulum)

$$s^2 = -\frac{mgl}{I}\cos 0 = -\frac{mgl}{I}$$
$$s_{1/2} = \pm i\sqrt{\frac{mgl}{I}}$$

Imaginary roots, oscillating response (constant, stable), $Re(s) = 0$.

For $\theta^{ss} = \pi$ (inverse pendulum)

$$s^2 = -\frac{mgl}{I}\cos\pi = +\frac{mgl}{I}$$

$$s_{1/2} = \pm\sqrt{\frac{mgl}{I}}$$

One positive real part, unstable

EXAMPLE 10.0-2: Inverse Pendulum + Controller

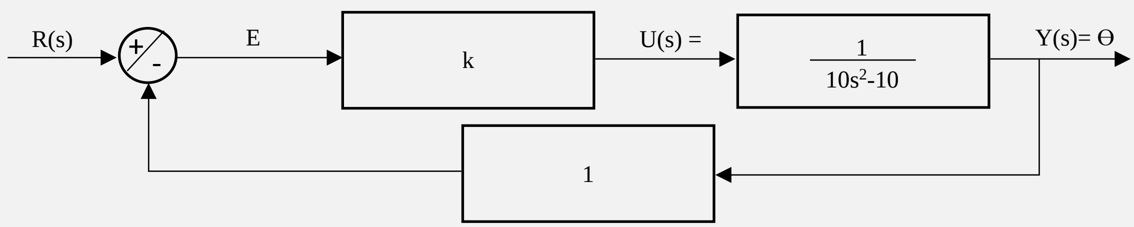


Figure 10-3. Control cycle.

$I = 20$ $mgl = 10$

New overall transfer function

$$\frac{Y}{R} = \frac{k\,\frac{1}{20s^2-10}}{1 + k\,\frac{1}{20s^2-10}} = \frac{k}{20s^2 - 10 + k}$$

New roots

$$20s^2 - 10 + k = 0$$

$$s_{1/2} = \pm\frac{\sqrt{50-5k}}{10}$$

for $k > 10$	purely imaginary roots, oscillating
for $k < 10$	1 positive, 1 negative root, unstable

The controller can change the dynamics of the inverse pendulum (originally unstable) to a stable (but oscillating) response.

11 Improved Controllers

Goal	P-Controller	I-Controller	D-Controller
Good final value, i.e. small steady state error	For large k	no steady state error (response increases with time when error persistent)	fast (reacts on slope of error, i.e. before error occurred)
Influence dynamics a) Faster b) Less oscillating	a) For large k b) for small k	slow (least negative root larger than for P-control)	sensitive to sensor noise
influence stability	for large k to some extent	for large T_I unstable	poor influence on stability

11.1 Integral-Controller

The integral controller is also called I-controller

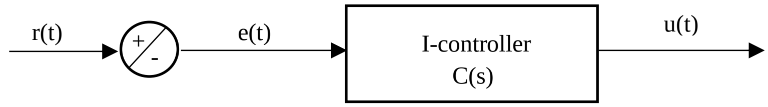


Figure 11-1. Part of a control cycle with an I-controller

The "transfer function" of this part is given by:

$$u(t) = \frac{k}{T_I}\int_0^t e(\tau)d\tau$$

Laplace transformation then gives:

$$U(s) = \frac{k}{T_I s}E(s)$$

with:

$$C(s) = \frac{k}{T_I s}$$

Where T_I is the integral or reset time (Nachstellzeit) and $\frac{1}{T_I}$ is the reset rate.

$u(t)$ changes until $\int e = 0$ and then $u(t)$ is constant and has reached its steady state. Therefore, for a controller with I portion the controller error will always go to zero. The steady state is reached when the error is zero.

Consider the following 2nd order system with an I-controller:

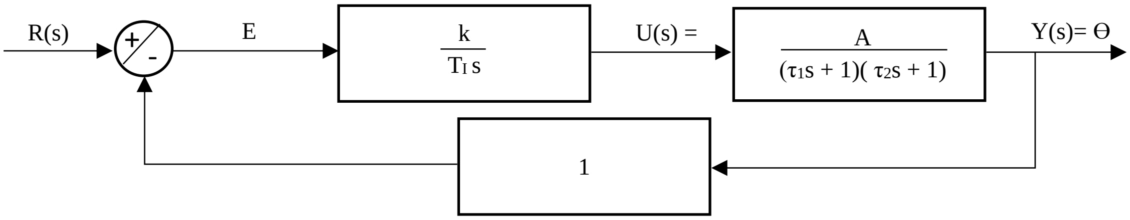


Figure 11-2. Control cycle of a second order system with integral control.

The transfer function of such a system is:

$$\frac{Y}{R} = \frac{\frac{k}{T_I}A}{\tau_1\tau_2 s^3 + (\tau_1 + \tau_2)s^2 + s + A\frac{k}{T_I}}$$

In comparison to the transfer function with the P-controller the characteristic equation has changed. The order has increased by 1 and also the roots and the root locus have changed.

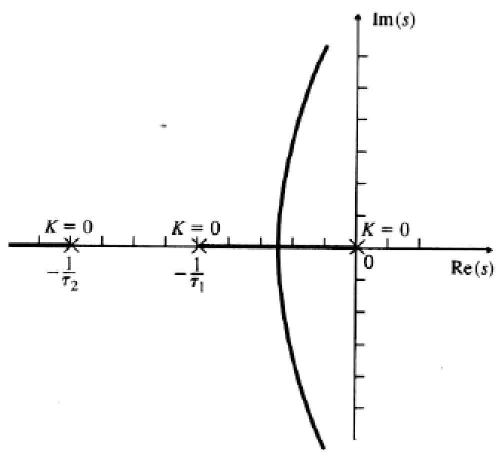


Figure 11-3. Root locus of a second order system with integral control.

There is an additional root for the controller at zero for k$= 0$. As k increases the roots move left in the plane (except for the middle one which will move right) and eventually a complex conjugate pair forms (at some point even with positive real parts). Thus, for increasing k the damping will be noticeably weaker until the system will eventually become unstable.

The advantage of the I-controller is that it has no steady state error but its response is slow. As we have seen it also tends to oscillate and get unstable with increasing k.

As the error for this kind of controller will always go to zero, we only have to look at the dynamics. The I-controller is not usually used on its own but coupled with a P-controller.

11.2 Differential Controller

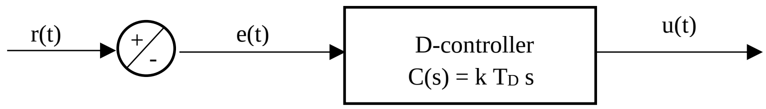


Figure 11-4. Part of a control cycle with differential control.

$$d(t) = kT_D\dot{e}(t)$$
$$D(s) = kT_D sE(s)$$
$$C(s) = kT_D s$$

Where T_D : Derivative time (Vorhaltezeit)

Differential controllers respond to the difference in slope. For steep slopes the response will be much stronger than for lower slopes, therefore they have a fast response.
As the response is very fast, the controller is sensitive to noise and is usually implemented in the feedback loop.

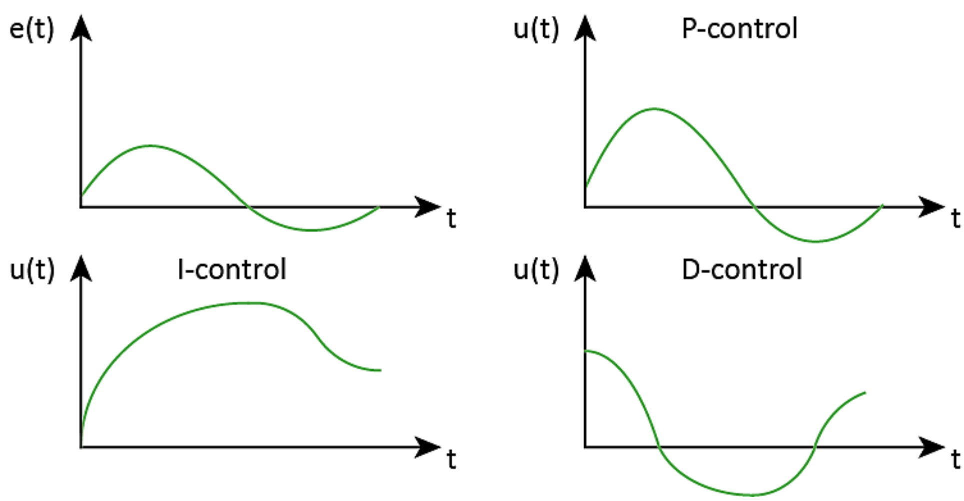


Figure 11-5. Comparison of different controller responses with e(t) being the assumed error.

11.3 Comparison of Controller Responses

→ P-controllers multiply the error by a constant (gain k) and result in a steady state error.
→ I-controllers are slow but adjust the steady state error.
→ D-controllers are fast but sensitive to background noise and tend to oscillate.

As all the controllers have different advantages and disadvantages is common to combine them into a **PID-**controller with three elements. The **P**roportional term closes the feedback look, the **I**ntegral term assures that there is no error remaining and the **D**erivative term improves the dynamic response and stability.

