Lecture 8

Physics of Electrostatics

In this lecture, we revise our knowledge of fundamental electrostatics. The formulae presented here will serve as important recipes for several interfacial phenomena as will be discussed in the following lectures.

The electrostatic force between two charges q and q' spaced by vector r in a medium is described by the famous Coulomb's law:

$$F_{qq'} = \frac{1}{4\pi\varepsilon_0\varepsilon_\mathrm{r}} \frac{qq'}{|r|^2} e_r \tag{8.1}$$

where ε_0=8.85×10^{-12} C^2·J^{-1}·m^{-1} is the vacuum permittivity (also as known as vacuum dielectric constant), ε_r is the relative permittivity of the medium which has unit of 1, and e_r is the unit vector along the direction of r. The electric field created by charge q is the force $F_{qq'}$ divided by q':

$$E_q = \frac{F_{qq'}}{q'} = \frac{1}{4\pi\varepsilon_0\varepsilon_r} \frac{q}{|r|^2} e_r \tag{8.2}$$

When the medium is vacuum, $\varepsilon_r = 1$, while in a **dielectric** material, $\varepsilon_r > 1$. We will discuss about the meaning of ε_r in a dielectric medium in the next section.

8.1 Electrostatics in a dielectric medium

A dielectric material is an electrical insulator that can be **polarized** by an applied electric field. The polarization refers to the process that the positive and negative charges in a material is displaced, and thus **electric dipoles** are created by the applied field.[1] The concept of dipole will be further explained in Lecture 9, while we give a simple view of the electric dipoles.

Consider the case where a free charge $+q$ immersed in a dielectric material as shown in Figure 9.2, The displaced charges create a local field that partially cancels out the external electric field, and the total electric field inside the dielectric material is screened (reduced).

The magnitude of such electrostatic screening is characterized by ε_r: the force between two charges in a dielectric material reduces to $1/\varepsilon_r$ compared with case in the vacuum. Therefore, the larger ε_r is, the more the electric field is screened. For instance, the forces

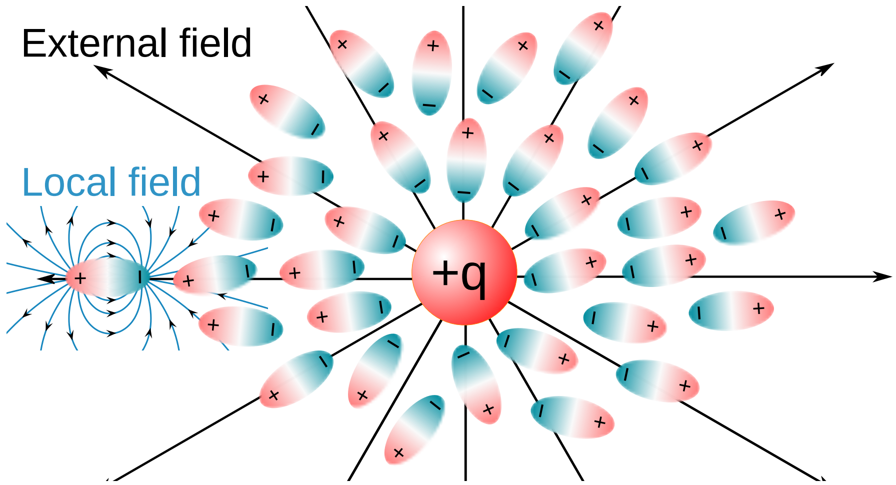

Figure 8.1: Scheme of a free charge $+q$ immersed in a dielectric medium. The dipole moments around the free charge creates a local field, and screens the electric field created by the charge.

between two charges in water is strongly screened since $\varepsilon_r \approx 80$. Therefore the solvated ions are very weakly bound and can move freely [2].

The vector electric field is the related to the gradient of the electric potential (scalar) ψ through:

$$E = -\nabla\psi \tag{8.3}$$

where $\nabla = (\frac{}{\partial x}, \frac{}{\partial y}, \frac{}{\partial z})$ is the vector differential operator. This gives the form of electric potential ψ of charge q as:

$$\psi = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{r} \tag{8.4}$$

where $r = |r|$. The scalar property of the electrical potential ensures that the work of translating a charge from position A to B, is independent of the path that the charge moves along:

$$\begin{aligned} W_{\mathrm{A}\to\mathrm{B}} &= \int_A^B F\cdot \mathrm{d}r \\ &= \int_A^B qE\cdot \mathrm{d}r \\ &= -q\int_A^B \nabla\psi\cdot \mathrm{d}r \\ &= -q(\psi_{\mathrm{B}} - \psi_{\mathrm{A}}) \end{aligned} \tag{8.5}$$

