Higher-order non-linear geometrically-exact beam theory with enhanced kinematics

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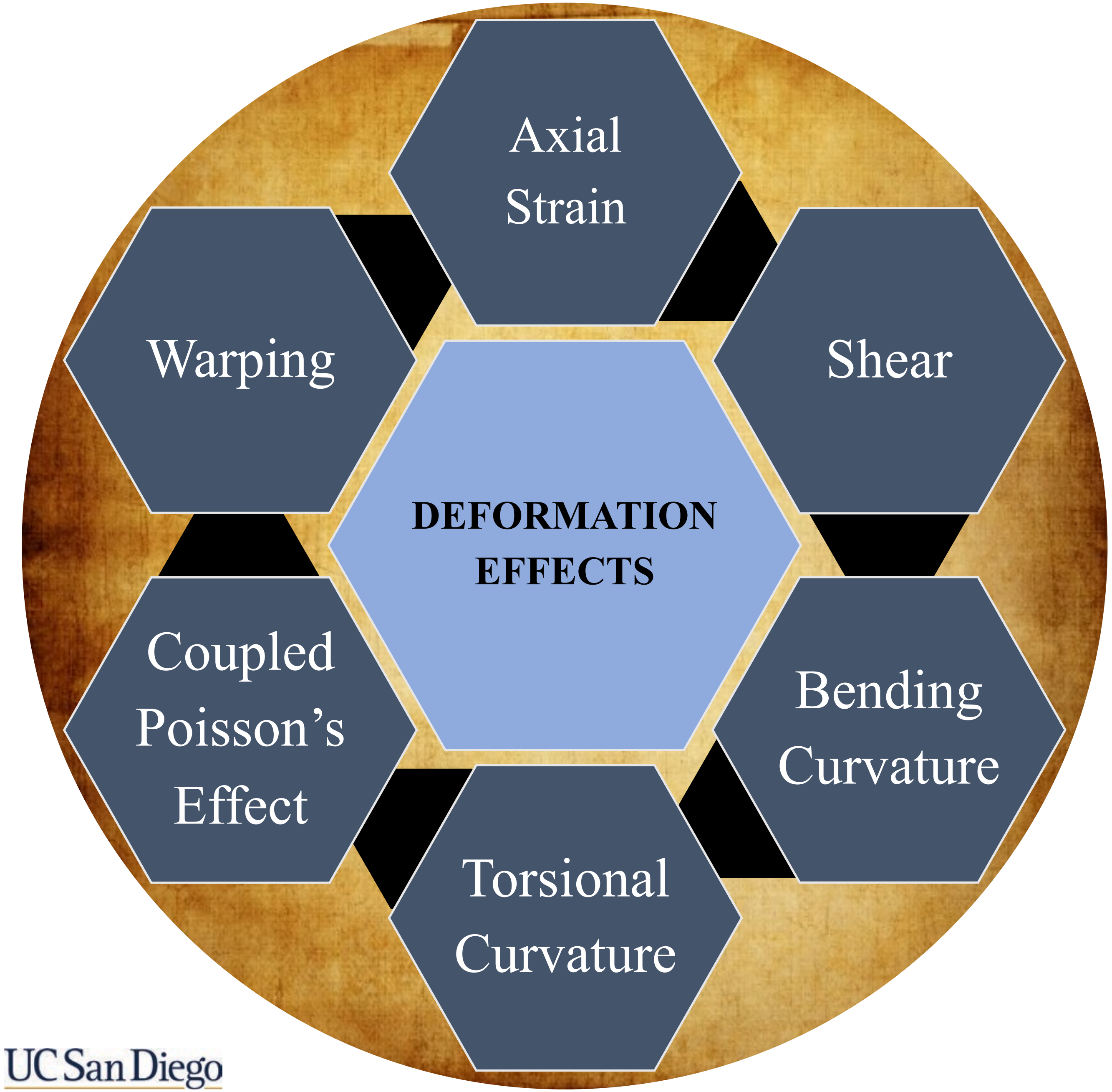
Objective and motivation

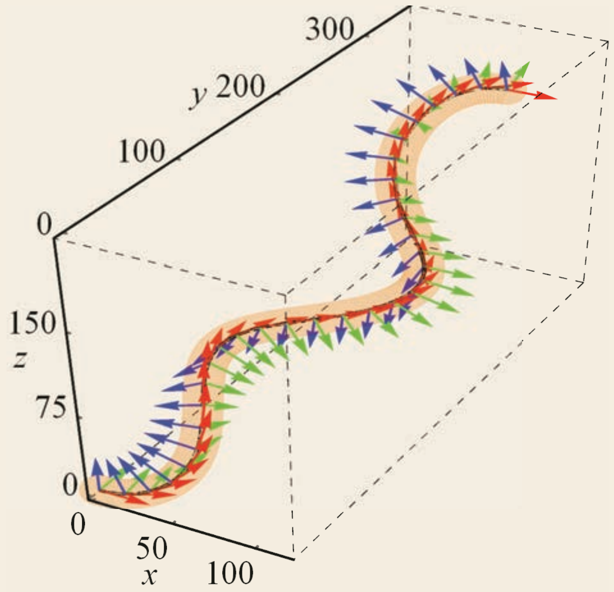


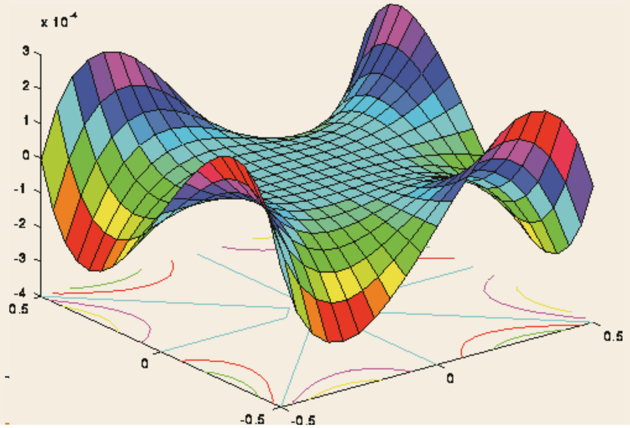
- The idea is to develop a comprehensive kinematics that incorporates all the deformation effects in the beam, **maintaining single manifold** nature of the problem.
- This geometric model is beneficial in obtaining **high accuracy** in **shape reconstruction methodology** and **analysis (FEM)** of such structures.

Presentation Outline

PART 1

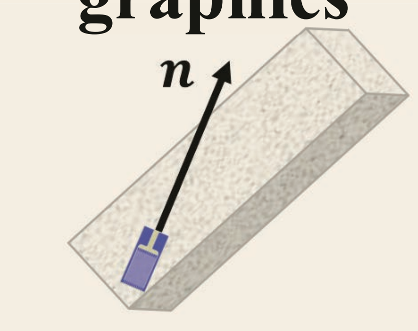
Comprehensive Kinematics

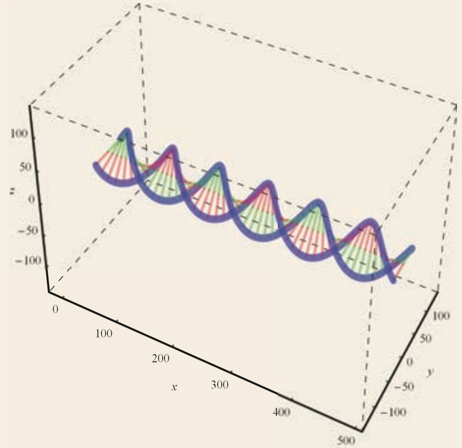




PART 2

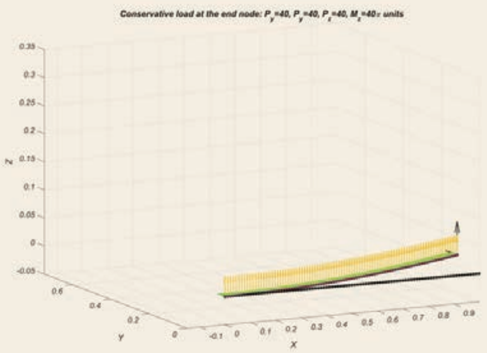
Measurement model for strain gauges, Shape sensing and graphics

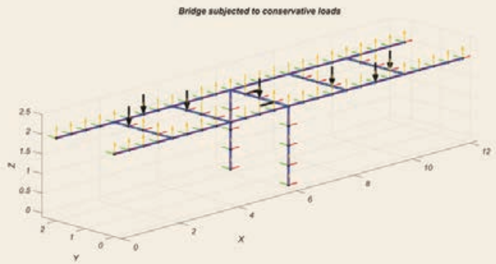




PART 3

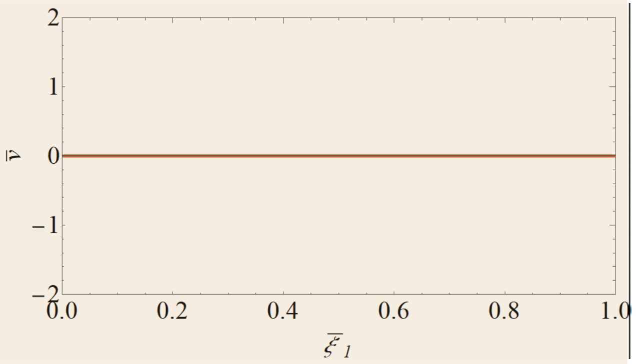
Kinetics, EOM, FEA for generalized kinematics