11.4 PID-Controller

Usually, a combination of P-, I- and D-control is used (industrial standard).

$$u(t) = k\left(e(t) + \frac{1}{T_I}\int_0^t e(\tau)d\tau + T_D\dot{e}(t)\right)$$
$$U(s) = k \cdot E(s)\left(1 + \frac{1}{T_I s} + T_D s\right)$$

T_I, T_D, k are selected to obtain an optimal response.
For a second order system all roots can be independently chosen with those three parameters. The three parameters have to be determined to get a good response. This is the aim of process control.
Sometimes PD or PI-controllers are used when the desired effect can be achieved with setting only two parameters.

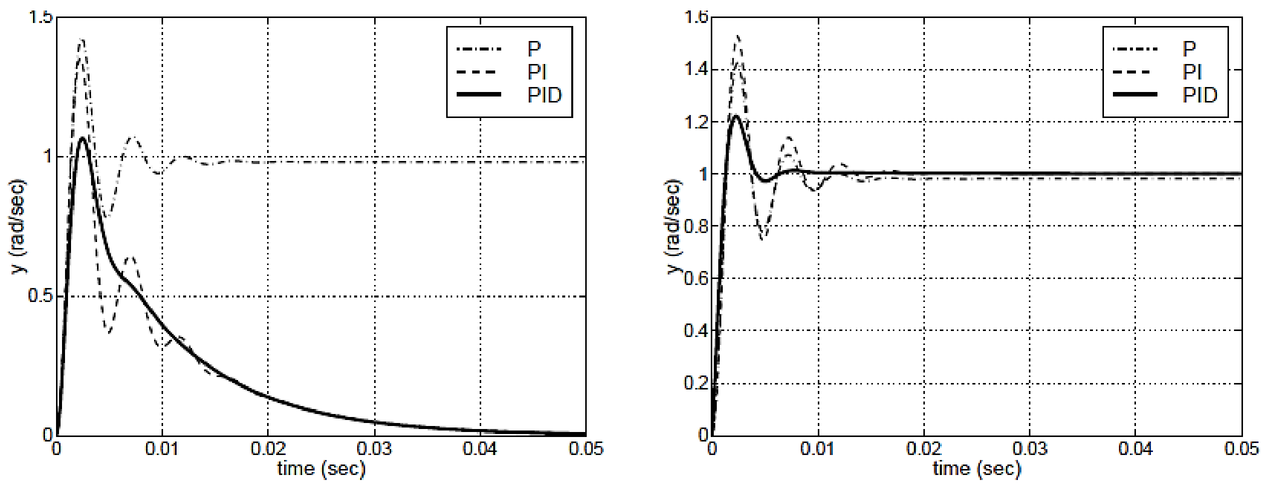


Figure 11-6. Transient behavior of different controllers after (a) a unit disturbance and (b) a reference step. Figure from [FrPE06]

In the left plot we can immediately recognize the controllers with an integral part as their steady state error goes to zero. When only the P-controller is used, y does not go to zero.
Comparing the PID and PI-controller we can say that the better response (not oscillating) must belong to the controller with more degrees of freedom when it is set efficiently.
On the right we can again spot the (small) steady state error of the P-controller and the smaller overshoot for the PID-controller when comparing it to the PI version.

11.5 Design of PID Controllers

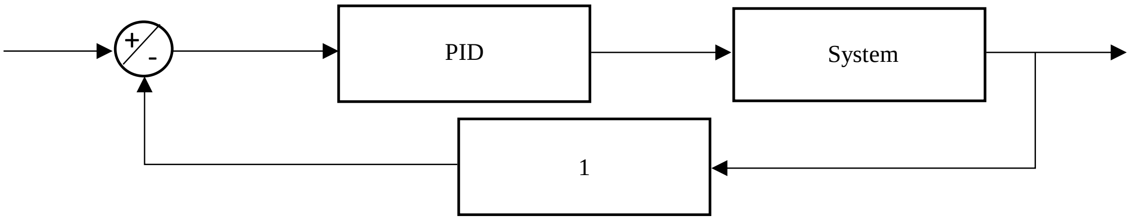


Figure 11-7. Control cycle of a system with PID control.

The main aim of control unit design is the stability. PID controllers are often used because their stability is well understood.

Experimental Methods

The methods below are used when models for the system and more advanced mathematical tools are not available. They were both developed by Ziegler and Nichols, two chemical engineers who were looking primarily at chemical systems.

Ziegler + Nichols (1942)

Most real systems have a "similar" response for a step function which is characterized by reaction rate R and lag L. The method aims at designing controllers for a system based on its step response and is also called step method.

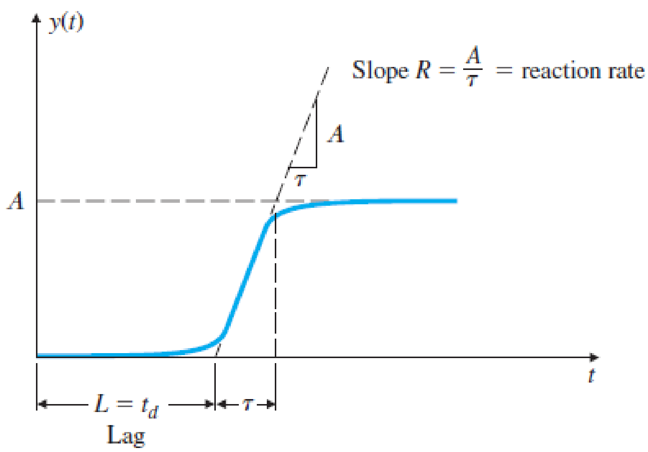


Figure 11-8. Process response. Figure from [FrPE06].

The response will have a shape as above and can be approximated by a step response with the transfer function

$$\frac{Y(s)}{U(s)} = \frac{A \exp(-st_d)}{s}$$

A step function is introduced into a process (open loop) without a controller

→ R and L can be determined from the response

→ the goal is to tune the control unit in such a way that the amplitude in each oscillation should decrease by 3/4 compared to the previous oscillation (see Figure 11-9) → decay ratio of 0.25

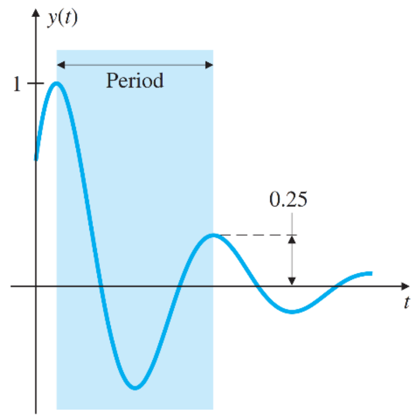


Figure 11-9. Aim of the Ziegler and Nichols method. Figure from [FrPE06].

→ Ziegler and Nichols developed rules to tune P, PI and PID controllers in order to achieve the desired response (see table below) and the parameters k, T_i and T_D can calculated from R and L.

→ The method is fast and reliable but only applicable when measurements are possible and when the step response has a shape as in Figure 11-8. It is suitable for slow processes when we aim mostly at correcting disturbances.

	k	T_i	T_D
P	$\frac{1}{RL}$	/	/
PI	$\frac{0.9}{RL}$	$\frac{L}{0.3}$	/
PID	$\frac{1.2}{RL}$	$2L$	$0.5\ L$

Ultimate Gain method

Ziegler and Nichols have developed a second method based on the experience that k shall not exceed a certain value k_u to prevent instability.

In the experiment a closed loop system with only P control is used and k is increased until the system becomes marginally stable. The oscillations are stable and constant.

→ this k is the ultimate gain k_u

→ the period at k_u is the ultimate period P_u of the oscillation at k_u

→ the period and gain are measured at the point of marginal stability

→ based on these values the parameters for the controllers can be determined from the table below

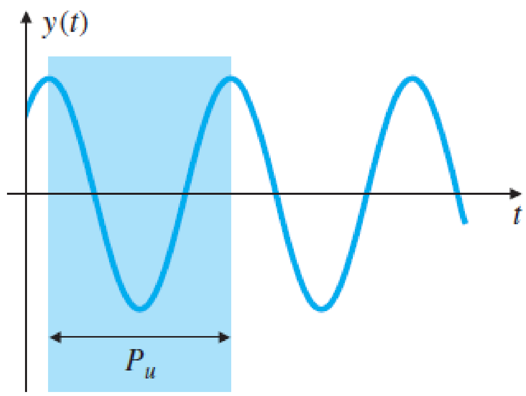


Figure 11-9. Ultimate gain method where P_u is the ultimate period. Figure from [FrPE06].

	k	T_i	T_D
P	$0.5\ k_u$	/	/
PI	$0.45\ k_u$	$\frac{1}{1.2} P_u$	/
PID	$0.6\ k_u$	$\frac{1}{2} P_u$	$\frac{1}{8} P_u$

Mathematical Method

Root-Locus

Disadvantage: necessary to compute roots of characteristic equation; has multiple solution and is difficult to solve for higher order systems

2nd order	binomial equation
3rd order	Cardano equation
nth order	no algebraic solution (n>4)

Alternative: Routh stability criterion (before computers were invented)

Characteristic equation

$$a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0 = 0$$

Necessary condition for stability: $a_i > 0$

Routh table

12 Frequency Response

This method is very useful to get a first impression of the system behavior and is often used in practice. The frequency response will allow us not only to predict the response for any sinusoidal input but will give qualitative information which is useful in determining the stability and robustness of a system.

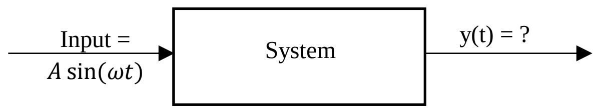


Figure 12-1. System with frequency input.