8.2 Gauss's law

As we have learned from fundamental electrostatics, the electrical potential $\psi(r)$ is the summation over all the charge q_i in the system of interest [1]:

$$\begin{aligned}\psi(r) &= \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_i}{|r_i - r|}\\ &= \frac{1}{4\pi\varepsilon_0}\int_\Omega \frac{\rho_{\text{tot}}(r')}{|r' - r|}\text{d}^3 r'\end{aligned} \tag{8.6}$$

where ρ_{tot} is the total charge density. Note Equation 8.6 does not contain ε_r, since the ρ_{tot} is the total charge density including both free charge density ρ_{f} and bound charge density ρ_{b}, such that $\rho_{\text{tot}} = \rho_{\text{f}} + \rho_{\text{b}}$. The definition for the two types of charges are:

- **Bound charge**

  The charges in a dielectric that cannot have *macroscopic* movement. They correspond to the electric dipoles (and induced dipoles) when the material is polarized. The external electric field polarizes the dielectric material and causes the positive and negative charges to displace a short distance.

- **Free charge**:

  The charges that cannot be categorized as bound charge, including free carriers in conductors, ions in aqueous solution, and charged impurities in semiconductors, etc. Note the name "free" is only due to historic reason to be contrasted from the bound charge, and does not essentially mean such charge can move freely in side the material (e.g. charged impurities in semiconductors). Free charges are still present even without external field.

A comparison between the free and bound charge densities of a dielectric material is illustrated in Figure 8.2. Combine Equations 8.3, 8.6 and the divergence theorem[3] in calculus, we can relate the surface integral of the electric field with the total charge density inside a domain of interest Ω:

$$\int_{\partial\Omega} E \cdot \text{d}S = \frac{1}{\varepsilon_0}\int_\Omega \rho_{\text{tot}}\text{d}V = \frac{1}{\varepsilon_0}\int_\Omega (\rho_{\text{f}} + \rho_{\text{b}})\text{d}V \tag{8.7}$$

where $\partial\Omega$ is the boundary (surface) of domain Ω. This is known as the Gauss's law or the first Maxwell equation. However this form of Gauss's law is usually not practical to use, since ρ_{b} also depends on ρ_{f} (see Figure 8.2). From Equation 8.2 we know $E = E_0/\varepsilon_{\text{r}}$, where E_0 is the electric field created by ρ_{r} alone in vacuum. Gauss's law for E_0 gives $\int_{\partial\Omega} E_0 \cdot \text{d}S = (\int \rho_{\text{f}}\text{d}V)/\varepsilon_0$, and compare with Equation 8.7 we know:

$$\begin{aligned}\rho_{\text{tot}} &= \frac{1}{\varepsilon_r}\rho_{\text{f}}\\ \rho_{\text{b}} &= (\frac{1}{\varepsilon_r} - 1)\rho_{\text{f}}\end{aligned} \tag{8.8}$$

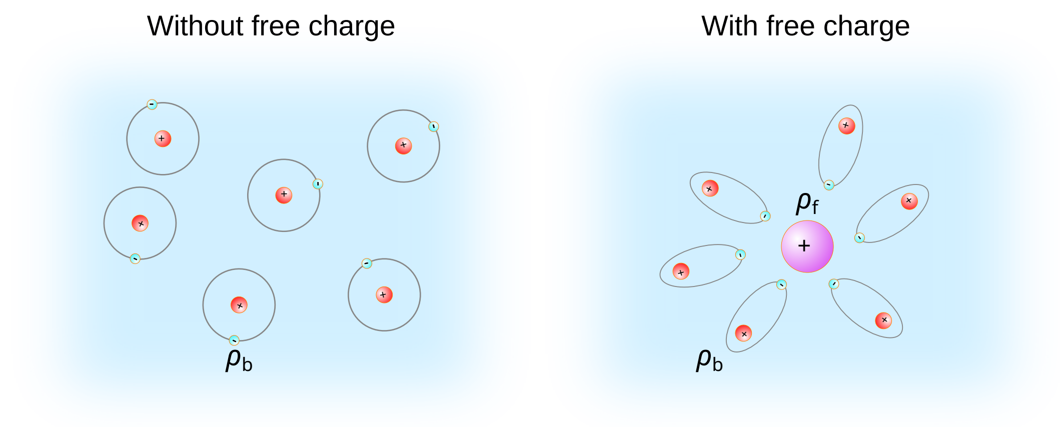

Figure 8.2: Difference between free and bound charge densities. Left: without free charges (and external field), the electrons move on normal orbitals around the atomic nuclei, and bound charge density ρ_b is 0 everywhere. Right, when free charges ρ_f are present, dipole moments are induced which creates non-zero spatial distribution of ρ_b.