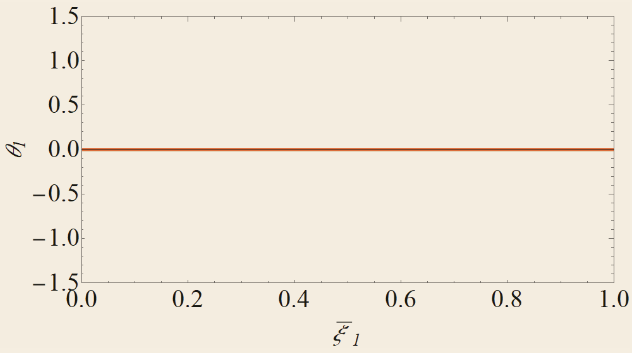




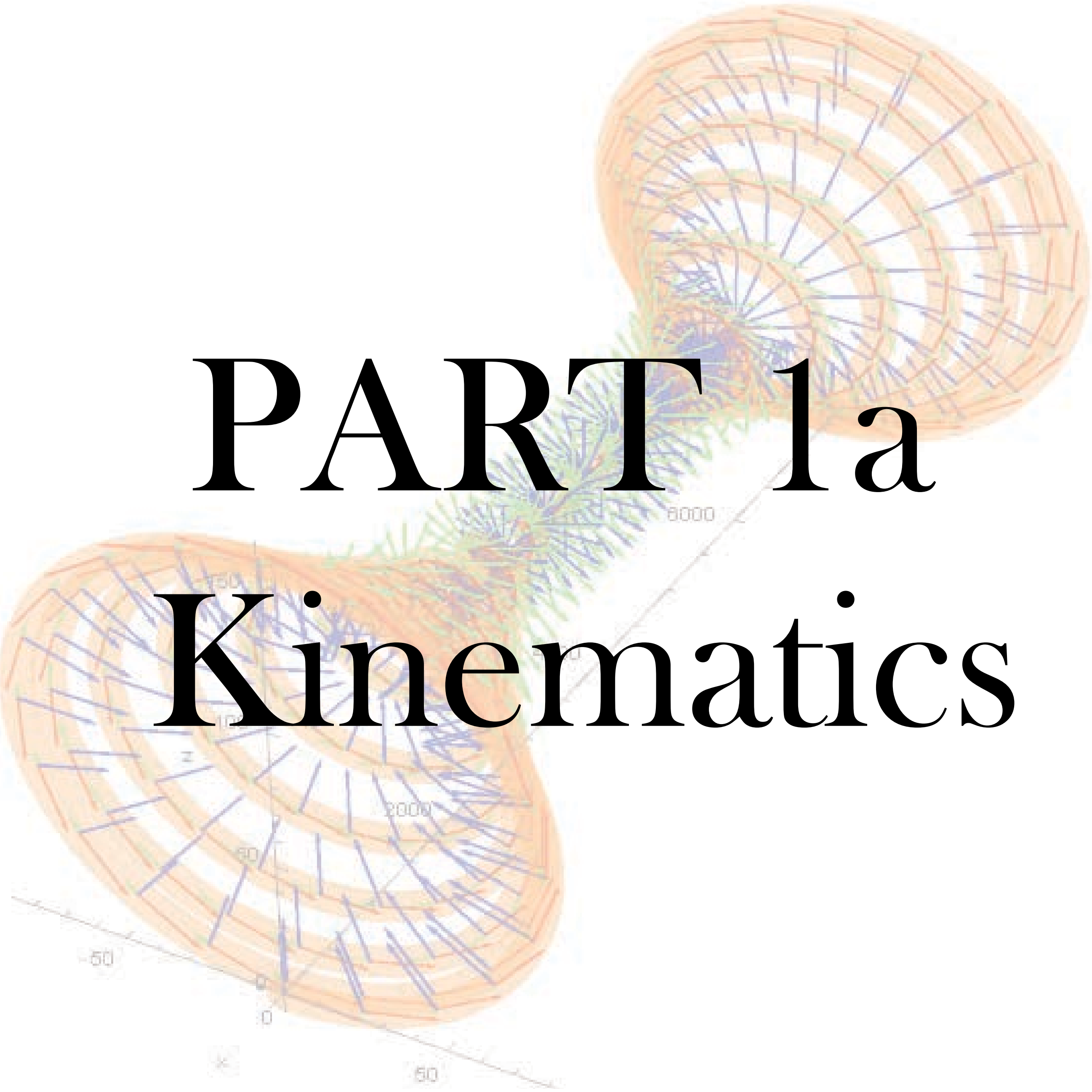
PART 4

Linear dynamics, decoupled EOM and modal analysis



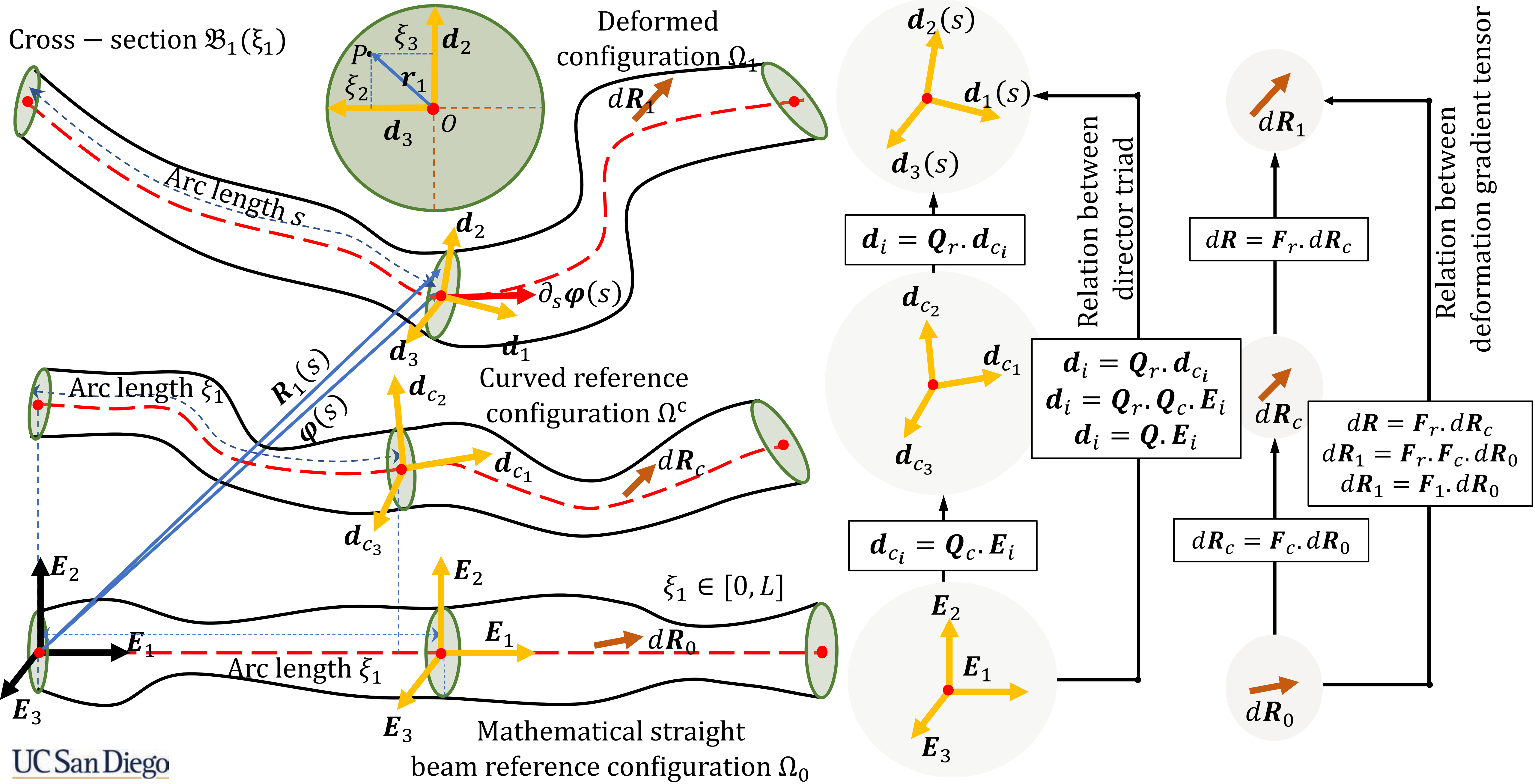


PART 1a
Kinematics



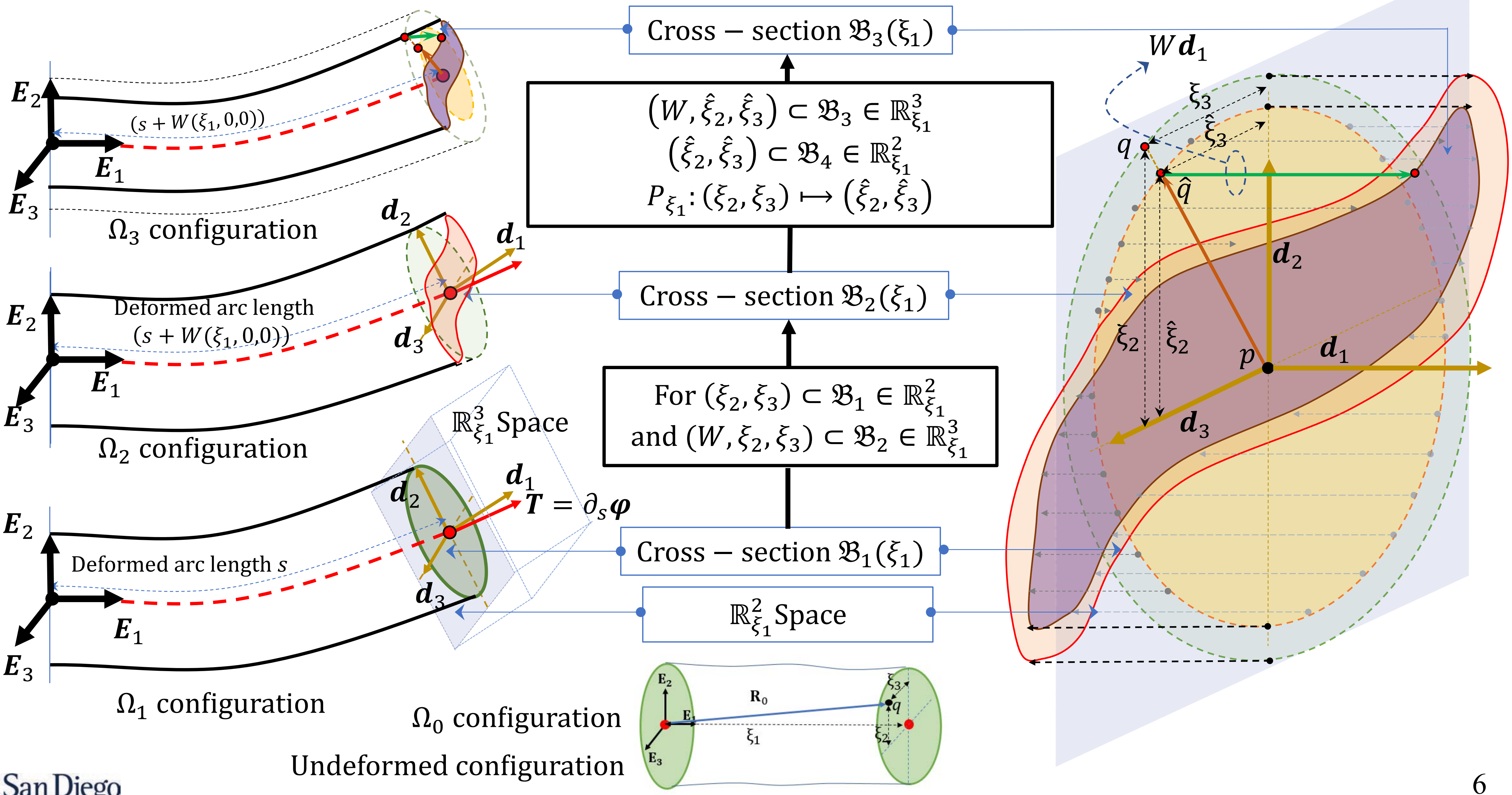
Kinematics of Cosserat Beam: Rigid cross-section

PART 1



Three deformed configurations

PART 1



Finite strain parameters-Curvatures

Consider the deformation from Ω_0 to Ω

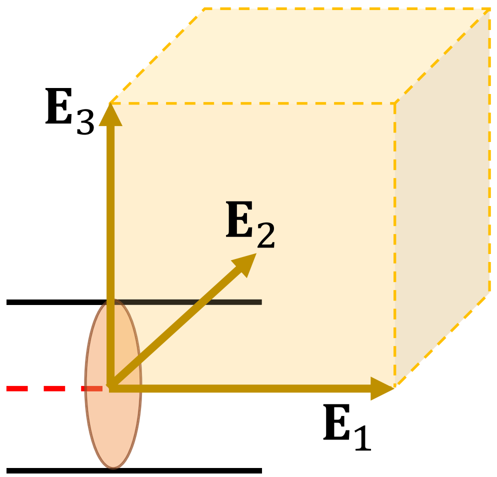
$$\mathbf{d}_{\mathrm{i}} = \mathbf{Q}.\mathbf{E}_{\mathrm{i}}, \text{where } \mathbf{Q} \in SO(3);$$

$$\mathbf{Q} = \sum_{i=1}^{3} \mathbf{d}_{\mathrm{i}} \otimes \mathbf{E}_{\mathrm{i}}$$

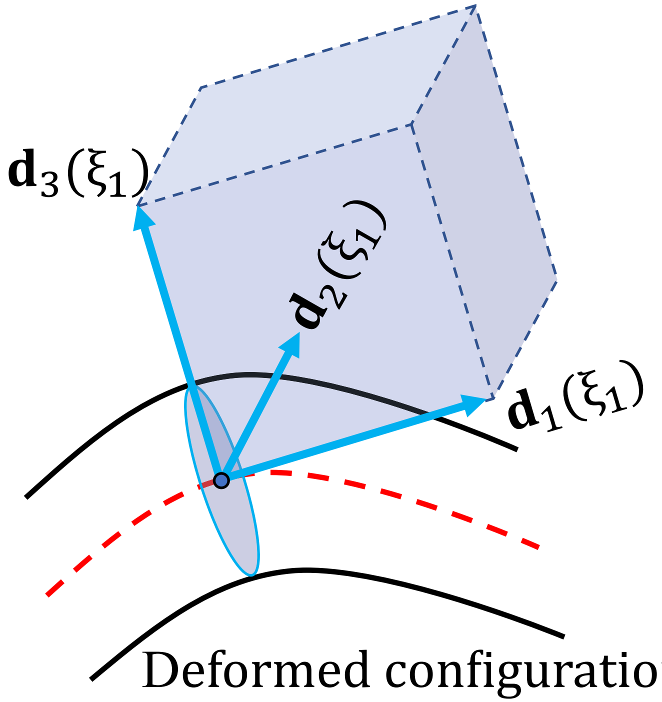
$$\mathbf{d}_{\mathrm{i},\xi_1} = \mathbf{Q}_{,\xi_1}\mathbf{E}_{\mathrm{i}} = \mathbf{Q}_{,\xi_1}\mathbf{Q}^{\mathrm{T}}\mathbf{d}_{\mathrm{i}} = \boldsymbol{\kappa}\times\mathbf{d}_{\mathrm{i}}$$

$$\begin{bmatrix} \partial_{\xi_1}\mathbf{d}_1 \\ \partial_{\xi_1}\mathbf{d}_2 \\ \partial_{\xi_1}\mathbf{d}_3 \end{bmatrix} = \begin{bmatrix} 0 & \bar{\kappa}_3 & -\bar{\kappa}_2 \\ -\bar{\kappa}_3 & 0 & \bar{\kappa}_1 \\ \bar{\kappa}_2 & -\bar{\kappa}_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix}$$

$$\partial_{\xi_1}\mathbf{d}_{\mathrm{i}} = \boldsymbol{\kappa}\times\mathbf{d}_{\mathrm{i}}, \text{where}, \boldsymbol{\kappa} = \bar{\kappa}_{\mathrm{i}}\mathbf{d}_1.$$



Undeformed configuration



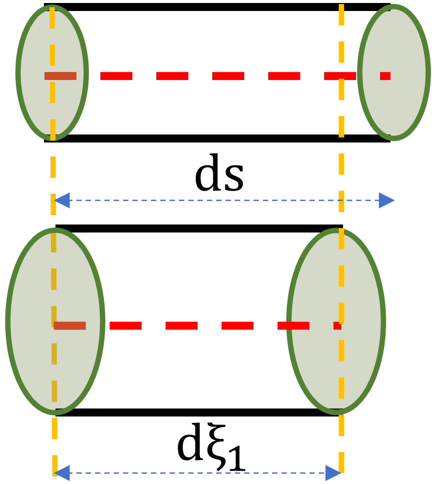
Deformed configuration

Such finite rotations can be parametrized by three scalar parameters. Three important approaches are,

1. Euler angle approach
2. Unit Quaternions
3. Rodriguez rotation formula

Finite strain parameters-Axial strain and shear angles

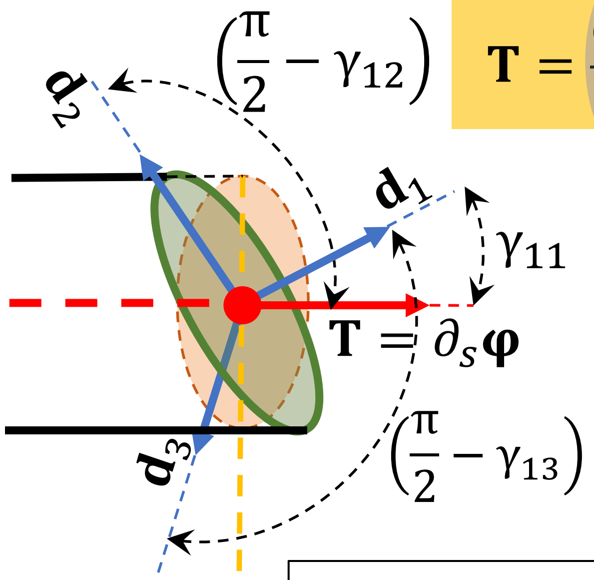
Infinitesimal portion of current beam configuration



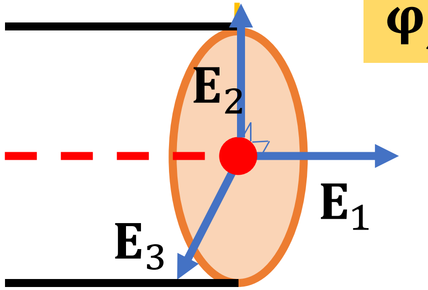
Infinitesimal portion of reference curved beam

$$e(\xi_1) = \frac{ds - d\xi_1}{d\xi_1}$$

$$\frac{ds}{d\xi_1} = 1 + e(\xi_1);$$



Deformed beam



Undeformed curved reference beam

$$\mathbf{T} = \frac{\partial \boldsymbol{\varphi}}{\partial s} = \cos(\gamma_{11})\mathbf{d}_1 + \sin(\gamma_{12})\mathbf{d}_2 + \sin(\gamma_{13})\mathbf{d}_3;$$