What happens at the output for $t \to \infty$ (in the stationary case)?
→ For a linear system we will have oscillation with the same frequency ω but a different amplitude A and phase ϕ

Input

$$u(t) = A\sin(\omega t)$$

Laplace transformation

$$U(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$Y(s) = U(s)G(s) = G(s)\frac{A\omega}{s^2 + \omega^2}$$

$$G(s) = \frac{1}{(s - s_1)(s - s_2)\dots}$$

Partial fraction expansion

$$Y(s) = \frac{\alpha_1}{s - s_1} + \frac{\alpha_2}{s - s_2} + \dots + \frac{\beta}{s + i\,\omega_0} + \frac{i\,\beta}{s - i\,\omega_0}$$

Inverse LT

$$y(t) = \alpha_1 e^{+s_1 t} + \alpha_2 e^{+s_2 t} + \dots + 2|\beta|\cos(\omega t + \phi)$$

For $t \to \infty$ and $p_i < 0$

$$y(t = \infty) = 2\,|\beta|\cos(\omega t + \phi)$$

→ System oscillates with the same frequency, new amplitude $M = 2\,|\beta|$ and phase shift ϕ.

To find the values of M and ϕ we find $y(t)$ for $u(t) = A\sin\omega t$ by means of convolution

$$y(t) = u(t) * g(t)$$

And we apply the relation for exponential and trigonometric functions $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

$$y(t) = \int_{-\infty}^{\infty} g(\tau)\, \frac{A}{2i}\left(e^{i\omega(t-\tau)} - e^{-i\omega(t-\tau)}\right) d\tau$$

This integral is not trivial to solve. We must find simplification for the convolution of $v(t) = e^{st}$

$$v(t) * g(t) = \int_{-\infty}^{\infty} g(\tau)\; e^{s(t-\tau)} d\tau = \int_{-\infty}^{\infty} g(\tau)\, e^{-s\tau}\, e^{st} d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} g(\tau)\, e^{-s\tau} d\tau}_{\text{Definition of the Laplace transform} \rightarrow G(s)} = G(s)\, e^{st}$$

We could show that $v(t) = e^{st}$ is the response in the time domain for the Laplace transform of the function. This can be applied to $s = i\omega$ which leads to

$$y(t) = \frac{A}{2i}\left(G(i\omega)\, e^{i\omega t} - G(-i\omega)\, e^{-i\omega t}\right)$$

Transform to radial coordinates $G(i\omega) = Me^{i\varphi}$

$$y(t) = \frac{A}{2i}\, M\left(e^{i(\omega t+\varphi)} - e^{-i(\omega t+\varphi)}\right)$$

And apply again the definition of the sine and compare with the expression found for $y(t = \infty)$

$$y(t) = AM \sin(\omega t + \varphi)$$

$$AM = 2|\beta| \qquad\qquad \varphi = \phi$$

Now we can find M and ϕ.

Amplitude ratio AR

$$AR = \frac{AM}{A} = M = |G(i\omega)| = |G(s)|_{i\omega}|$$

Phase shift ϕ

$$\phi = \tan^{-1}\left(\frac{Im\left(G(i\omega)\right)}{Re\left(G(i\omega)\right)}\right)$$

With the amplitude ratio and phase shift we can find the outlet for any sine or cosine by plugging in $i\omega$ instead of s.

EXAMPLE 12-1: Frequency response

For a system with the transfer function

$$G(s) = \frac{1}{s+k}$$

We plug in $i\omega$

$$G(i\omega) = \frac{1}{i\omega + k}$$

And we can calculate the amplitude ratio

$$AR = \frac{1}{\sqrt{\omega^2 + k^2}}$$

As well as the phase shift at the outlet

$$\phi = \tan^{-1}\left(-\frac{\omega}{k}\right)$$

12.1 Bode Diagram

For a 1st order system the transfer function is given by:

$$G = TF = \frac{k}{\tau s + 1}$$

The input of the system is again a wave function as in Fig 12-1. The oscillation of the output will have the same frequency as we have seen, however amplitude and phase change. When the input frequency is low, the corresponds to slow changes, and we are likely to still have the same amplitude at the outlet (only amplified by k). As the frequency increases, the changes occur faster and we approach the time constant of the system. The behavior changes to a lower response amplitude and phase shift (see figure 12-2).

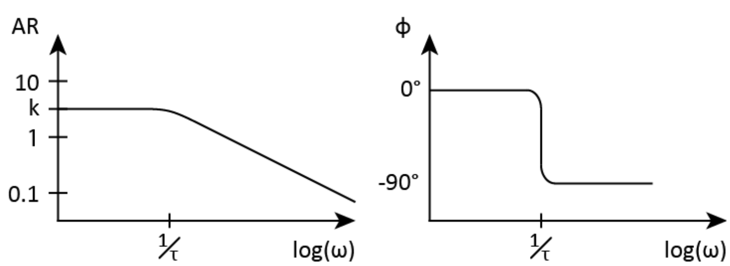


Figure 12-2. Bode diagram for a 1st order system.

Bode plots depict the amplitude ratio and phase shift as a function of the input frequency. Both AR and ω are on a log scale (AR most often in dB), ϕ is drawn in degrees.

The plots can be determined experimentally or constructed by hand (see example 12.1-1) in a quick and yet sufficiently accurate way. Bode plots of more complex transfer functions can be constructed in an additive way, where the transfer function is split up into simpler components. The plots for these parts can be drawn easily and are added up to give the Bode plot of the more complex system.

Simple Graphical Bode Plot Design

Computing of $|G(i\omega)|$ is not necessary as certain classes of functions behave the same. Below the plotting of these classes will be discussed.

Standard classes of TF:

a) $\mathrm{K}s^n = \mathrm{K}(\mathrm{i}\omega)^{\mathrm{n}}$
   $\rightarrow \log(K|i\omega|)^n = \log(K) + n\log(|i\omega|)$
   $\rightarrow AR$: line with slope n through K at $\omega = 1$
   $\rightarrow \phi = n \cdot 90°$

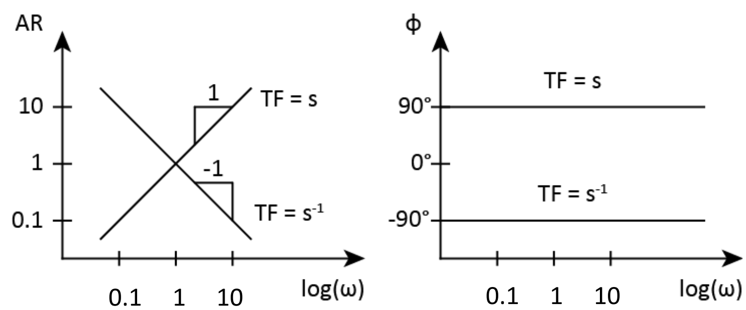


Figure 12-3. Simple Bode diagram general cases of $\mathbf{K}\boldsymbol{s^n}$.

b) $(\tau s + 1)^n = (\tau(i\omega) + 1)^n$
   AR: constant until $\omega = 1/\tau$ (break point), then slope n
   $\phi = 0$ until $\omega = 1/\tau$, then $\phi = n \cdot 90°$

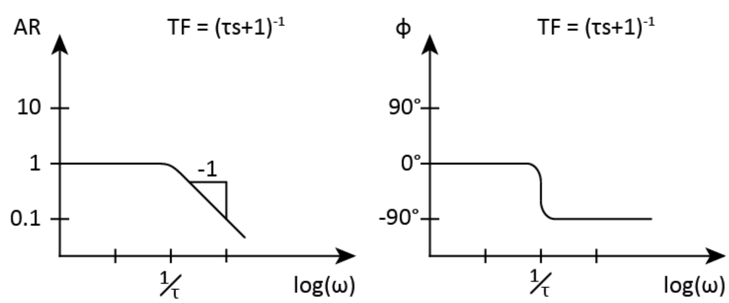


Figure 12-4. Simple Bode diagram general cases of $(\boldsymbol{\tau s} + \mathbf{1})^{\boldsymbol{n}}$.

The plots above are only asymptotically correct and would have rounded edges and smooth transitions.

EXAMPLE 12.1-1: Drawing Bode Diagrams

The following transfer function is given:

$$TF = \frac{2000\,(s+0.5)}{s\,(s+10)(s+50)}$$

Rearranging to have $\tau s + 1$ in the denominator:

$$TF = \frac{2(2\,s+1)}{s\left(\frac{1}{10}\,s+1\right)\left(\frac{1}{50}\,s\,+1\right)}$$

We can identify four components, one of standard class a) (of form Ks^n) and three of standard class b) (of form $(\tau s+1)^n$) with time constants:

$$\tau_1 = 2$$
$$\tau_2 = \frac{1}{10}$$
$$\tau_3 = \frac{1}{50}$$

Let us first look at the magnitude plot. The term of class a) is first order and in the denominator will lead to a slope of $n_a = -1$ throughout the entire ω range. The terms of class b) will identify the behavior only after their break point ω_i which we can find by the inverse of the time constant. The effect on the slope will be positive or negative depending on whether the term is in the nominator or denominator.

$$\omega_1 = 0.5 \qquad n_1 = +1$$
$$\omega_2 = 10 \qquad n_2 = -1$$
$$\omega_3 = 50 \qquad n_3 = -1$$

As the slopes behave additive, we can construct the magnitude plot piecewise. Initially the slope will be -1 until after $\omega = 0.5$ it will be horizontal as the $n_1 = +1$ effect of the $2s+1$ in the nominator cancels the -1 slope. When ω reaches a value of 10 the effect of the $\frac{1}{10}\,s+1$ term in the denominator will come in and decrease the slope to -1 and the last b) type term further to -2 after $\omega = 50$.