By introducing an auxiliary quantity $D = \varepsilon_0\varepsilon_r E$, called the electrical displacement field, we can write Gauss's law with only ρ_f:

$$\int_{\partial\Omega} D \cdot \mathrm{d}S = \int_{\Omega} \rho_f \mathrm{d}V \tag{8.9}$$

and in its differential form:

$$\nabla \cdot D = \rho_f \tag{8.10}$$

We can see such treatment buries all information about ρ_b into ε_r. The relation between the two forms of the Gauss's law is shown in Figure 8.3:

From now on all the charges we refer to are free charges unless otherwise specified. Equation 8.10 can also be written in the form of Poisson equation, by replacing D with ψ:

$$\nabla^2\psi = -\frac{\rho}{\varepsilon_0\varepsilon_r} \tag{8.11}$$

The electrical potential in the system can be uniquely determined from Equation 8.11 by knowing the distribution of ρ as well as ε_r.

At the boundary between two domains 1 and 2, the continuity equations are:

- Potential

  $\psi_1 = \psi_2$

- Tangential electrical field

  $E_1 \cdot \tau = E_2 \cdot \tau$ where τ is the tangent vector of the interface.

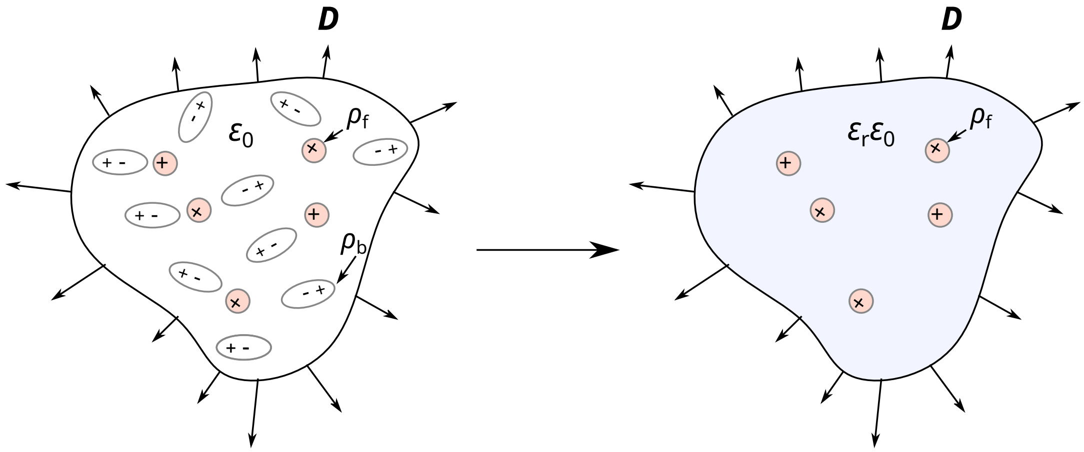

Figure 8.3: The Gauss's law that considers the explicit distribution of ρ_b (left) and treating the dielectric material as ε_r (right).

- Normal displacement field

  $D_1 \cdot n - D_2 \cdot n = \sigma_{1/2}$ where n is normal vector pointing from 1 to 2, and $\sigma_{1/2}$ is the interfacial charge density between 1 and 2 (has unit of $C{\cdot}m^{-2}$).

8.3 Capacitors

Dielectric materials are widely used to make capacitors. The simplest case is the parallel-plate capacitor, consists of two parallel conducting plates (A and B) and a dielectric material with thickness d sandwiched between the plates. By transferring an electron from A to B, the capacitor gains charge on both plates, and the electric potentials ψ_A and ψ_B change (Figure 8.4). The ratio between the charge transferred, Δq, with the electric potential difference $\Delta\psi = \psi_A - \psi_B$, is the capacitance C of the capacitor:

$$C = \frac{\Delta q}{\Delta \psi} \tag{8.12}$$

which has unit of F. C is also frequently expressed as the capacitance per unit area, and has the unit of $F{\cdot}m^{-2}$. The work to put charge q into a capacitor is:

$$\begin{aligned} W(q) &= \int_0^q \Delta\psi \mathrm{d}q' \\ &= \int_0^q \frac{q'}{C}\mathrm{d}q' \\ &= \frac{q^2}{2C} \end{aligned} \tag{8.13}$$

We can easily solve C using Gauss's law. In the dielectric, since there is no free charges