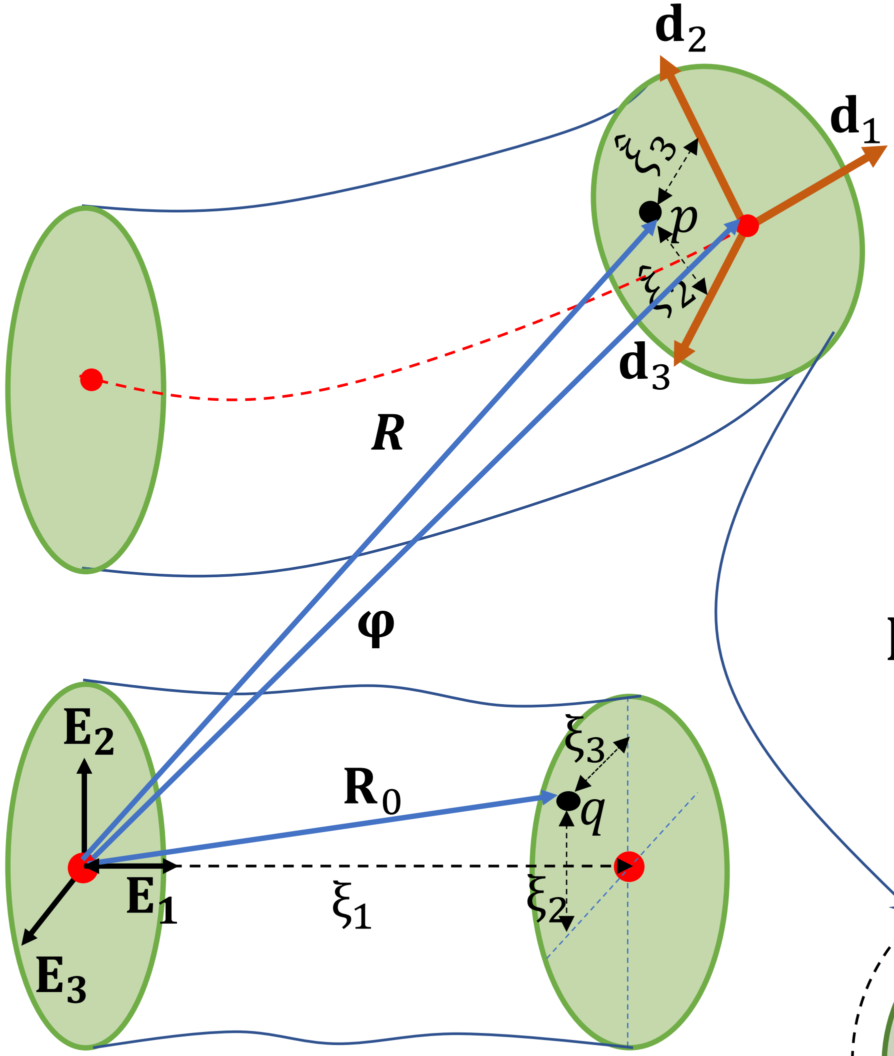
$$\frac{\partial \boldsymbol{\varphi}}{\partial \xi_1} = \frac{\partial \boldsymbol{\varphi}}{\partial s} \cdot \frac{ds}{d\xi_1}$$

$$\boldsymbol{\varphi}_{,\xi_1} = (1+e)(\cos(\gamma_{11})\mathbf{d}_1 + \sin(\gamma_{12})\mathbf{d}_2 + \sin(\gamma_{13})\mathbf{d}_3)$$

$$\boldsymbol{\varepsilon} = \bar{\varepsilon}_j \mathbf{d}_j = \partial_{\xi_1}\boldsymbol{\varphi} - \boldsymbol{d}_1$$

Beam kinematics referenced to straight beam

What is W, $\hat{\xi}_2$ and $\hat{\xi}_3$?



$\mathbf{R}_0 = \xi_i \mathbf{E}_i$

$\mathbf{R}_1 = \boldsymbol{\varphi} + \xi_2 \mathbf{d}_2 + \xi_3 \mathbf{d}_3$ (Rigid cross-section)

$\mathbf{R}_2 = \boldsymbol{\varphi} + \xi_2 \mathbf{d}_2 + \xi_3 \mathbf{d}_3 + W(\xi_1, \xi_2, \xi_3)\mathbf{d}_1$ (State Ω_2)

$\mathbf{R} = \boldsymbol{\varphi} + \hat{\xi}_2 \mathbf{d}_2 + \hat{\xi}_3 \mathbf{d}_3 + W(\xi_1, \xi_2, \xi_3)\mathbf{d}_1$ (Final state Ω_3)

Deformation gradient tensor of the general configuration Ω

$$\mathbf{F} = \sum_{i=3} \partial_{\xi_i}\mathbf{R} \otimes \mathbf{E}_i$$

Contribution due to rotation

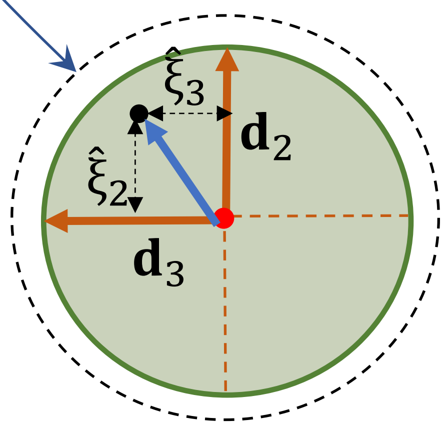
$$\partial_{\xi_i}\mathbf{R} = \boldsymbol{\lambda}_i + \mathbf{d}_i$$

Strain vector in i^{th} direction

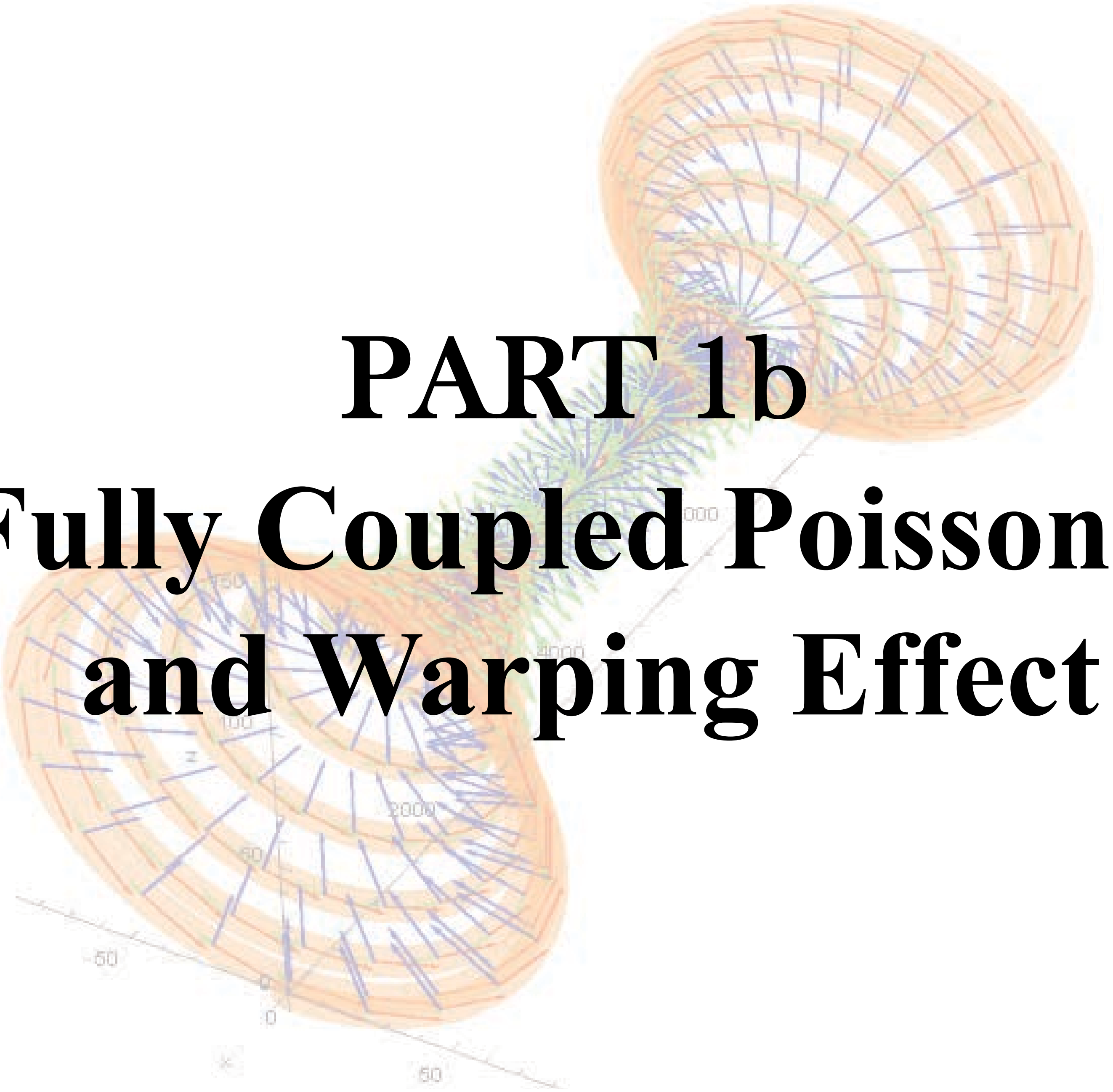
Contribution due to rotation

$$\mathbf{F} = \sum_{i=3} \boldsymbol{\lambda}_i \otimes \mathbf{E}_i + \mathbf{Q}$$

Strain vector in i^{th} direction



PART 1b
Fully Coupled Poisson's and Warping Effect



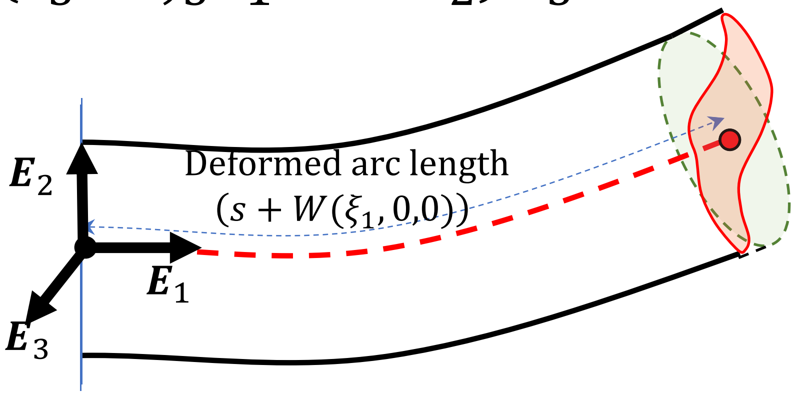
Coupled Poisson's effect for geometrically exact beam

Consider the deformed configuration Ω_2, the corresponding strain vectors are,

$$\boldsymbol{\lambda}_1^{\Omega_2} = \left(\bar{\varepsilon}_1 + \xi_3\bar{\kappa}_2 - \xi_2\bar{\kappa}_3 + W_{,\xi_1}\right)\boldsymbol{d}_1 + (\bar{\varepsilon}_2 - \xi_3\bar{\kappa}_1 + W\bar{\kappa}_3)\boldsymbol{d}_2 + (\bar{\varepsilon}_3 + \xi_3\bar{\kappa}_1 - W\bar{\kappa}_2)\boldsymbol{d}_3$$

$$\boldsymbol{\lambda}_i^{\Omega_2} = \partial_{\xi_1} W \boldsymbol{d}_1 \text{ for } i = 2,3$$

Axial strain along $\boldsymbol{d}_1 = \varepsilon_l = \boldsymbol{\lambda}_1^{\Omega_2} . \boldsymbol{d}_1$



Define Poisson's transformation as:

$$P_{\xi_1} : (\xi_2, \xi_3) \longmapsto (\hat{\xi}_2, \hat{\xi}_3)$$

$$\hat{\xi}_2 = (1 - \nu\varepsilon_l)\xi_2$$

$$\hat{\xi}_3 = (1 - \nu\varepsilon_l)\xi_3$$

$$[\boldsymbol{F}_j]_{\boldsymbol{d}_p \otimes \boldsymbol{E}_q} = [\boldsymbol{F}_j]_{\boldsymbol{E}_p \otimes \boldsymbol{E}_q} = \overbrace{\begin{bmatrix} \langle \boldsymbol{\lambda}_1^j, \boldsymbol{d}_1 \rangle & \langle \boldsymbol{\lambda}_2^j, \boldsymbol{d}_1 \rangle & \langle \boldsymbol{\lambda}_3^j, \boldsymbol{d}_1 \rangle \\ \langle \boldsymbol{\lambda}_1^j, \boldsymbol{d}_2 \rangle & \langle \boldsymbol{\lambda}_2^j, \boldsymbol{d}_2 \rangle & \langle \boldsymbol{\lambda}_3^j, \boldsymbol{d}_2 \rangle \\ \langle \boldsymbol{\lambda}_1^j, \boldsymbol{d}_3 \rangle & \langle \boldsymbol{\lambda}_2^j, \boldsymbol{d}_3 \rangle & \langle \boldsymbol{\lambda}_3^j, \boldsymbol{d}_3 \rangle \end{bmatrix}}^{\text{displacement gradient tensor } [\nabla_{\Omega_0} \boldsymbol{u}_j]_{\boldsymbol{d}_p \otimes \boldsymbol{E}_q}} + \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\boldsymbol{I}_3};$$

$$[F_{j_{pq}}]_{\boldsymbol{d}_p \otimes \boldsymbol{E}_q} = \langle \boldsymbol{\lambda}_q^j, \boldsymbol{d}_p \rangle + \delta_{pq}.$$

$$\mathbf{R}_3 \equiv \boldsymbol{R} = \boldsymbol{\varphi} + \hat{\xi}_2 \mathbf{d}_2 + \hat{\xi}_3 \mathbf{d}_3 + W(\xi_1, \xi_2, \xi_3)\mathbf{d}_1$$

$$\boldsymbol{F}_3 \equiv \boldsymbol{F} = \sum_{i=3} \lambda_i^{\Omega_3} \otimes \mathbf{E}_i + \mathbf{Q}$$

Generalized warping function for geometrically exact beam

Assumed small displacement field

$$u_1 = W(\xi_1,\xi_2,\xi_3) - \xi_2\left(\int \overline{\varkappa}_3(\xi_1)\mathrm{d}\xi_1 + C_1\right) + \xi_3\left(\int \overline{\varkappa}_2(\xi_1)\mathrm{d}\xi_1 + C_2\right) + \left(\int e(\xi_1)d\xi_1 + C_3\right);$$

$$u_2 = \left[\int\int \overline{\varkappa}_3(\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_1 + C_1\xi_1 + C_4\right] - \xi_3\left(\int \overline{\varkappa}_1(\xi_1)\mathrm{d}\xi_1 + C_5\right) - \nu e(\xi_1)\xi_2;$$

$$u_3 = -\left[\int\int \overline{\varkappa}_2(\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_1 + C_2\xi_1 + C_6\right] + \xi_2\left(\int \overline{\varkappa}_1(\xi_1)\mathrm{d}\xi_1 + C_5\right) - \nu e(\xi_1)\xi_3.$$