To draw the plot we should start with the first asymptote before the effect of the class b) terms. This asymptote has a slope of -1 (or 20 dB per decade) and we can locate it as it would pass through the value 2 (the value of k in this case) at $\omega = 1$. The asymptotes can then be constructed by changing the slope at the break points and the amplitude plot is then obtained by connection of the lines.

For the phase plot the same break points play a role for type b) terms and a) terms will again have an influence over the entire ω range. The effect can also be read from the value of n_i and will be $n_i \cdot 90°$. Phase shifts also show additive behavior. We therefore start at -90°, go to 0°, to -90° and finally -180°. The obtained staircase like function will be only approximate with the actual shape being much smoother.

This procedure yields the final Bode diagram in figure 12-5 and 12-6.

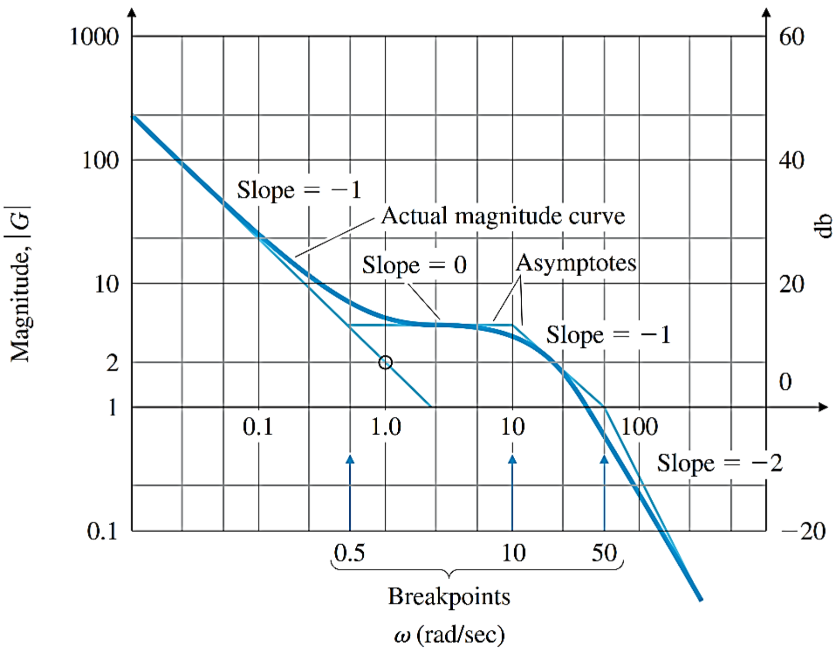


Figure 12-5. Magnitude plot for example 12.1-1. Figure from [FrPE06].

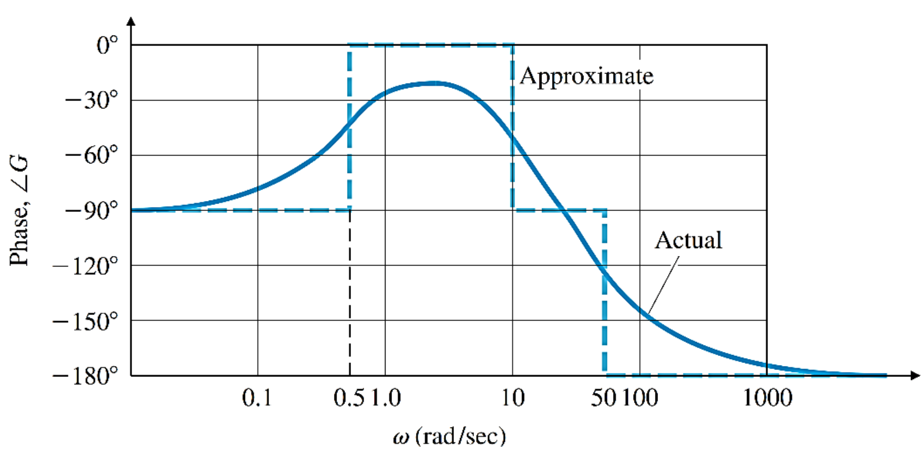


Figure 12-6. Phase plot for example 12.1-1. Figure from [FrPE06].

Bode Diagram of PID Controllers

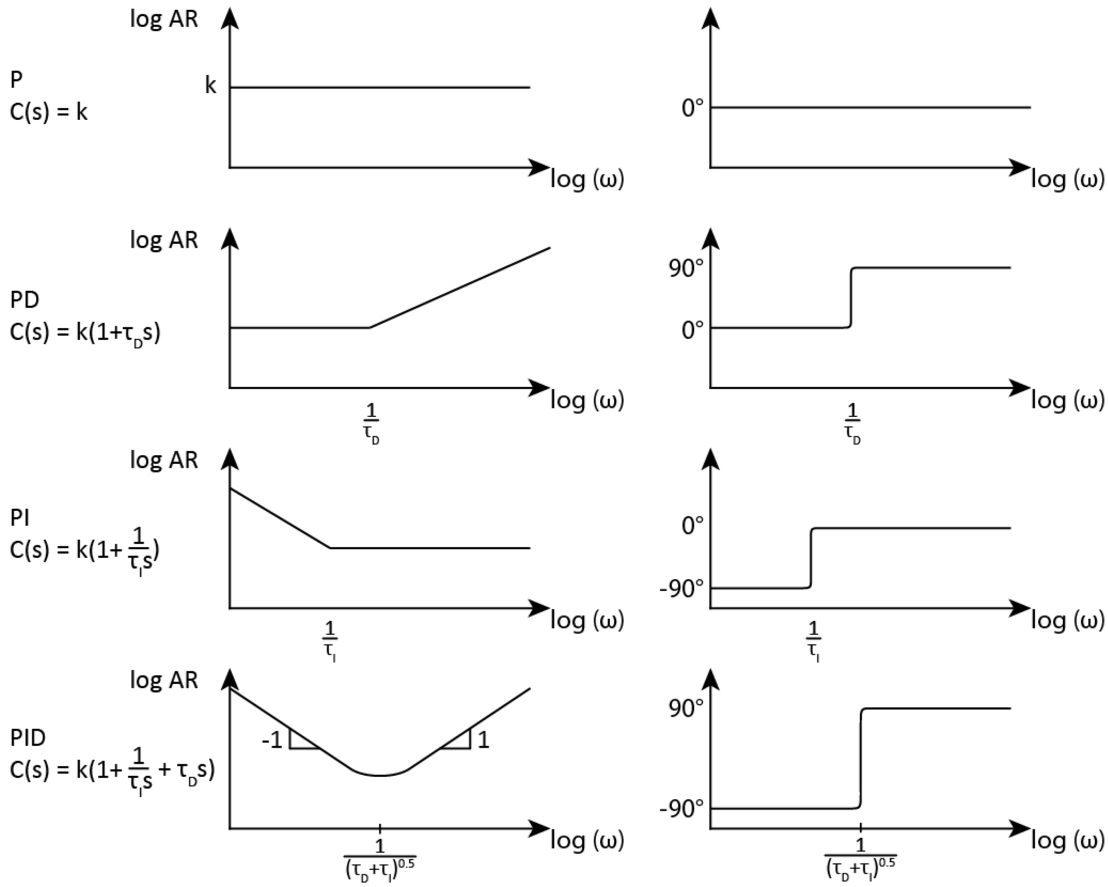


Figure 12-7. Bode diagrams for P, PD, PI and PID controllers with different C(s).

Bode Plots of 2nd Order Systems

$$G(s) = \frac{k}{\tau^2 s^2 + 2\,\xi\,\tau s\ + 1}$$

With damping factor $0 < \xi < 1$
→ oscillating response to step input
→ maximum amplitude (if it exists) at the natural frequency ω_n is the greater, the smaller the damping factor ξ
→ medium phase of -90° at the natural frequency ω_n and the steeper the smaller the damping factor ξ

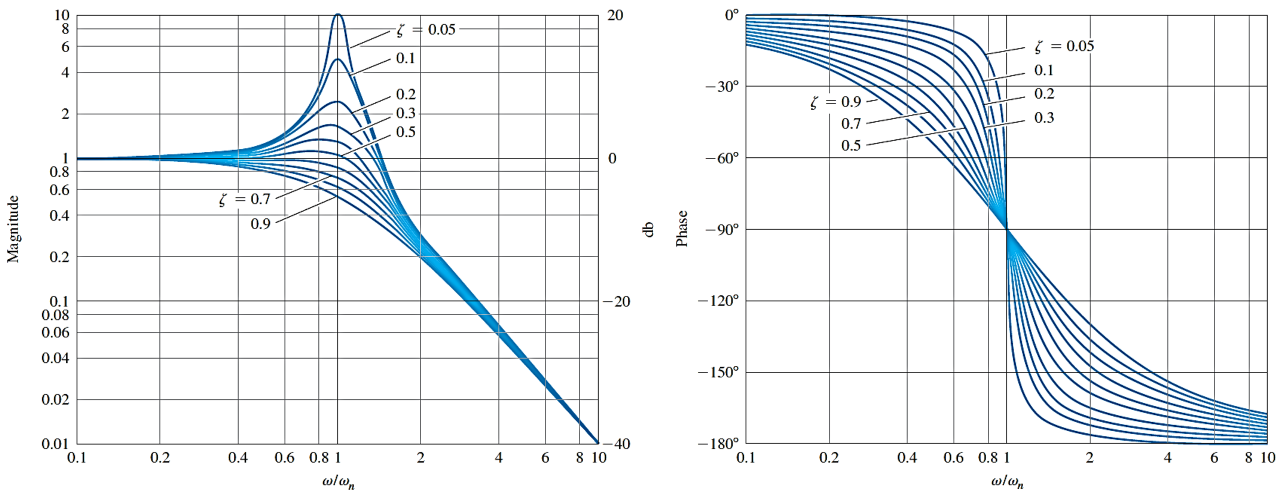


Figure 12-8. Bode diagrams for oscillating responses for different values of ξ. Figure from [FrPE06].