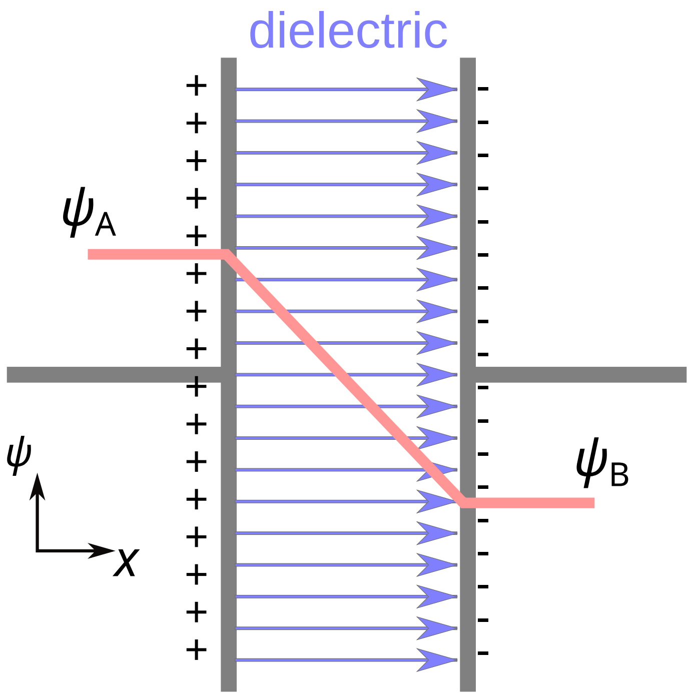

Figure 8.4: Scheme of a parallel-plate capacitor. The distribution of potential ψ is superimposed with the scheme.

within the dielectric layer, the Poisson equations become the Laplace equation:

$$\nabla^2\psi = \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = 0 \tag{8.14}$$

which gives an linear correlation between ψ and d. The magnitude electrical displacement field is then $D = -\varepsilon_0\varepsilon_r\frac{\mathrm{d}\psi}{\mathrm{d}x} = \varepsilon_0\varepsilon_r\Delta\psi/d$. Consider a closed box near one of the plate, the surface charge on the plate is then $\sigma = D = \varepsilon_0\varepsilon_r\Delta\psi/d$. Therefore we get the capacitance the parallel-plate capacitor as:

$$C = \frac{\Delta q}{\Delta\psi} = \frac{\sigma S}{\Delta\psi} = \frac{\varepsilon_0\varepsilon_r}{d}S \tag{8.15}$$

where S is the surface are of the plates. Here are some exercises for calculating the capacitance of a capacitor with different geometries that you can try out. The relative permittivity of the dielectric is ε_r in all cases.

- Cylindrical capacitor

  The cylindrical capacitor consists two concentric cylinder conductors with radii r_A and r_B ($r_\mathrm{B} > r_\mathrm{A}$), respectively (Figure 8.5a). Both cylinders have length of L. The capacitance is $C = 2\pi L\varepsilon_0\varepsilon_\mathrm{r}/\ln(r_\mathrm{B}/r_\mathrm{A})$

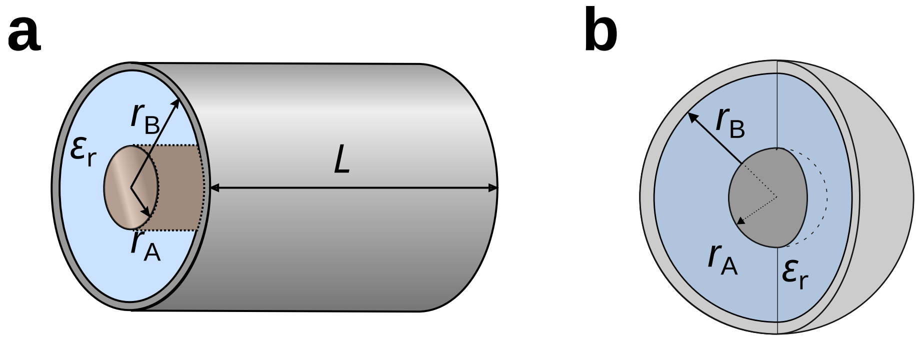

Figure 8.5: Capacitors of different geometries: a. cylindrical capacitor and b. spherical capacitor.

- Spherical capacitor

  The spherical capacitor consists two concentric conducting spheres with radii r_A and r_B ($r_B > r_A$), respectively (Figure 8.5b). The capacitance is $C = 4\pi\varepsilon_0\varepsilon_r/(r_A^{-1} - r_B^{-1})$

- Parallel capacitors

  2 capacitors C_1 and C_2 are connected parallel to each other. The effective capacitance $C_{eff} = C_1 + C_2$

- Series capacitors

  2 capacitors C_1 and C_2 are connected in series. The effective capacitance $C_{eff} = (1/C_1 + 1/C_2)^{-1}$

References

(1) Griffiths, D. J., *Introduction to electrodynamics*, 4th ed.; Cambridge University Press: 2005.

(2) Israelachvili, J. N., *Intermolecular and surface forces*, Third edition; Elsevier, Academic Press: Amsterdam, 2011.

(3) Spiegel, M.; Lipschutz, S., *Schaum's Outline of Vector Analysis, 2ed*; Schaum's Outline Series; McGraw-Hill Education: 2009.