Governing differential equation for warping

$$\sigma_{1j,j} = 0 \Rightarrow \nabla^2 W + \frac{\tilde{\lambda}}{G}\left(W_{,\xi_1\xi_1} - \xi_2\overline{\varkappa}_{3,\xi_1} + \xi_3\overline{\varkappa}_{2,\xi_1}\right) + \overline{\lambda}e_{,\xi_1} = 0 \quad \text{on } \mathfrak{B}(\xi_1)$$

$$W_{,n} = \overline{\varkappa}_1\overbrace{\left\langle [\boldsymbol{n}\times(\xi_2\boldsymbol{E}_2+\xi_3\boldsymbol{E}_3)],\boldsymbol{E}_1\right\rangle}^{-t} + e_{,\xi_1}\nu\overbrace{\left\langle \boldsymbol{n},(\xi_2\boldsymbol{E}_2+\xi_3\boldsymbol{E}_3)\right\rangle}^{\tilde{t}} \quad \text{on } \Gamma(\xi_1).$$

Integrate across $\mathfrak{B}$

$$\int_{\mathfrak{B}} \nabla^2 W \mathrm{d}\xi_2\mathrm{d}\xi_3 = -\frac{\tilde{\lambda}}{G}\int_{\mathfrak{B}} W_{,\xi_1\xi_1}\mathrm{d}\xi_2\mathrm{d}\xi_3 - \overline{\lambda}Ae_{,\xi_1}.$$

$$\int_{\mathfrak{B}} \nabla^2 W \mathrm{d}\xi_2\mathrm{d}\xi_3 = \oint W_{,n}\mathrm{d}\Gamma.$$

Integrate along Γ

$$\oint W_{,n}\mathrm{d}\Gamma = -\overline{\varkappa}_1\oint t\,\mathrm{d}\Gamma + e_{,\xi_1}\nu\oint \tilde{t}\,\mathrm{d}\Gamma = 2\nu Ae_{,\xi_1}.$$

Inconsistency condition

$$P_{1,\xi_1} = (\tilde{\lambda} - 2\nu\lambda)Ae_{,\xi_1} + \tilde{\lambda}\int_{\mathfrak{B}} W_{,\xi_1\xi_1}\mathrm{d}\xi_2\mathrm{d}\xi_3 = 0$$

OR

$\boldsymbol{P}_1$ = Reduced axial force = Constant

Generalized warping function for geometrically exact beam

Consistent warping equation

$$\nabla^2 W + C_1\left(W_{,\xi_1\xi_1} - \xi_2\overline{\varkappa}_{3,\xi_1} + \xi_3\overline{\varkappa}_{2,\xi_1}\right) + C_2\overline{\lambda}\int_{\mathfrak{B}} W_{,\xi_1\xi_1}\,\mathrm{d}\xi_2\mathrm{d}\xi_3 = 0 \text{ on } \mathfrak{B}(\xi_1)$$

$$W_{,n} = -\overline{\varkappa}_1 t + \left\{\nu C_2 \int_{\mathfrak{B}} W_{,\xi_1\xi_1}\,\mathrm{d}\xi_2\mathrm{d}\xi_3\right\}\tilde{t} \text{ on } \Gamma(\xi_1),$$

Assumed solution

$$W(\xi_1,\xi_2,\xi_3) = \sum_{r=0}^{\infty}\left(\overline{\varkappa}_1^{(r)}\Psi_{1r} + \overline{\varkappa}_2^{(r)}\Psi_{2r} + \overline{\varkappa}_3^{(r)}\Psi_{3r} + e^{(r)}\Psi_{4r}\right)$$

Final form of solution

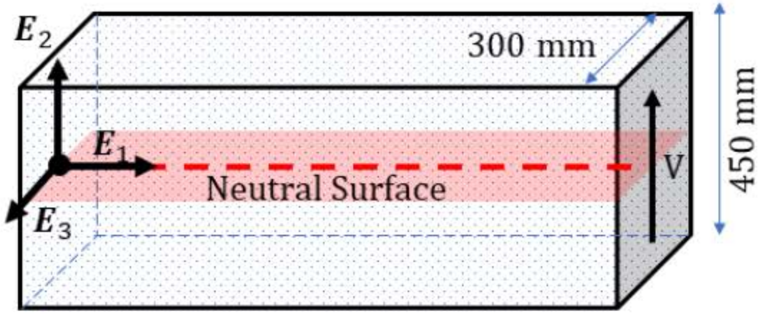
$$W(\xi_1,\xi_2,\xi_3) = \left(\overline{\varkappa}_1\Psi_{10} + \overline{\varkappa}_1^{(2)}\Psi_{12} + \overline{\varkappa}_1^{(4)}\Psi_{14} + \ldots\right) + \left(\overline{\varkappa}_2^{(1)}\Psi_{21} + \overline{\varkappa}_2^{(3)}\Psi_{23} + \overline{\varkappa}_2^{(5)}\Psi_{25} + \ldots\right) + \left(\overline{\varkappa}_3^{(1)}\Psi_{31} + \overline{\varkappa}_3^{(3)}\Psi_{33} + \overline{\varkappa}_3^{(5)}\Psi_{35} + \ldots\right),$$

No explicit contribution to warping due to axial strain!!

Physical interpretation of the warping function Ψ_{31} or Ψ_{21}

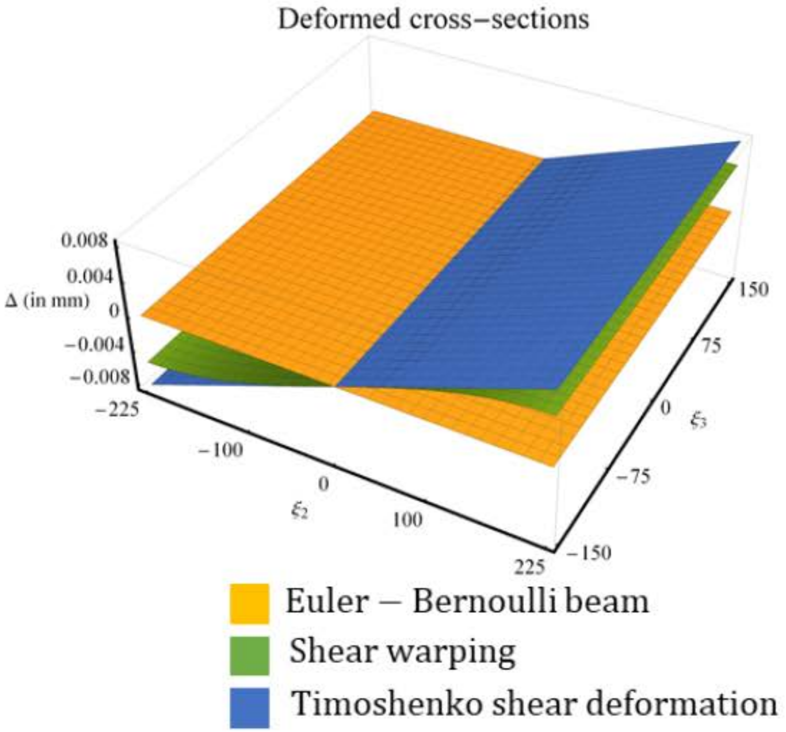
Claim: The warping contribution $\bar{\kappa}_3^{(1)}\Psi_{31}$ represents the out-of-plane deformation of the cross-section due to the non-uniform shear stress field induced by bending about $\boldsymbol{E}_3$.

This implies that the slope $\bar{\kappa}_3^{(1)} \dfrac{\partial \Psi_{31}}{\partial \xi_2}$ is the shear strain profile of the cross-section



Material Properties assumed: E = 30GPa, G = 15Gpa
Assumed Shear force: V = 50kN

$$\Delta = \begin{cases} \bar{\kappa}_3^{(1)}\Psi_{31}, & \text{Shear warping} \\ \bar{\kappa}_3^{(1)}\Psi_{31}^{t}, & \text{Timoshenko shear deformation} \\ 0, & \text{Euler – Bernoulli beam} \end{cases}$$



To do FE formulation, we assume simplified warping function as

$W = p\Psi_{10} + \bar{\kappa}_3^{(1)}\Psi_{31} + \bar{\kappa}_2^{(1)}\Psi_{21}$, where p is warping amplitude.

Physical interpretation of $\lambda_1^{\Omega_3}$

$\mathbf{E}_1$ — Pure axial strain → $(1+\mathrm{e})\mathbf{d}_1$ — Pure axial strain followed by shear →

$$(1+\mathrm{e})\cos(\gamma_{11})\mathbf{d}_1 + (1+\mathrm{e})\sin(\gamma_{12})\mathbf{d}_2 + (1+\mathrm{e})\sin(\gamma_{13})\mathbf{d}_3 = \partial_{\xi_1}\boldsymbol{\varphi}$$

Bending about $\mathbf{d}_2$ and $\mathbf{d}_3$ →

$$\{(1+\mathrm{e})\cos(\gamma_{11}) - \hat{\xi}_2\bar{\kappa}_3 + \hat{\xi}_3\bar{\kappa}_2\}\mathbf{d}_1 + (1+\mathrm{e})\sin(\gamma_{12})\mathbf{d}_2 + (1+\mathrm{e})\sin(\gamma_{13})\mathbf{d}_3$$

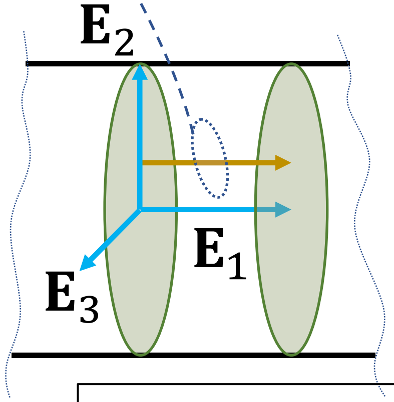
Torsion about $\mathbf{d}_1$ →

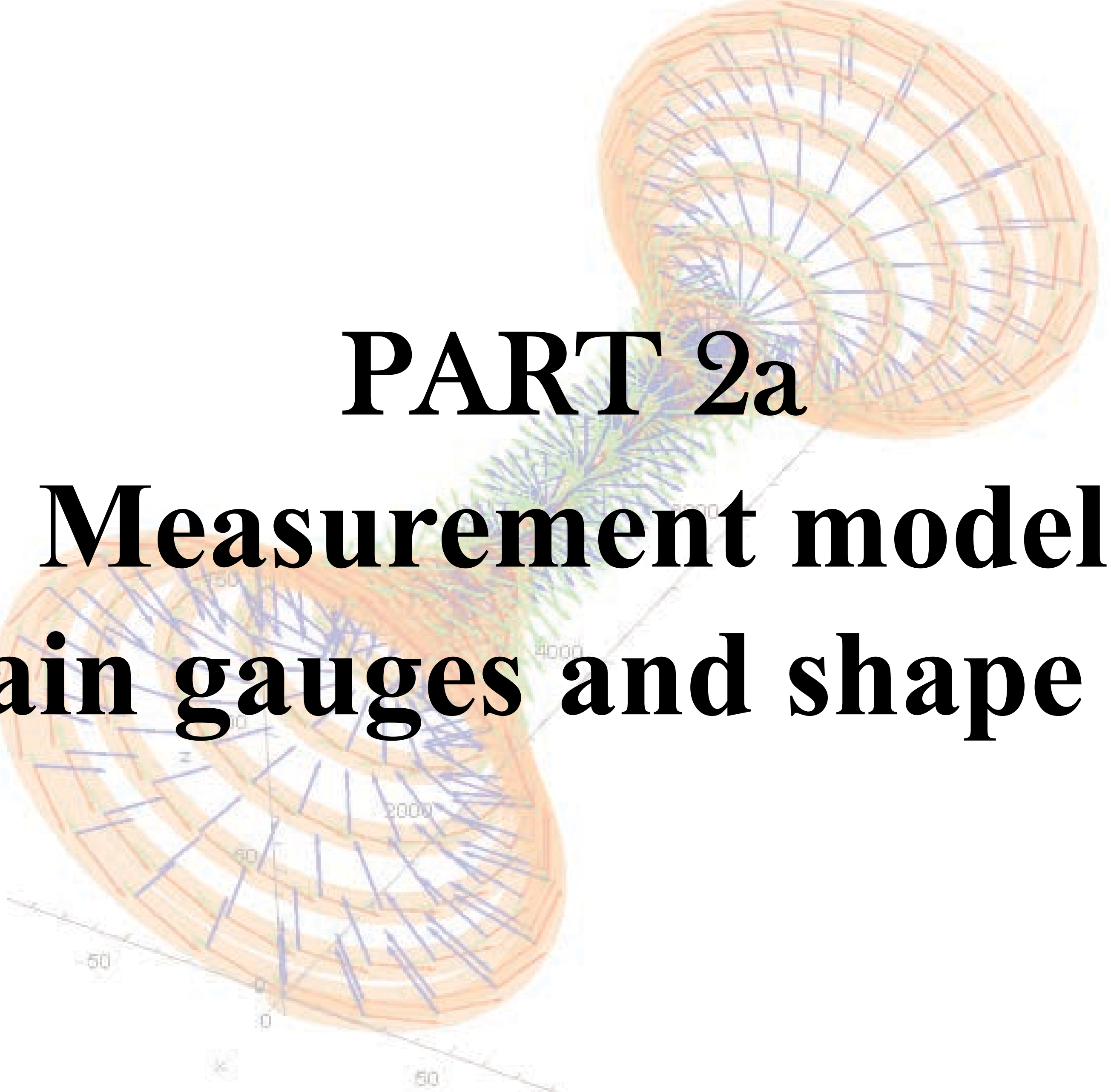
$$\{(1+\mathrm{e})\cos(\gamma_{11}) - \hat{\xi}_2\bar{\kappa}_3 + \hat{\xi}_3\bar{\kappa}_2\}\mathbf{d}_1 + \{(1+\mathrm{e})\sin(\gamma_{12}) - \hat{\xi}_3\bar{\kappa}_1\}\mathbf{d}_2 + \{(1+\mathrm{e})\sin(\gamma_{13}) + \hat{\xi}_2\bar{\kappa}_1\}\mathbf{d}_3$$