Bode Stability Criterion

The Bode stability criterion is a simple method to analyze the stability of feedback control systems. Instead of the mathematically more complicated closed loop, the stability of the open loop is analyzed.

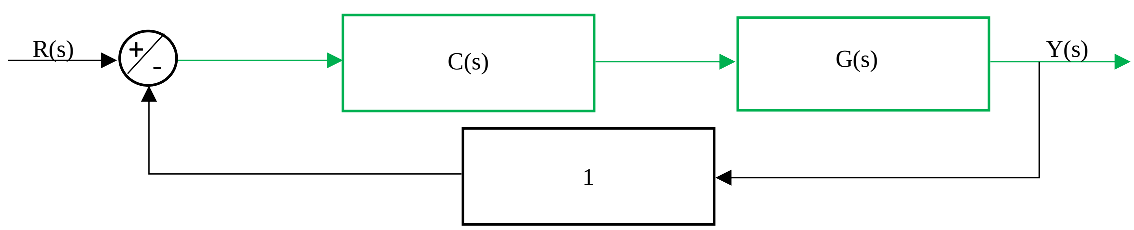


Figure 12-9. Control cycle with an open loop (green) and a closed loop (green and black).

If the open loop is stable, the closed loop will also be stable if at the critical frequency ω_c the condition $AR(\omega_c) < 1$ holds, where ω_c is the frequency when $\phi = -180°$.

The criterion can be applied only to overall decreasing Bode plots and when the open loop is stable.

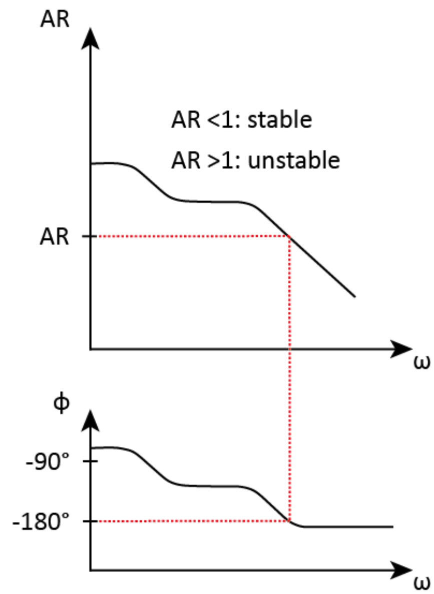


Figure 12-10. Bode diagram with phase shifts up to -180°

Why is a phase shift of -180° critical?

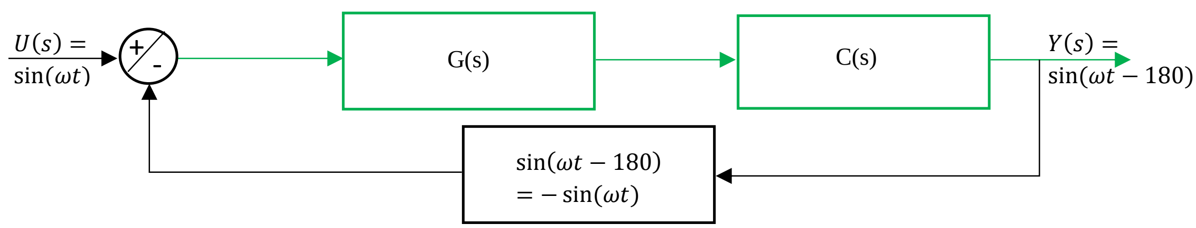


Figure 12-11. Control cycle with phase shift of – 180°

→ For a phase shift of -180° and the same amplitude the output will be exactly the opposite of the input
→ As it is subtracted this is the same as adding again the input signal
→ So after the back coupling there will be constructive interference! (If the amplitude ratio is greater than unity is will amplify, otherwise it disappears)

The Bode stability criterion can also be used for controller design. When we introduce another controller into the system or change the parameters of the implemented controllers, the Bode plot will change. The new controller/parameters should change it in such a way, that the stability criterion is fulfilled and we can find it by construction. This procedure is illustrated in the following example.

EXAMPLE 12.3-1: Controller Design using Bode Plots

The following transfer function is examined in this exercise

$$G(s) = \frac{1}{(0.2s+1)(2s+1)(s+1)^2}$$

The Bode diagrams for the system with a P and a PI-controller are drawn below. When only a P-controller with $k = 10$ is introduced, the function becomes $G(s) \cdot k$ and according to the Bode stability criterion will be slightly unstable.

With a PI-controller with $\tau_I = 1$ we can achieve stability when the criterion is considered.

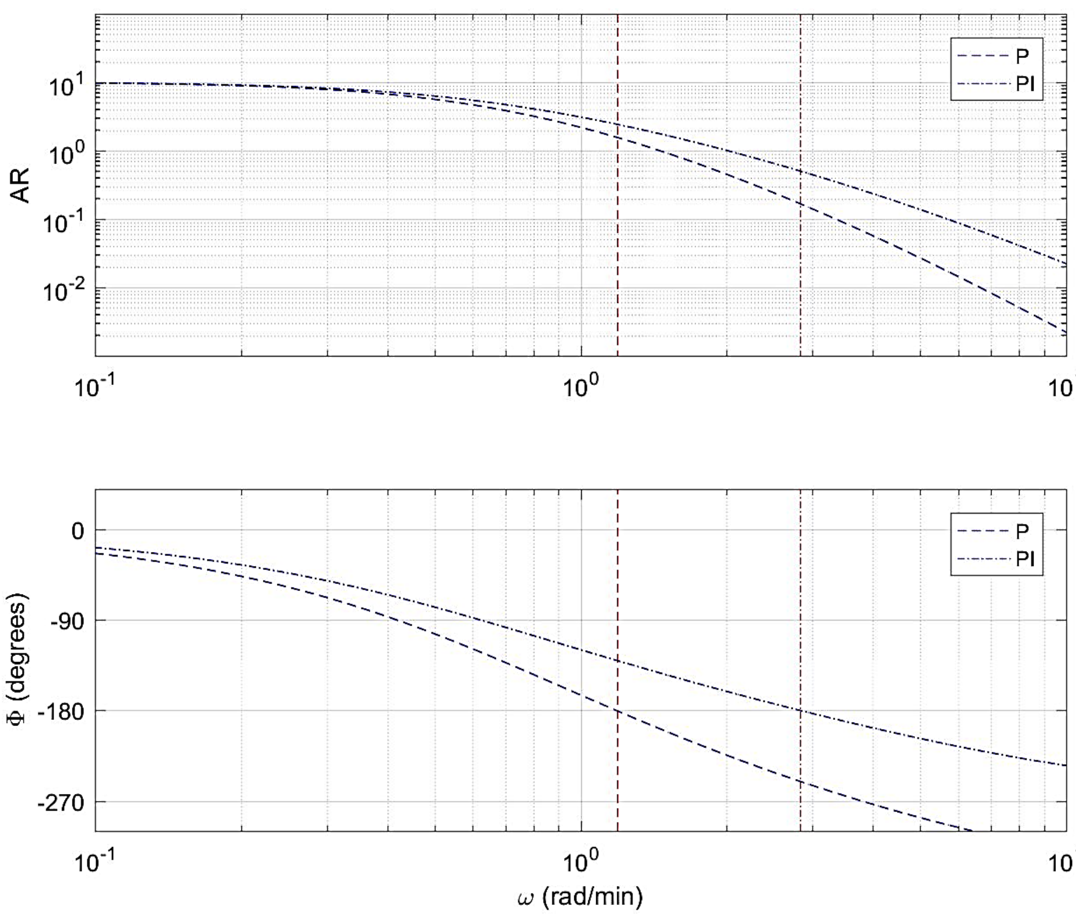


Figure 12-12. Bode diagram of the system with a P- and a PI-controller. The critical frequencies for both set ups are indicated in red.

Stability Margin

The Bode criterion can be extended to analyse the stability also in a quantitative way and to determine not only whether the system is stable or not but how stable. This described by the gain and phase margin.

As a chemical process is rarely unchanged as the process conditions may vary or it might be disturbed, this is very useful. The margins indicate how likely such changes will result in instability.