Poisson's deformation and Warping →

$$\{(1+\mathrm{e})\cos(\gamma_{11}) - \hat{\xi}_2\bar{\kappa}_3 + \hat{\xi}_3\bar{\kappa}_2 + \partial_{\xi_1}W\}\mathbf{d}_1 + \{(1+\mathrm{e})\sin(\gamma_{12}) - \hat{\xi}_3\bar{\kappa}_1 + \partial_{\xi_1}\hat{\xi}_2 + W\bar{\kappa}_3\}\mathbf{d}_2 + \{(1+\mathrm{e})\sin(\gamma_{13}) + \hat{\xi}_2\bar{\kappa}_1 + \partial_{\xi_1}\hat{\xi}_3 - W\bar{\kappa}_2\}\mathbf{d}_3$$

$$\mathbf{F}.\mathbf{E}_1 = \boldsymbol{\lambda}_1 + \boldsymbol{d}_1$$

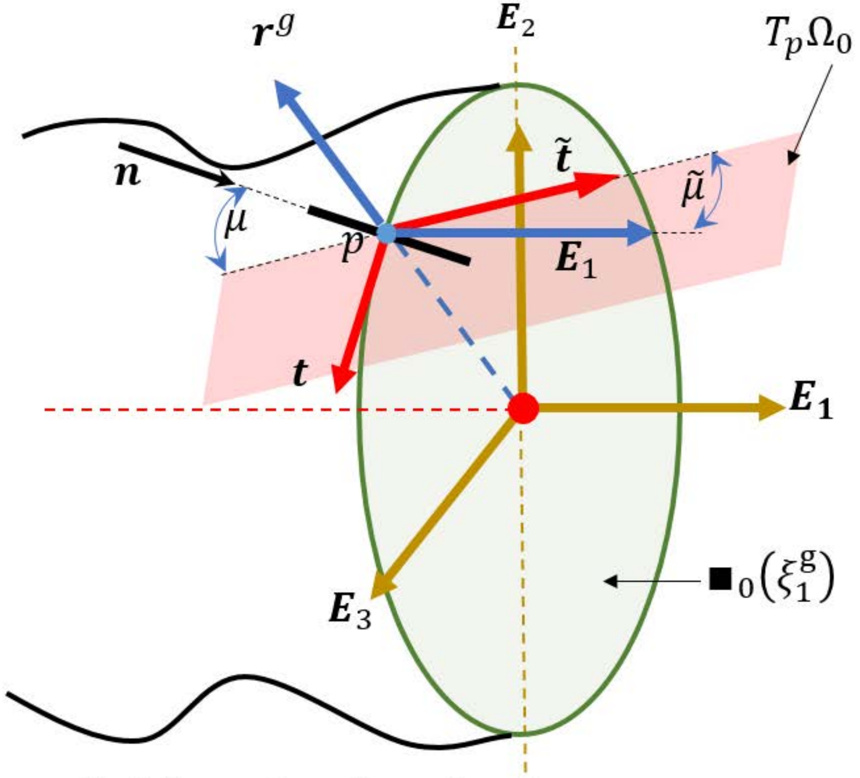




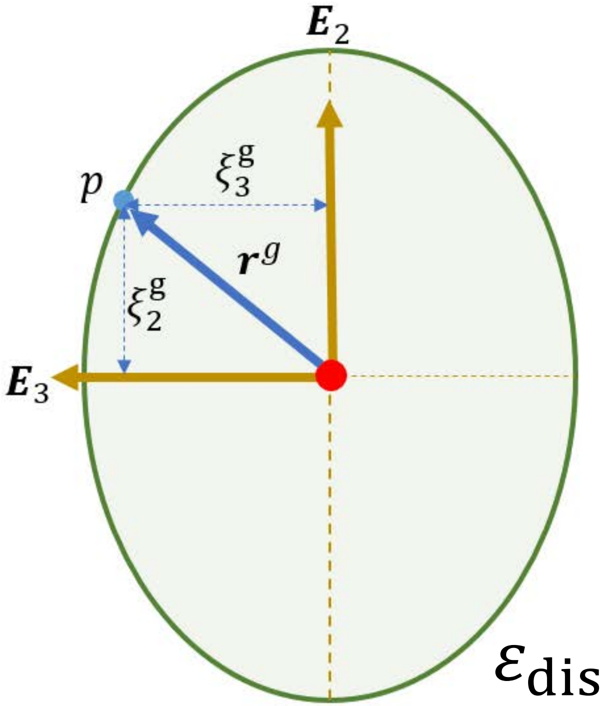
PART 2a
Measurement model for strain gauges and shape sensing

Discrete strain gauge

Discrete strain gauge can be treated as an infinitesimally small vector $l_g \boldsymbol{n} \in T_p\Omega_0$, where,



Undeformed configuration Ω_0



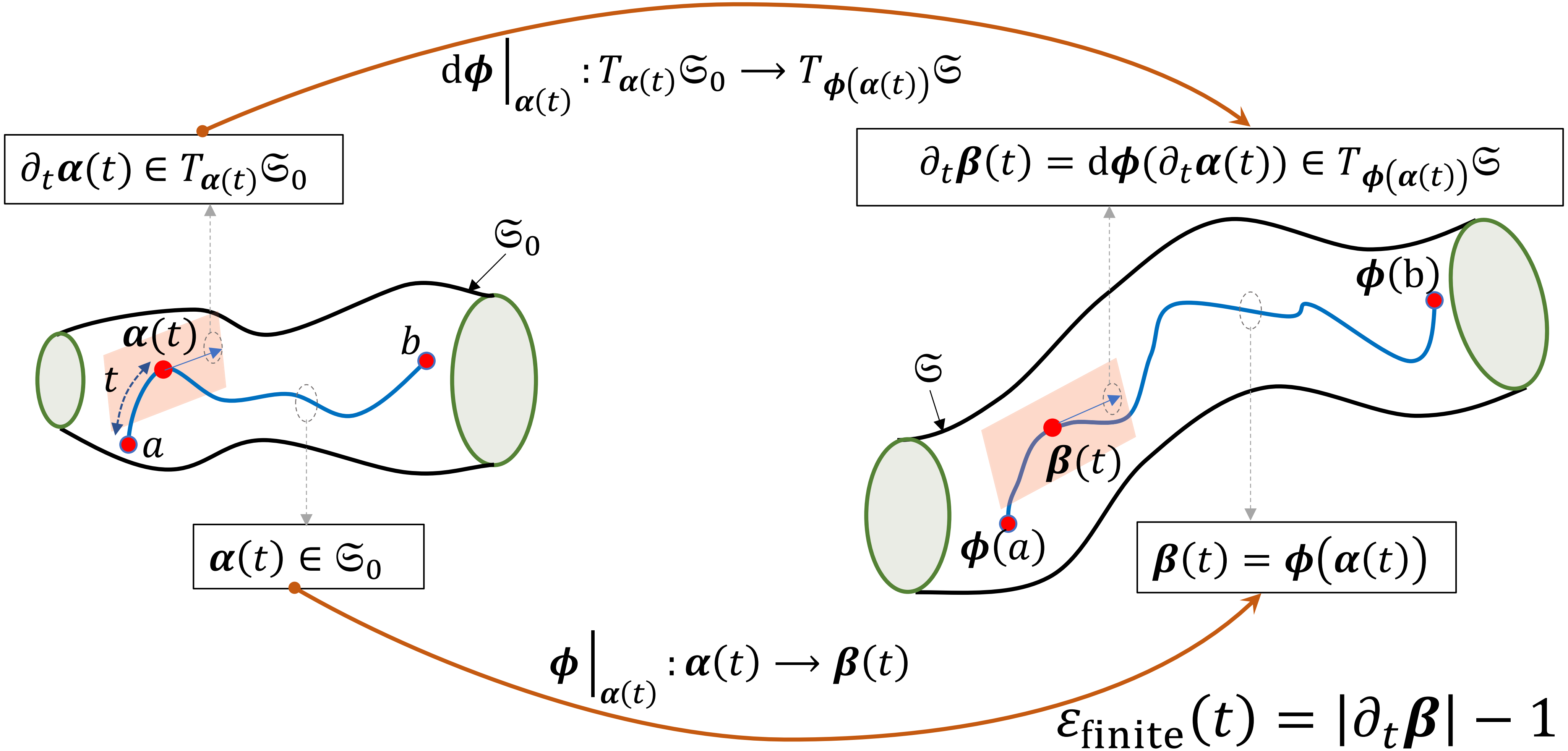
Cross – section $\blacksquare_0(\xi_1^g)$

$$\boldsymbol{n} = \cos\mu\,\tilde{\boldsymbol{t}} + \sin\mu\,\boldsymbol{t}$$

$$\varepsilon_{\text{discrete}} = \sqrt{(\mathbf{F}(p)\boldsymbol{n}).(\mathbf{F}(p)\boldsymbol{n})} - 1$$

$$\varepsilon_{\text{discrete}} = \sqrt{\boldsymbol{n}.\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{n}} - 1$$

Finite length strain gauge



$$\varepsilon_{\text{finite}}(t) = |\partial_t \boldsymbol{\beta}| - 1$$

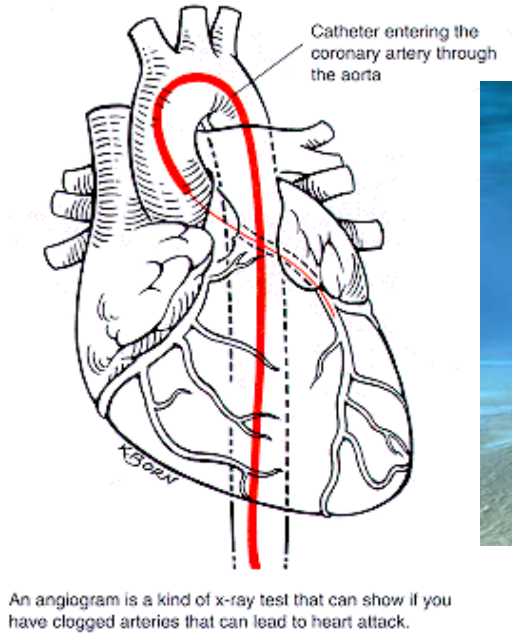
$$\partial_t \boldsymbol{\beta} = \boldsymbol{F}.\partial_t \boldsymbol{\alpha}$$

$$|\partial_t \boldsymbol{\beta}| = \sqrt{\partial_t \boldsymbol{\alpha}.\boldsymbol{F}^T \boldsymbol{F} \partial_t \boldsymbol{\alpha}}$$

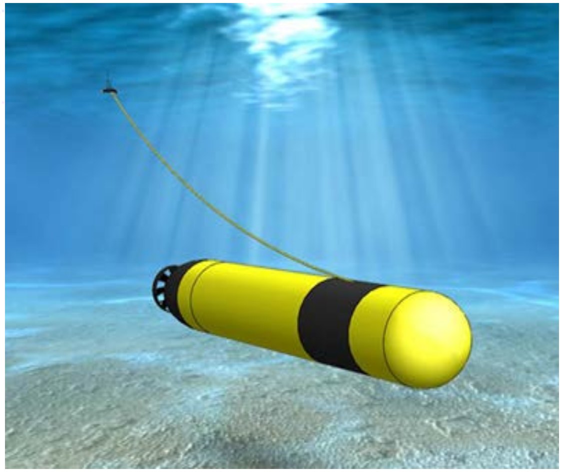
$$\varepsilon_{\text{finite\_avg}} = \frac{1}{l_o}\int_0^{l_o} |\partial_t \boldsymbol{\beta}|\, dt - 1$$

Shape sensing: motivation

There are multiple instances where it is desirable to reconstruct the full-field deformed shape of a very long, slender object such as pipelines, suspension cables, tethers, surgical tubing, catheters, and others.

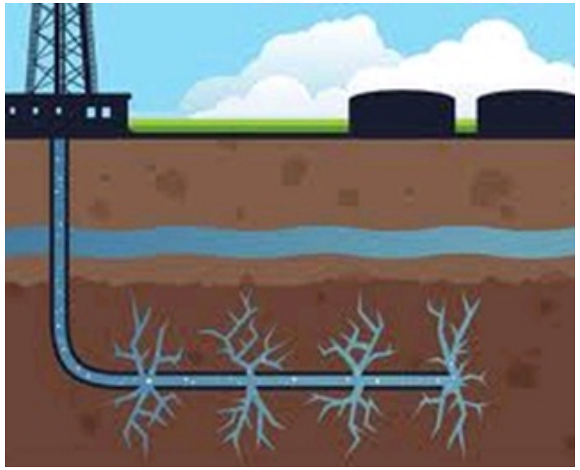


Cardiac catheter monitoring ($12B industry in 2013)



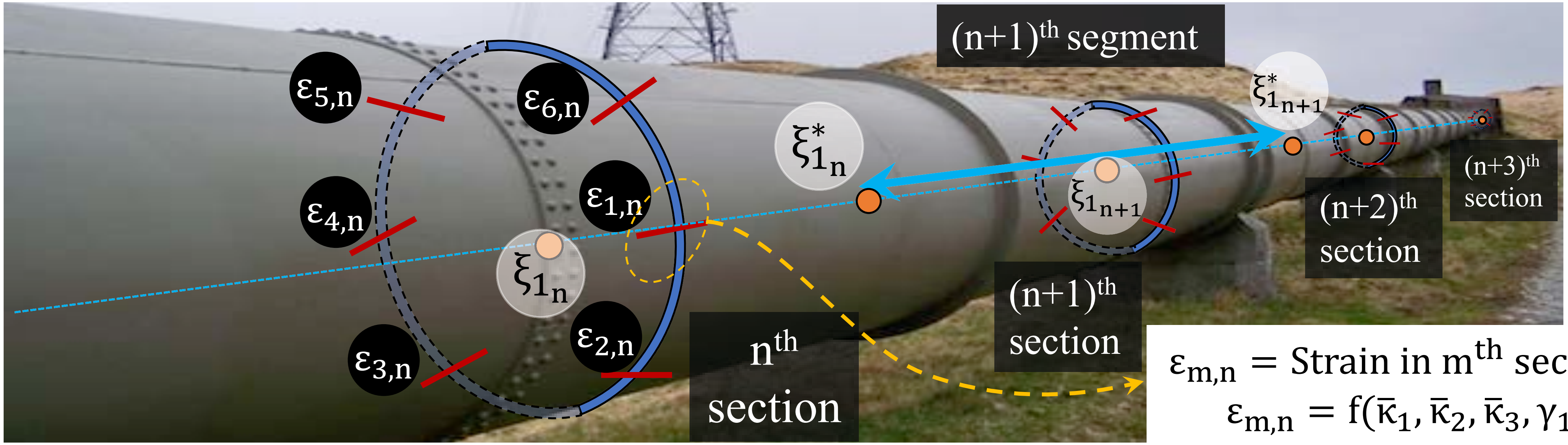
Smart tethers for target Localization and marine exploration

Structural health monitoring applications



Application in the oil industry

Overview of the shape reconstruction methodology



$\varepsilon_{m,n}$ = Strain in m^{th} section in n^{th} gauge

$\varepsilon_{m,n} = f(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \gamma_{11}, \gamma_{12}, \gamma_{13}, e)$

INPUT: We obtain 6 values of uniaxial surface strain measurements $\varepsilon_{m,n}$ at each discretized section.

Then we solve for the finite strain terms from the kinematics.

SYSTEM: Governing differential equations

$$\begin{bmatrix} \partial_{\xi_1}\boldsymbol{\varphi} \\ \partial_{\xi_1}\mathbf{d}_1 \\ \partial_{\xi_1}\mathbf{d}_2 \\ \partial_{\xi_1}\mathbf{d}_3 \end{bmatrix} = \begin{bmatrix} 0 & (1+e)\cos\gamma_{11} & (1+e)\sin\gamma_{12} & (1+e)\sin\gamma_{13} \\ 0 & 0 & \bar{\kappa}_3 & -\bar{\kappa}_2 \\ 0 & -\bar{\kappa}_3 & 0 & \bar{\kappa}_1 \\ 0 & \bar{\kappa}_2 & -\bar{\kappa}_1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix}$$

Forward model

OUTPUT: Deformed shape

Shape sensing algorithm

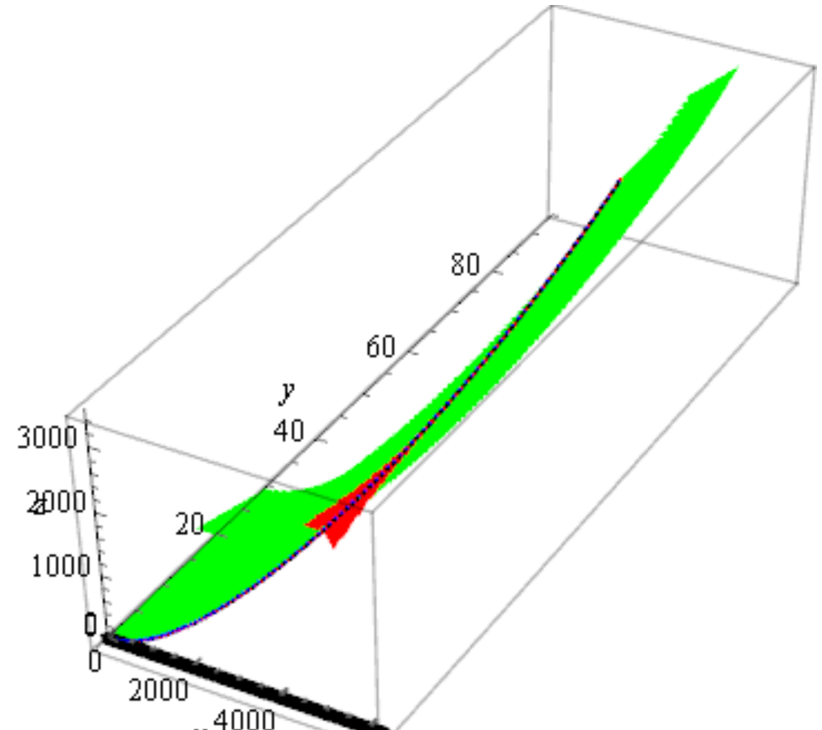
Inverse model

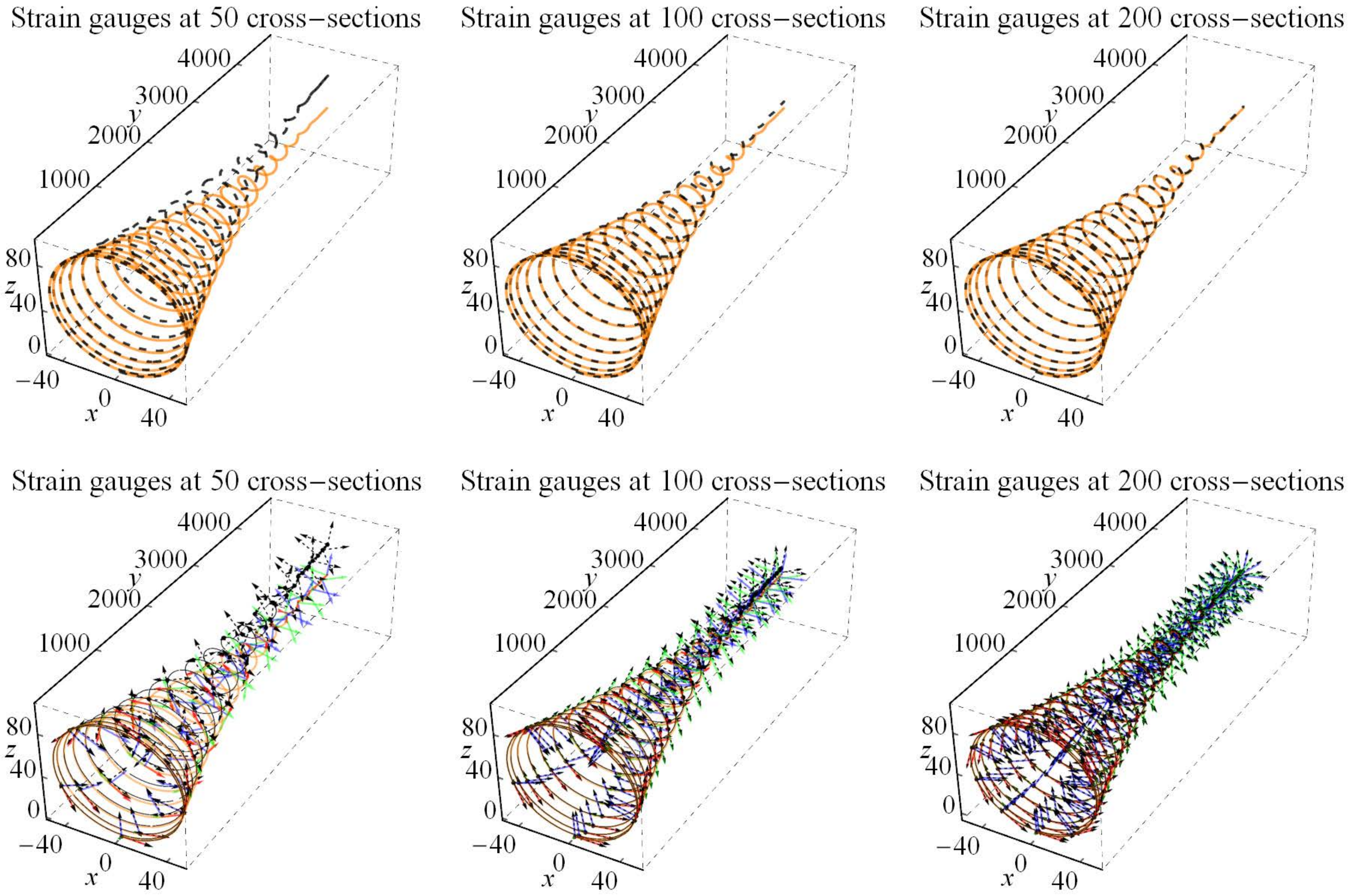
Shape sensing for pure bending curvature

Simulation 1:

- Length: 6300m
- Deformations: **Curvature.**

No. of cross-section	Spacing of sections	RMS Error
50	160 m	15.9 m
200	31.5 m	0.78 m



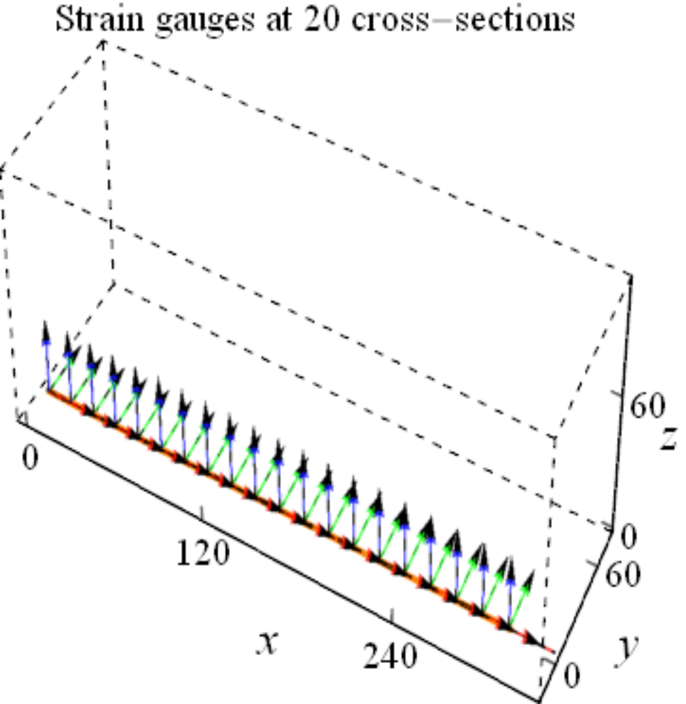


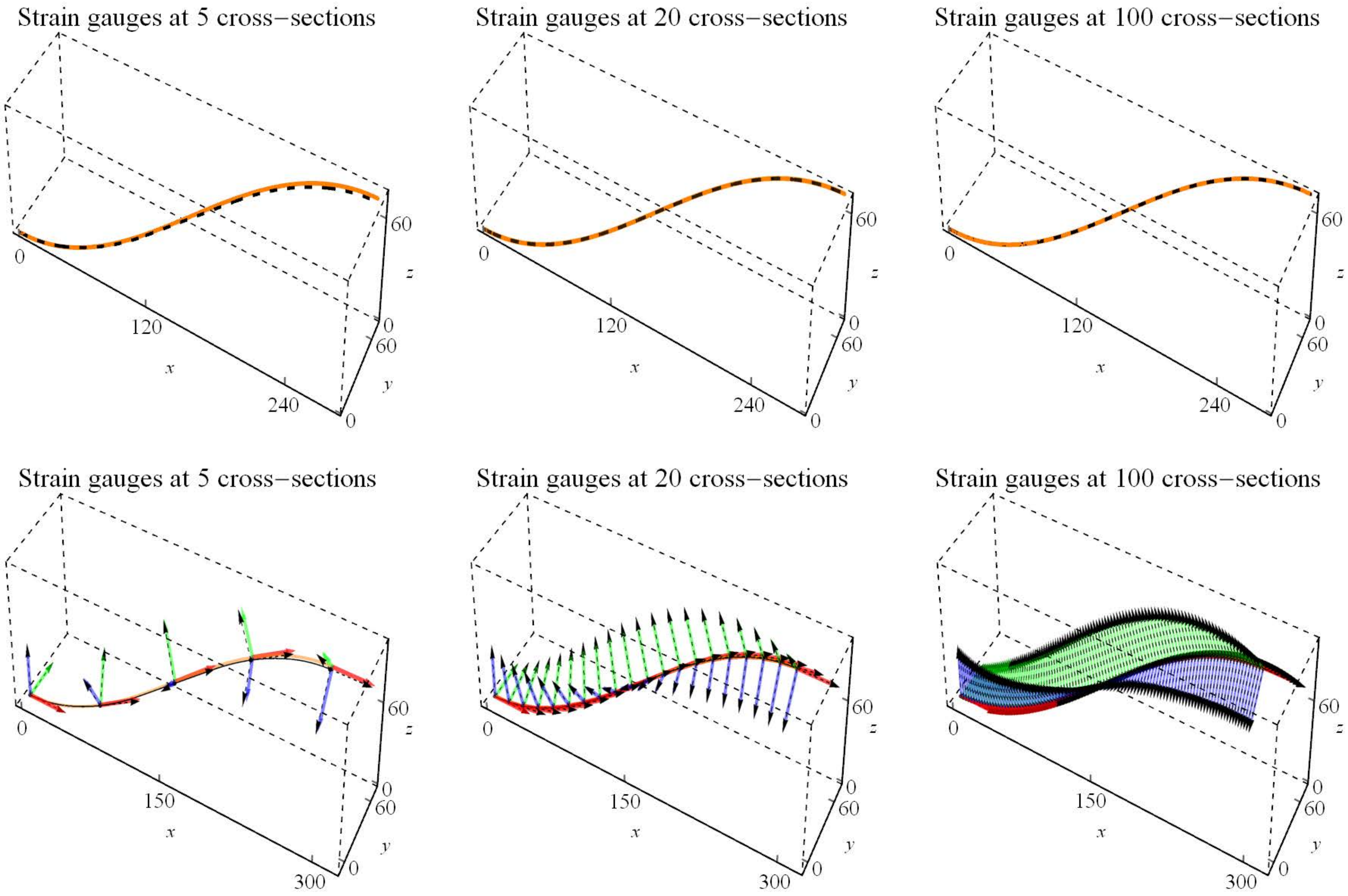
Shape sensing example 2

Simulation 1:

- Length: 300m
- Deformations: **Curvature, torsion, elongation..**

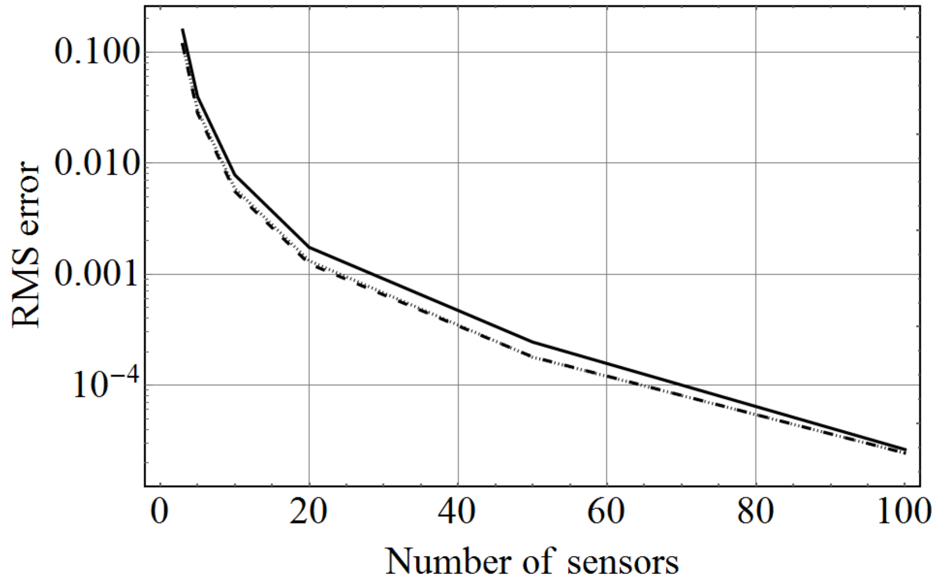
No. of cross-section	Spacing of sections	RMS Error
5	60 m	5.2 m
10	30 m	1.03 m