Gain margin

at $\phi = -180°$, how far away is AR of 1

Phase margin

at $AR = 1$, how far away is φ of 180°

Bode stability is only valid for a system with stable open loops
→ Nyquist stability criterion for closed loops with unstable open loops and complex systems

12.2 Nyquist Diagram

Nyquist Stability Criterion

The imaginary and real for all ω between -∞ and ∞ are plotted against each other. The amplitude ratio and phase shift can be read from the plot but not the corresponding ω value of the frequency.

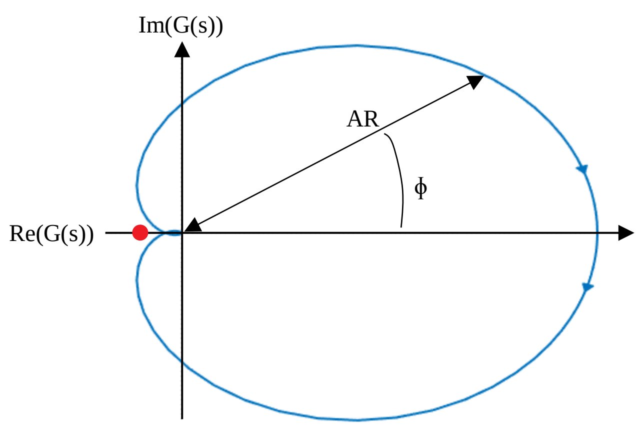


Figure 12-12. Nyquist diagram of G(s). The red dot is called Nyquist point.

$AR = |G(i\omega)|$ = distance from origin

$\varphi = \tan^{-1}\left(\frac{Im(G(i\omega))}{Re(G(i\omega))}\right)$ = angle

● Nyquist point: $(-1,0)$

Criterion: If G(s) in the open loop is unstable, the closed loop is stable if the Nyquist plot encircles $(-1,0)$ in counter-clockwise direction once per unstable root.

Remember: Positive angles are left-turning.

EXAMPLE 12.2-1: Inverse Pendulum

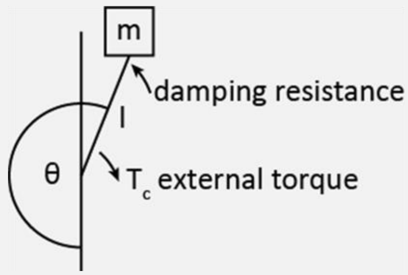


Figure 12.10. Problem description.

$$\ddot{\theta} + \alpha\dot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}$$

with α as a damping constant

Steady states for no external force $T_c = 0$

$$0 + 0 + \frac{g}{l}\sin\theta = 0$$

steady states for $\sin\theta = 0$

$$\theta = 0$$
$$\theta = \pi$$

Linearized system

$$f = \ddot{\theta} + \alpha\dot{\theta} = \frac{T_c}{ml^2} - \frac{g}{l}\sin\theta$$
$$f \approx \left.\frac{\mathrm{d}f}{\mathrm{d}T_c}\right|_{ss}\Delta T_c + \left.\frac{\mathrm{d}f}{\mathrm{d}\theta}\right|_{ss}\Delta\theta$$

$$f \approx \frac{1}{ml^2}\Delta T_c + \left(-\frac{g}{l}\cos\theta^{ss}\right)\Delta\theta$$

with $\cos\pi = -1$

$$f = \Delta\ddot{\theta} + \alpha\Delta\dot{\theta} = \frac{\Delta T_c}{ml^2} + \frac{g}{l}\Delta\theta$$

Laplace transform

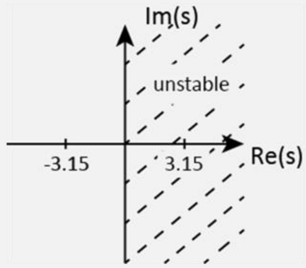
$$s^2\bar{\Theta} + \alpha s\bar{\Theta} = \frac{\bar{T_c}}{ml^2} + \frac{g}{l}\bar{\Theta}$$
$$\frac{\bar{\Theta}}{\bar{T_c}} = \frac{1}{ml^2s^2 + ml^2\alpha s - mlg} = G(s)$$

With $m = 1\ kg$, $l = 1\ m$, $\alpha = 0.01$ and $g = 10\frac{m}{s^2}$

Roots

$$s^2 + 0.01s - 10 = 0$$
$$s_{1/2} = \pm 3.15$$

Root locus



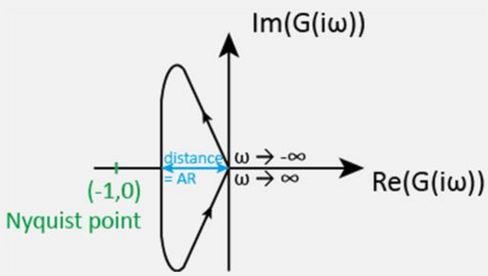
It is not possible to tell from the open loop if the closed loop is stable or not.

Nyquist plot of $\frac{1}{s^2+0.01s-10}$

$$G(i\omega) = \frac{1}{-\omega^2 + 0.01i\omega - 10}$$

for $\omega \in \mathbb{R}$

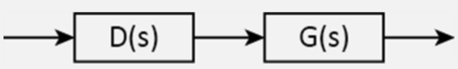
Plotting $Re\big(G(i\omega)\big)$ versus $Im\big(G(i\omega)\big)$ gives the Nyquist plot:

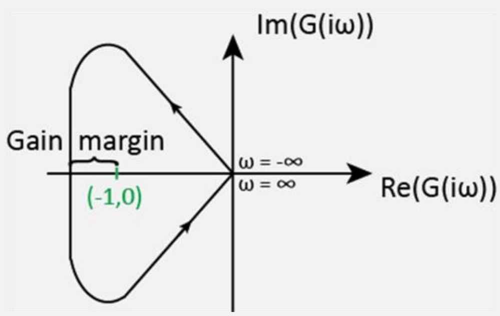


Stability criterion: Encircle $(-1, 0)$ counterclockwise once per unstable root of the open loop to obtain stability for closed loop.

This criterion can be fulfilled by adding a controller to move/scale the behavior of the system.

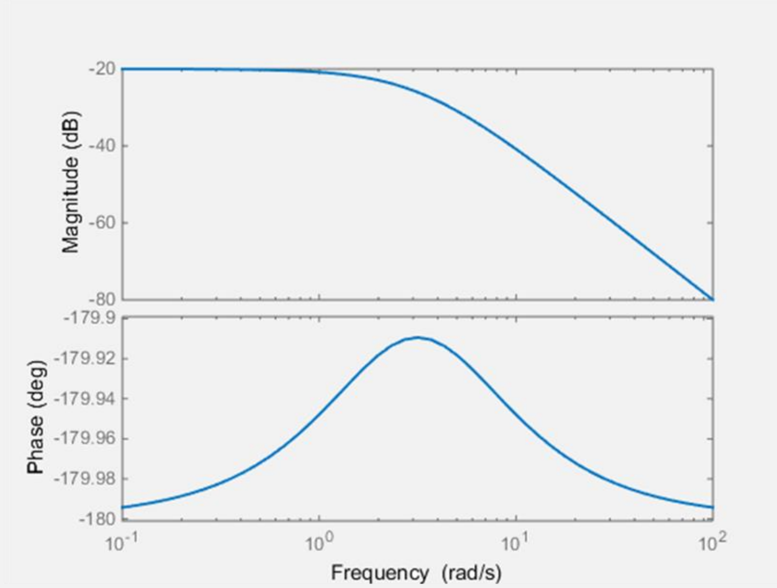
Nyquist plot for P-control





For $D(s) = k$ and $k > 10$ → stability

Bode plot of $\frac{1}{s^2+0.01s-10}$



12.3 Time Delay

In many chemical systems the problem of time delay occurs and affects the controller.

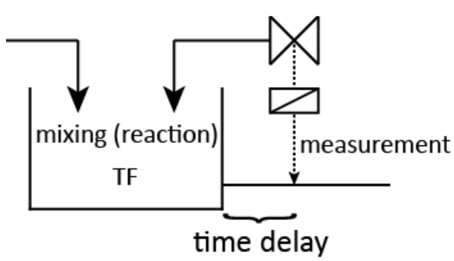


Figure 12-13. Problem description.

$$TF = \frac{k}{\tau s + 1}$$

TF with time delay λ

$$TF_{TD} = \frac{k}{\tau s + 1} e^{(-s\lambda)}$$

To illustrate the effect of time delay, we consider the following system:

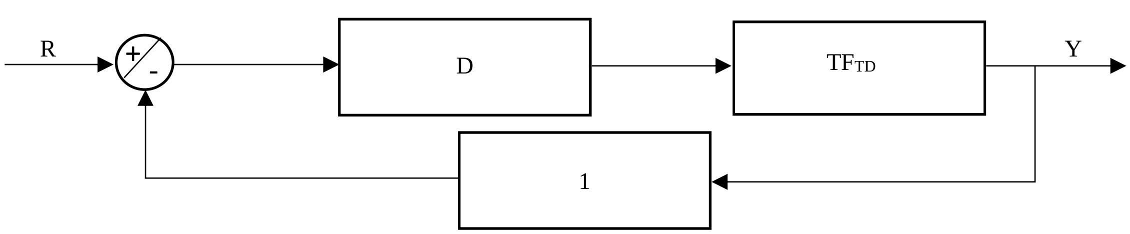


Figure 12-12. Control Cycle of a system with time delay.