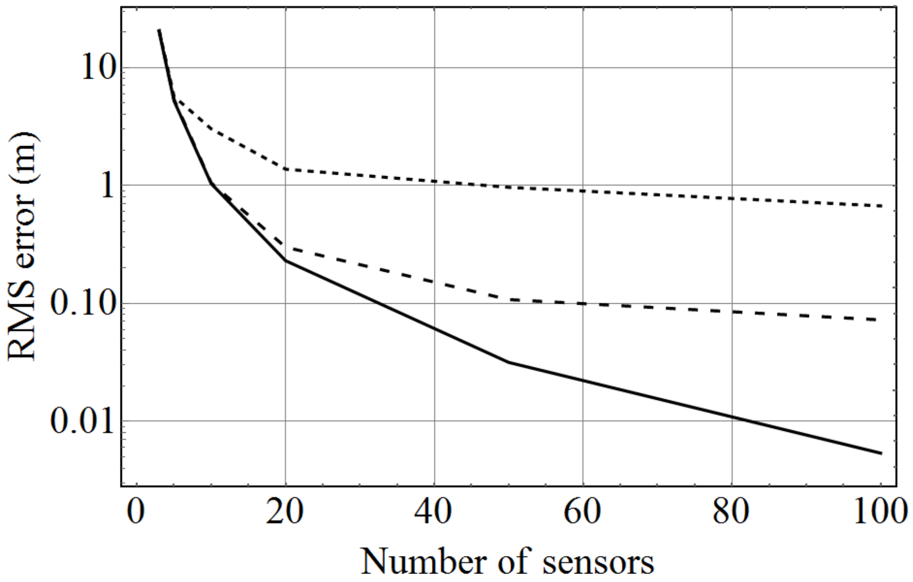


Measurement and estimation error

RMS error in $\boldsymbol{d}_i(\xi_1)$



RMS error in $\boldsymbol{\varphi}(\xi_1)$



Legend for the plot of RMS error in $\boldsymbol{d}_i$

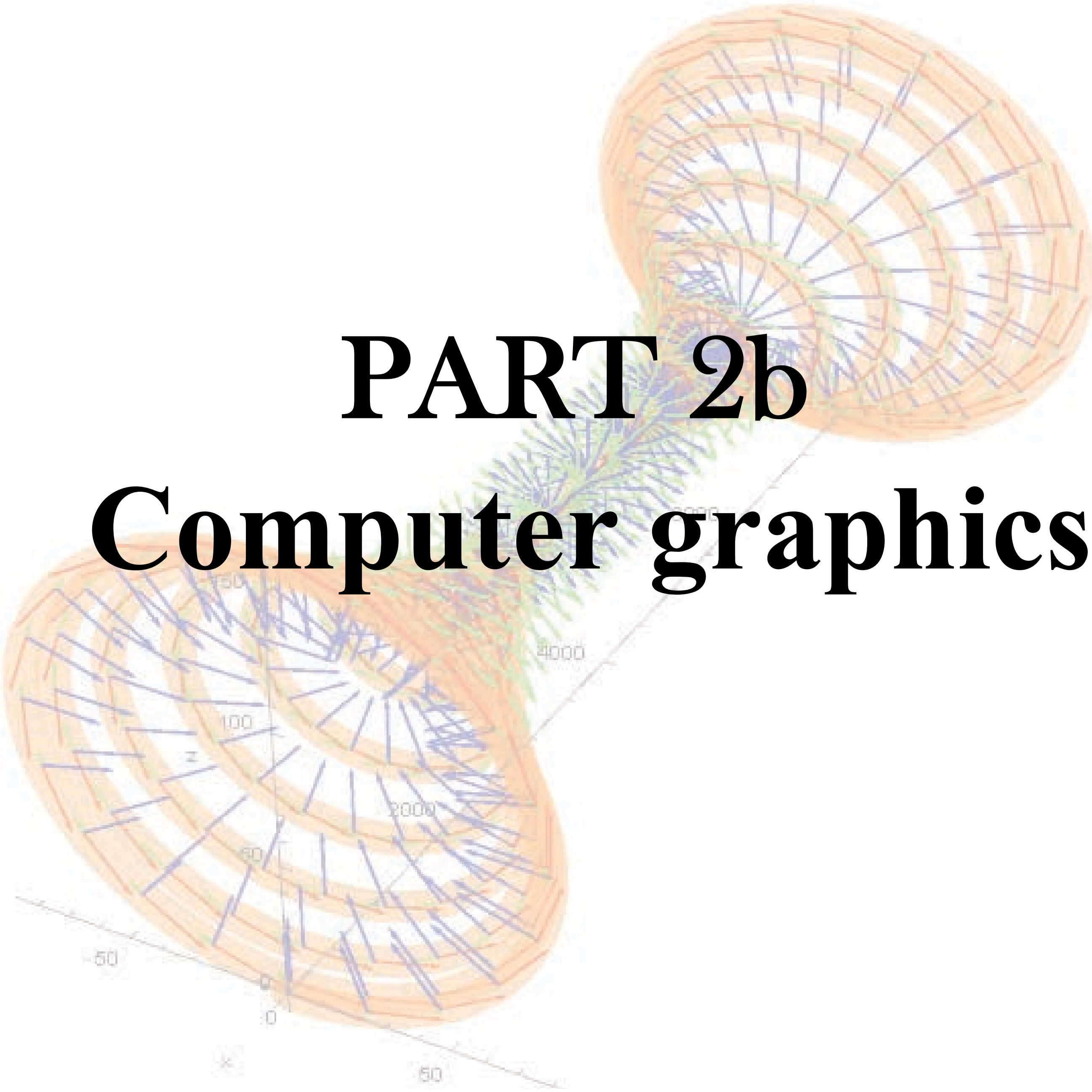
- (solid) RMS error in $\boldsymbol{d}_1$ for no noise case
- (dashed) RMS error in $\boldsymbol{d}_2$ for no noise case
- (dotted) RMS error in $\boldsymbol{d}_3$ for no noise case

Legend for the plot of RMS error in $\boldsymbol{\varphi}$

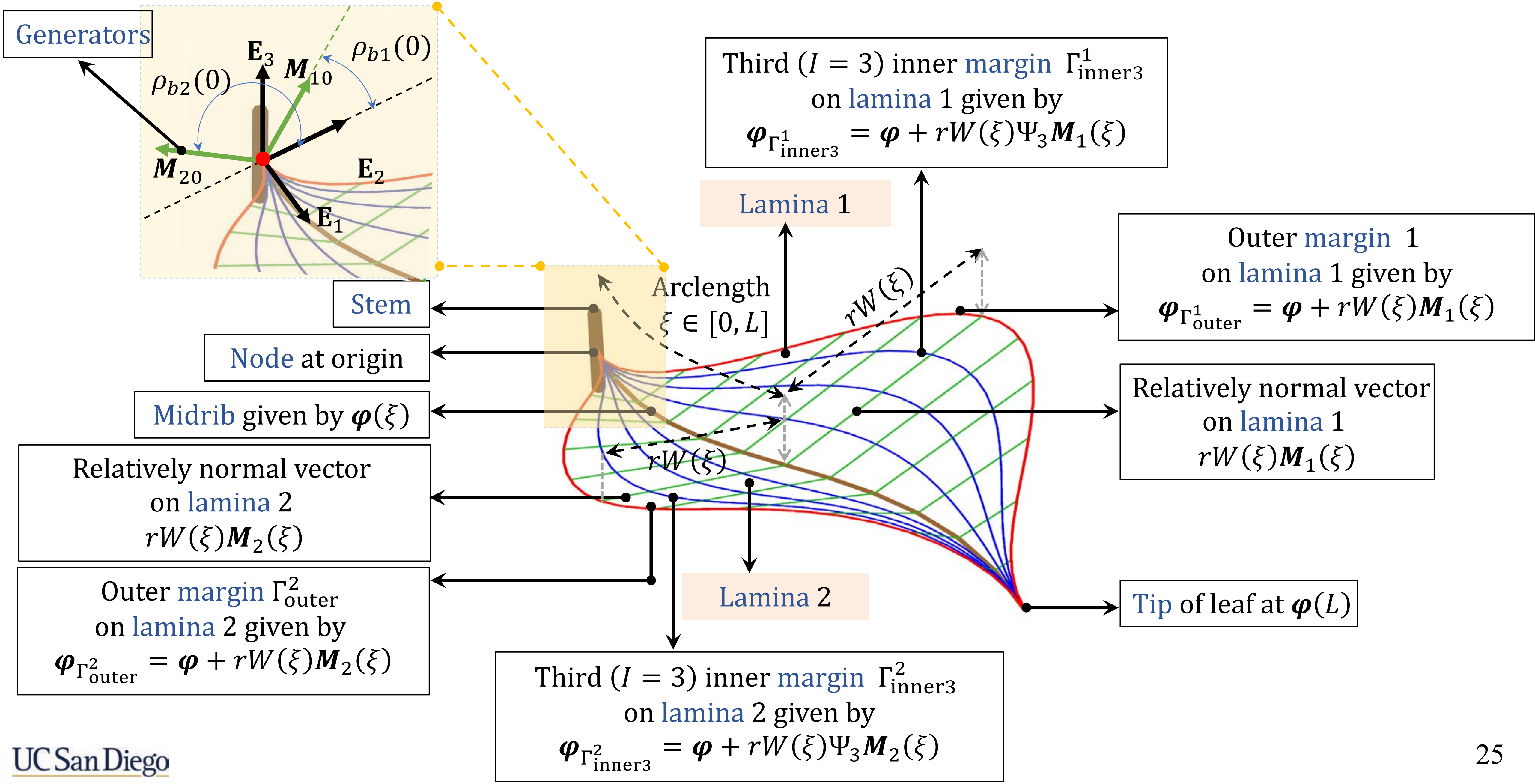
- (solid) RMS error in $\boldsymbol{\varphi}$ for no noise case
- (dashed) RMS error in $\boldsymbol{\varphi}$ for $[-5,5]$ microstrain uniform noise
- (dotted) RMS error in $\boldsymbol{\varphi}$ for $[-50,50]$ microstrain uniform noise

- The error reduces with increase of the sensor count.
- The error depends on the noise level and the complexity of deformation.
- Error propagates from the region of high certainty (proximal end) to region of uncertainty (farther end here)