With the transfer function

$$\frac{Y}{R} = \frac{D\frac{k}{\tau s+1}e^{(-s\lambda)}}{1 + D\frac{k}{\tau s+1}e^{(-s\lambda)}} = \frac{Dke^{(-s\lambda)}}{\tau s + 1 + Dke^{(-s\lambda)}}$$

Time delay makes it hard to use root locus method to evaluate stability. We can still apply the criterion for Bode plots.

Bode Plot with Time Delay

Without time delay

$$TF = \frac{k}{\tau s + 1}$$

With time delay

$$TF_{TD} = \frac{k}{\tau s + 1}e^{(-s\lambda)}$$

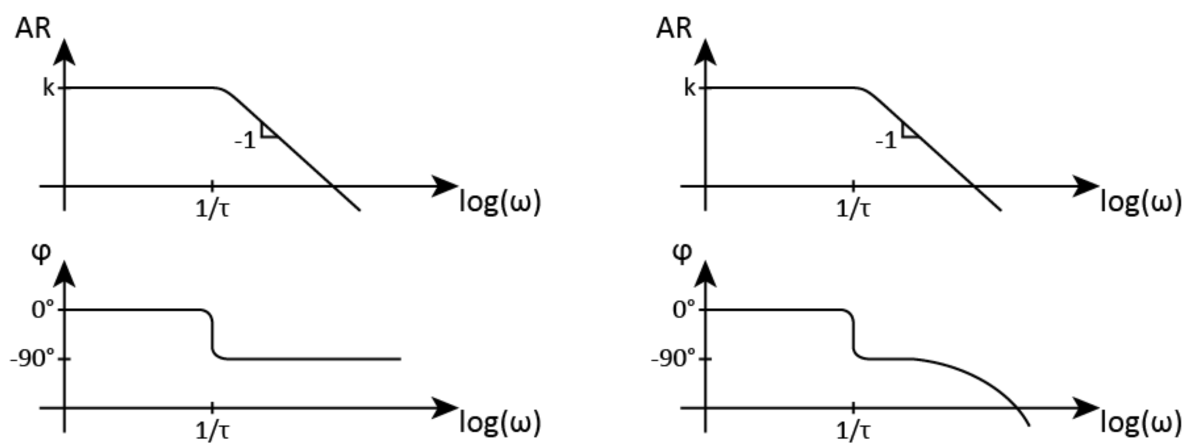


Figure 12-13. Bode diagrams of systems with time delay.

The Bode plot can still be used to assess stability of a system with time delay (stability criterion: $AR < 1$ for $\varphi = -180°$).

When time delay occurs the AR will be unaffected but the phase shift will be unbounded and decrease by $-\omega\lambda$ and the point of $\varphi = -180°$ is approached sooner. This is detrimental to the stability of closed loop systems.

EXAMPLE 12.3-1: Time Delay and Instability

For a system with the transfer function

$$G(s) = \frac{10}{(0.2s+1)(2s+1)(s+1)} e^{-0.5s}$$

With a time constants in minutes and time delay of 0.5 minutes (in the exponential term).

The Bode plots of the system with (left) and without time delay (on the right) are drawn below.

The amplitude ratios of the two systems are the same. However, the system with time delay approaches the critical phase shift of -180° much sooner. The critical frequency is indicated with a red line.

One can see that the system without time delay is stable, as the amplitude ratio is below 1 at the critical frequency. The phase shift of the delayed system decreases much faster and reaches the critical phase shift earlier, when the amplitude ratio is still above 1, indicating that the system will be unstable.

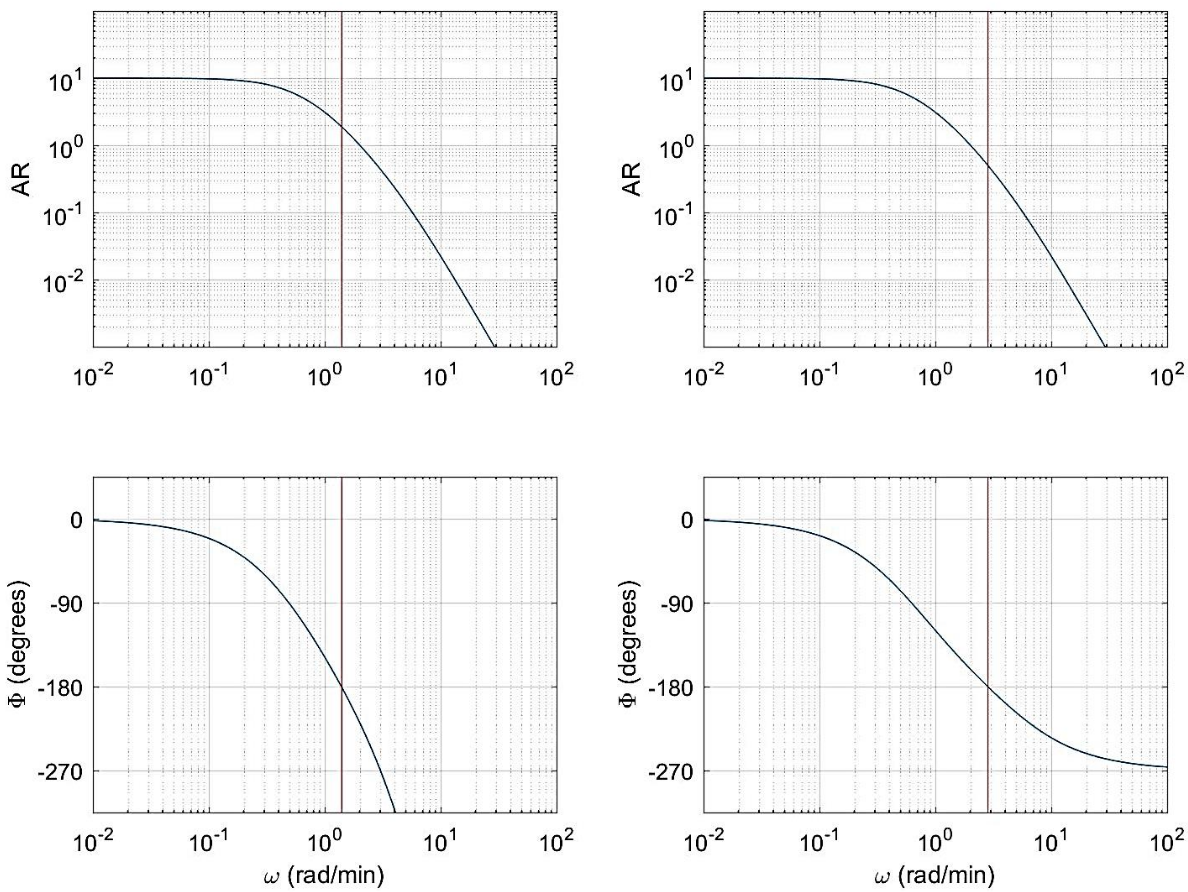


Figure 12-14. Left: Bode Plot of system with time delay of 0.5 min
Right: Bode Plot of system without time delay

13 Problems with Current Control Loops

So far:

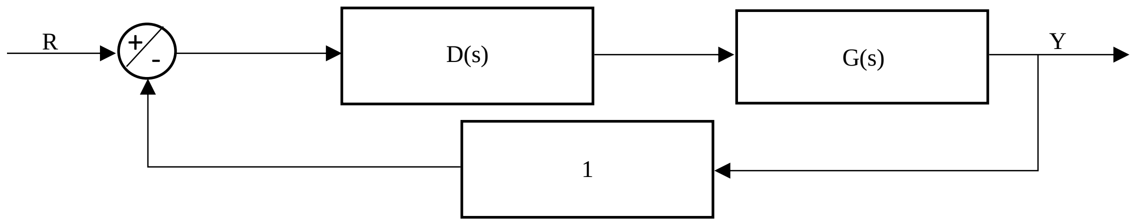


Figure 13-1. Control cycle.

find $D(s)$, i.e. PID
- → experimental
- → root locus
- → Bode/Nyquist stability

Problems:
- → only for single input, single output
- → slow: first error, then reaction

13.1 Feed Forward Control

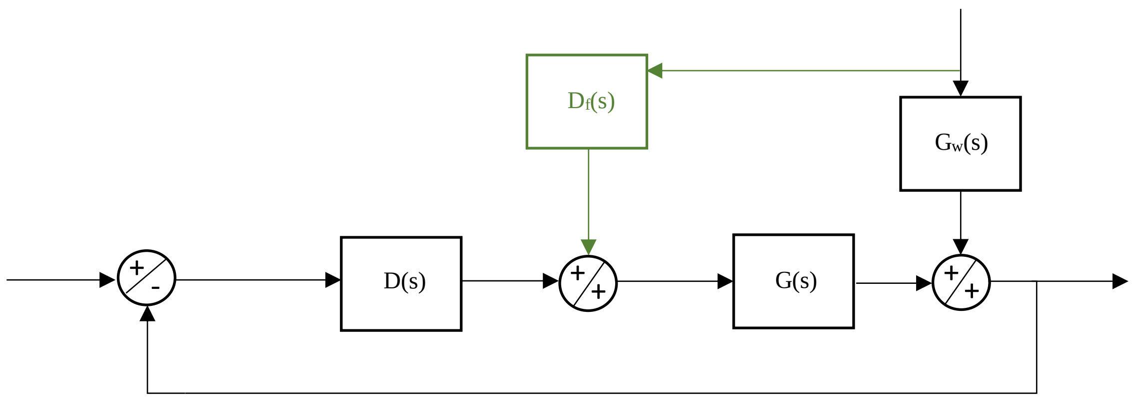


Figure 13-2. Control cycle of a system with feed forward control.

$$Y = \frac{DG}{1+DG}R + \frac{D_f G + G_w}{1+DG}W$$

Idea: D_f predicts the disturbance and reacts to it before it has altered the system response (significantly)

Goal: Y independent of disturbance W

Remove 2nd term

Choose $D_f = -\frac{G_w}{G}$ such that W has no effect on system

Prerequisites
- W measurable
- G and G_w have to be stable
- D_f has to be stable

13.2 Cascade Control

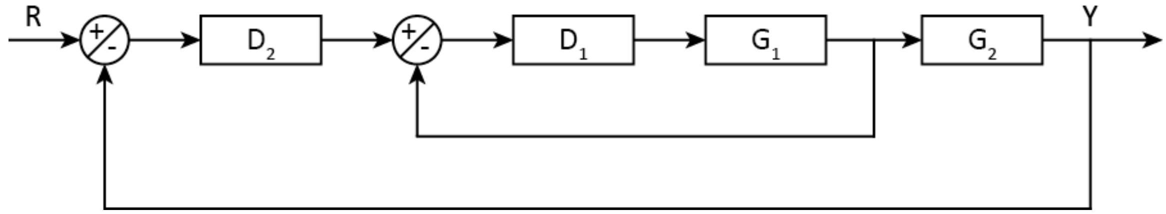


Figure 13-3. Control cycle of a system with cascade control.

In cascade control an inner closed loop G_2 is added to the system which compensates for disturbances within the inner system. This system has to be faster than the outer system G_1 in order to be effective.

Cascade control is used for example in heating to correct for disturbances in the valve before their effect becomes measurable by a change in room temperature.

EXAMPLE 13.1-1: pH Control

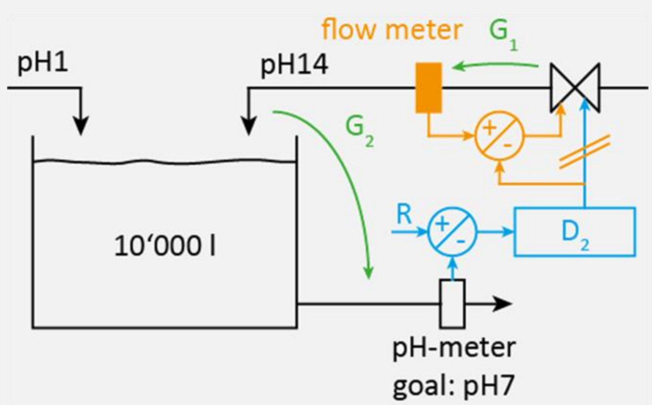


Figure 13-4. Problem description.

Inner loop: flow control Outer loop: pH control

13.3 Multivariable Control

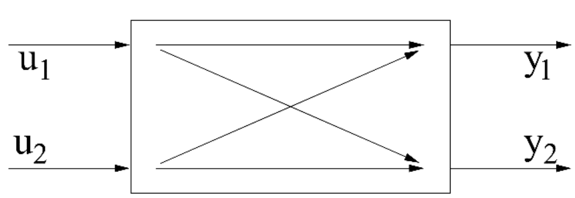


Figure 13-5. Multivariable system (MIMO)

When there are multiple input variables which influence more than one output variable in a system, we call this system multivariable system or MIMO (multiple input, multiple output).
In some cases u_1 might have no or negligible effect on y_2 and u_2 might not influence y_1 significantly. Then the system can be treated as two single variable systems as there is no coupling.

When the coupling is stronger, decoupling compensators are used to make up for the effects. The transfer functions in front of the process are chosen in such a way that they cancel out the coupling and again the system can be treated as two single variable systems. This is achieved with a control system as drawn in the figure below.

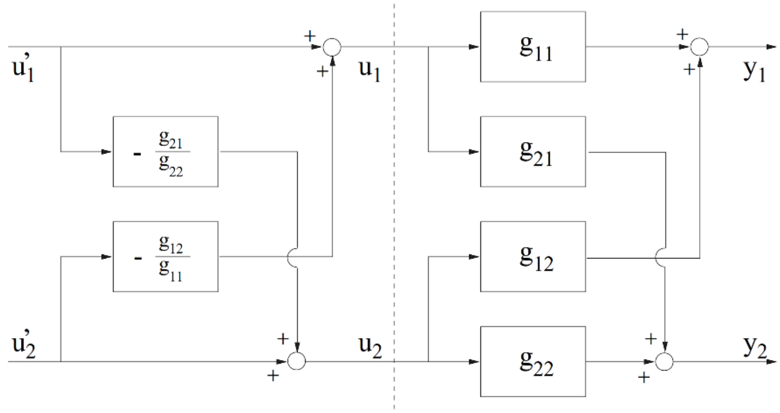


Figure 13-6. Decoupling compensator.

The strength of the coupling is quantified through the relative gain measured along the loop represented in green in the setup below. The effect of g_{c1} on $\frac{y_2}{u_2}$ is determined and the relative gain m is defined such that

$$m = \frac{\frac{y_2}{u_2}|g_{c1} = 0}{\frac{y_2}{u_2}|g_{c1} \neq 0}$$

and will have values between 0 (no coupling) and ∞ (very strong coupling). It is often normalized with the maximum gain so that $0 < m < 1$.

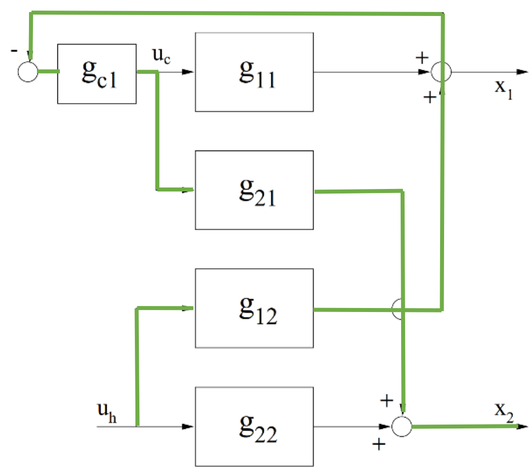


Figure 13-7. Relative gain array.

13.4 Model Predictive Control

Control systems today are mostly based on model predictive control.

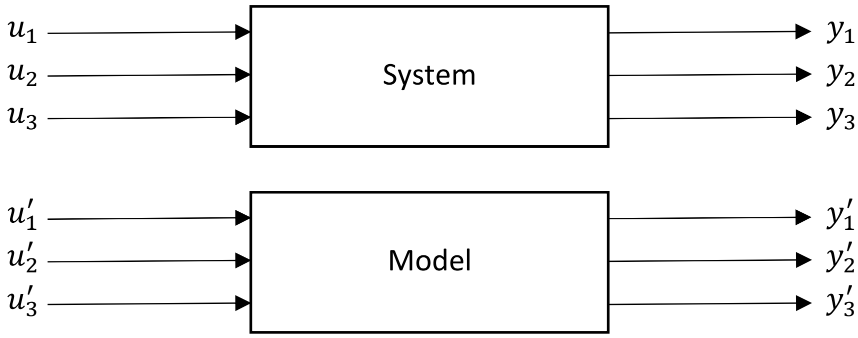


Figure 13-8. Model Predictive Control.

The system is limited by the precision of the model that predicts the system behavior. It is based on differential equations that must be solved fast enough.

Procedure:

- → Measure current values
- → Solve differential equations of the model to get path which reaches target
- → Implement only the first step
- → Repeat measurements and calculations

In order to compensate for the fact that the model is not able to represent the system precisely, the whole path to reach the target is simulated but only the first step is implemented. This compensates for uncertainties but a reasonable good model is still necessary as well as large computing power.

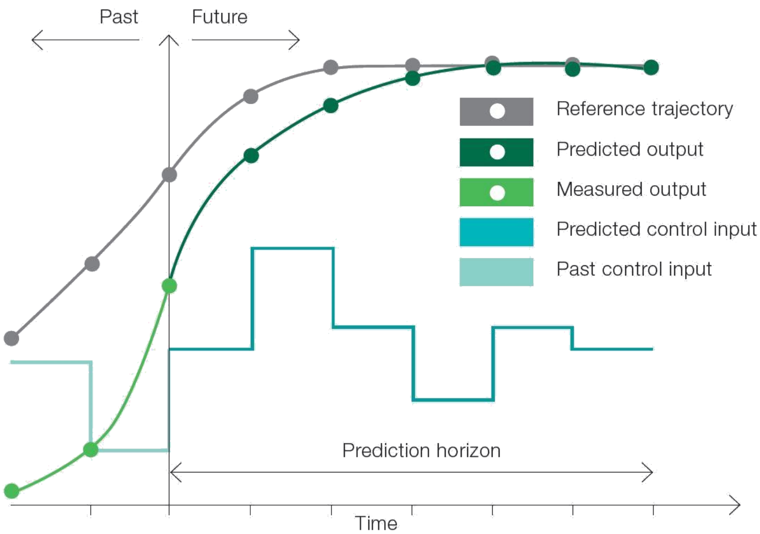


Figure 13-9. Model Predictive Control over time, here the target is the reference trajectory.
Figure from http://new.abb.com/control-systems/features/model-predictive-control-mpc, 8.6.16