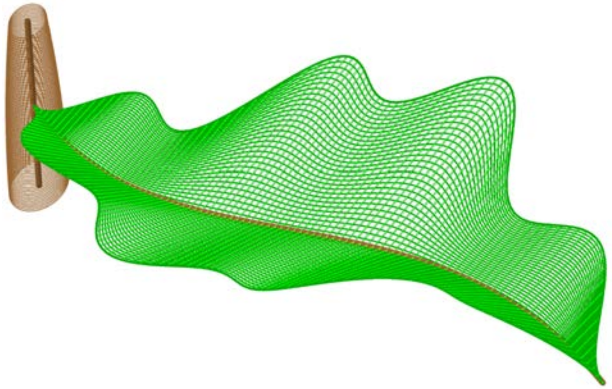
PART 2b
Computer graphics

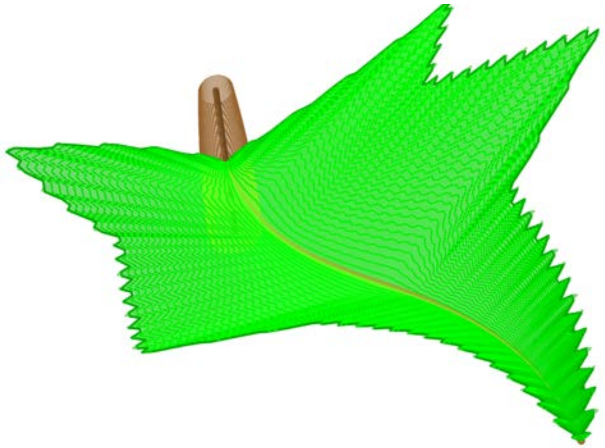


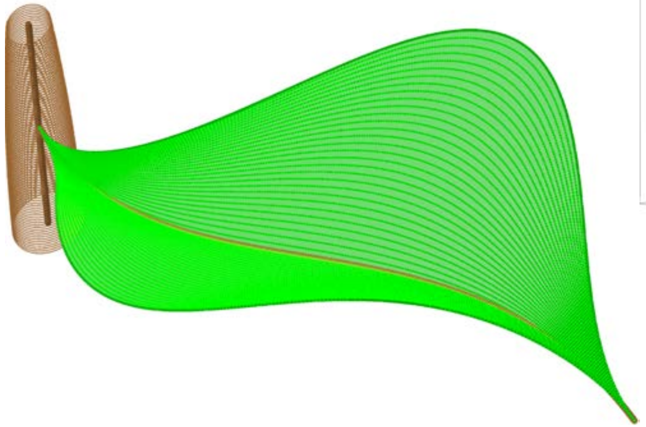
Computer graphics: Leaf and plant

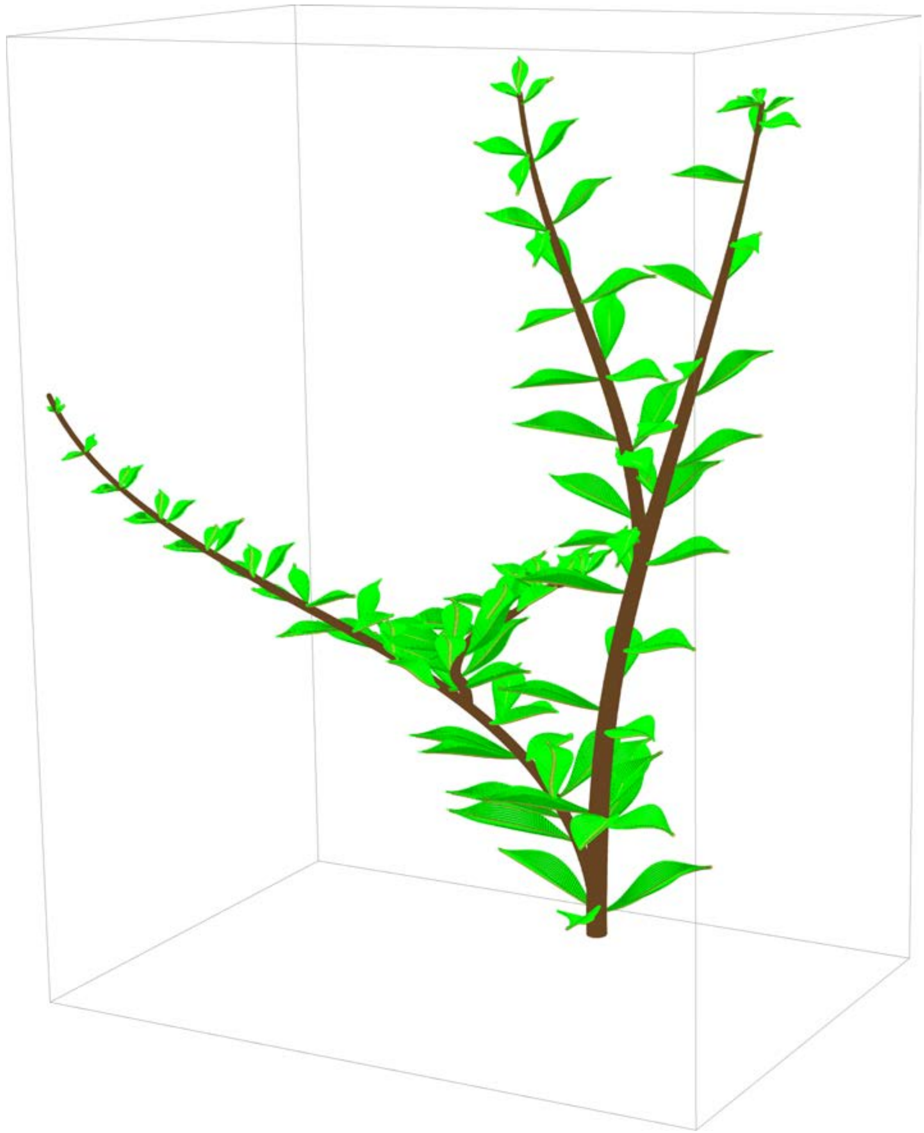


Computer graphics: Leaf and plant





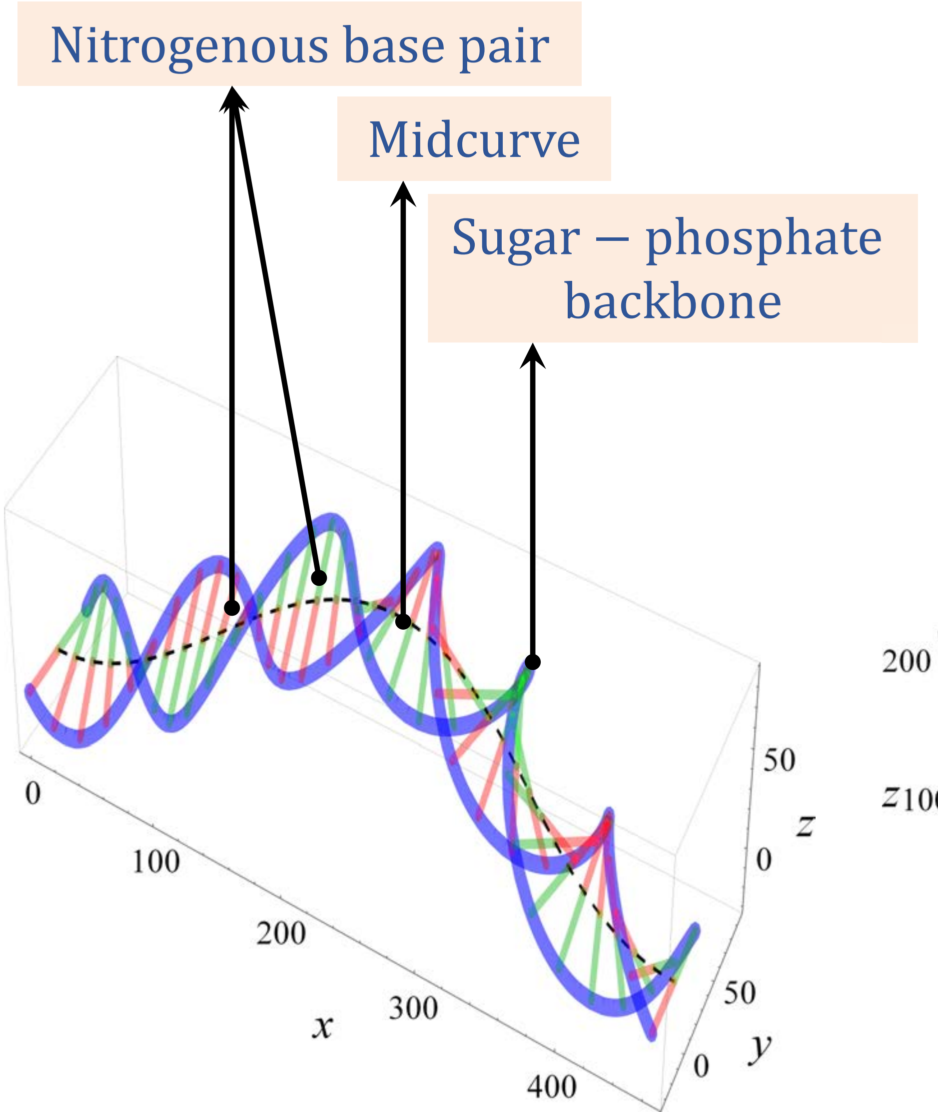


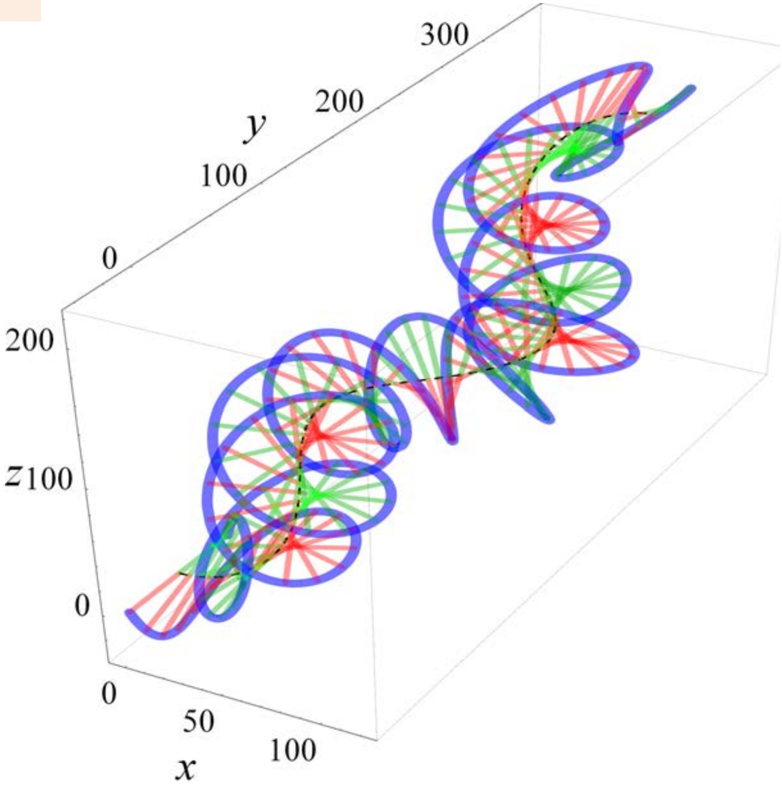


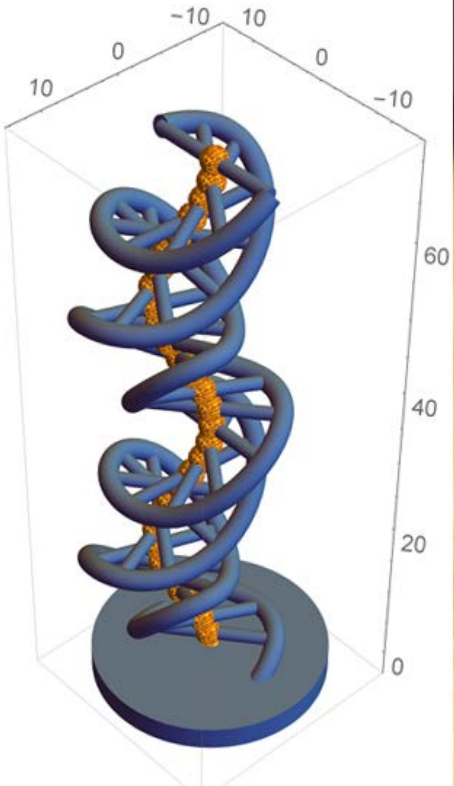
Double helix and DNA modeling

PART 2

The material frames can also be used to model double helix or DNA molecule where the nitrogenous base pairs can be represented by the cross-sectional director triads.

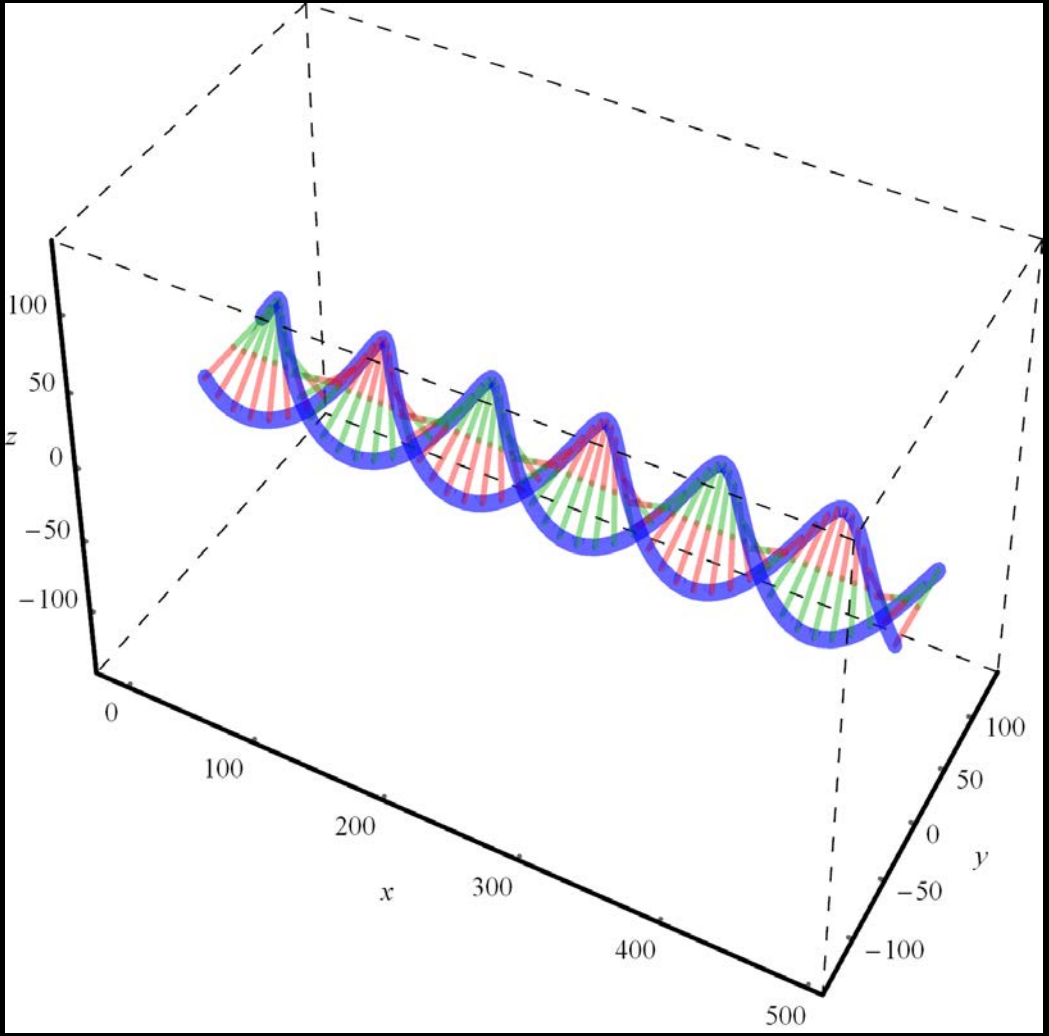


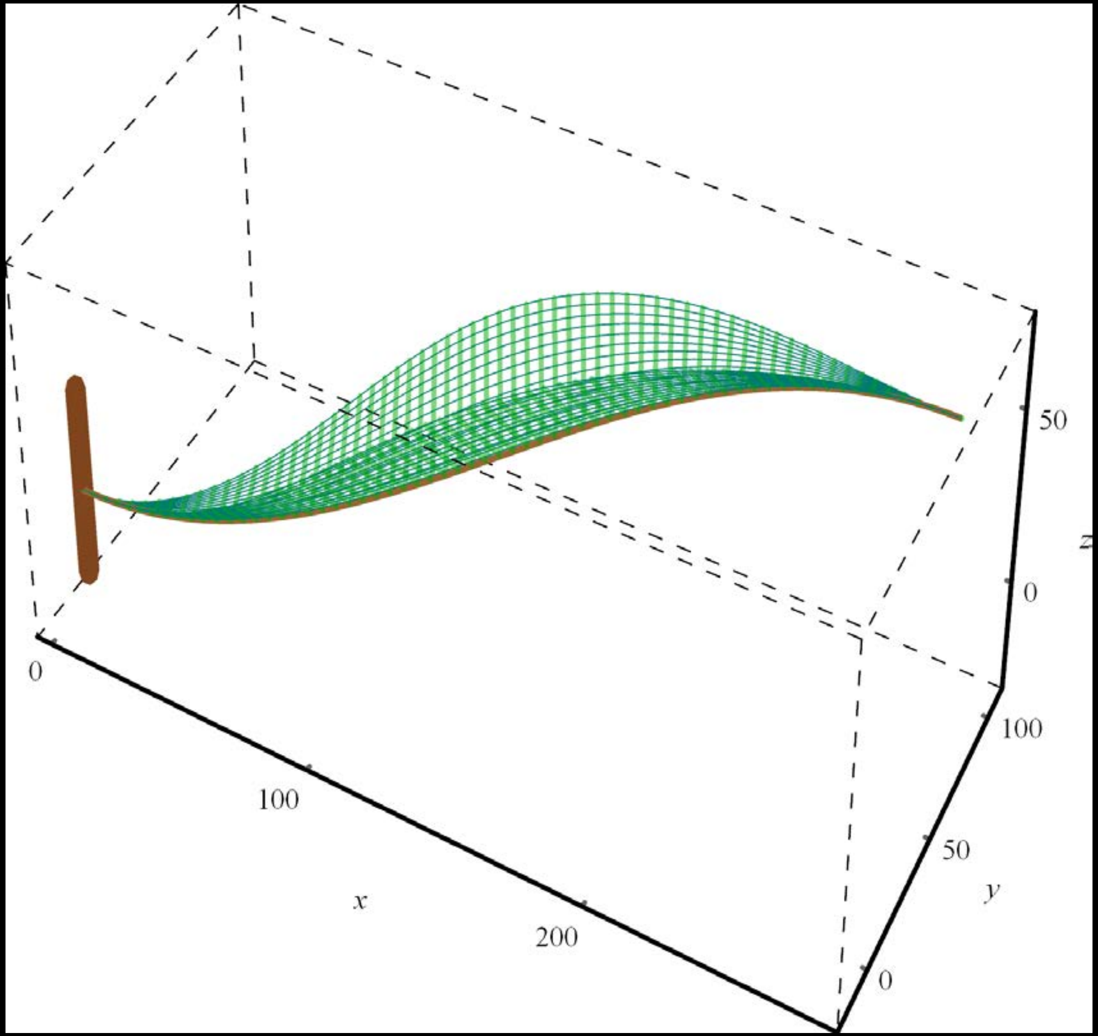




Dynamic effects in computer graphics

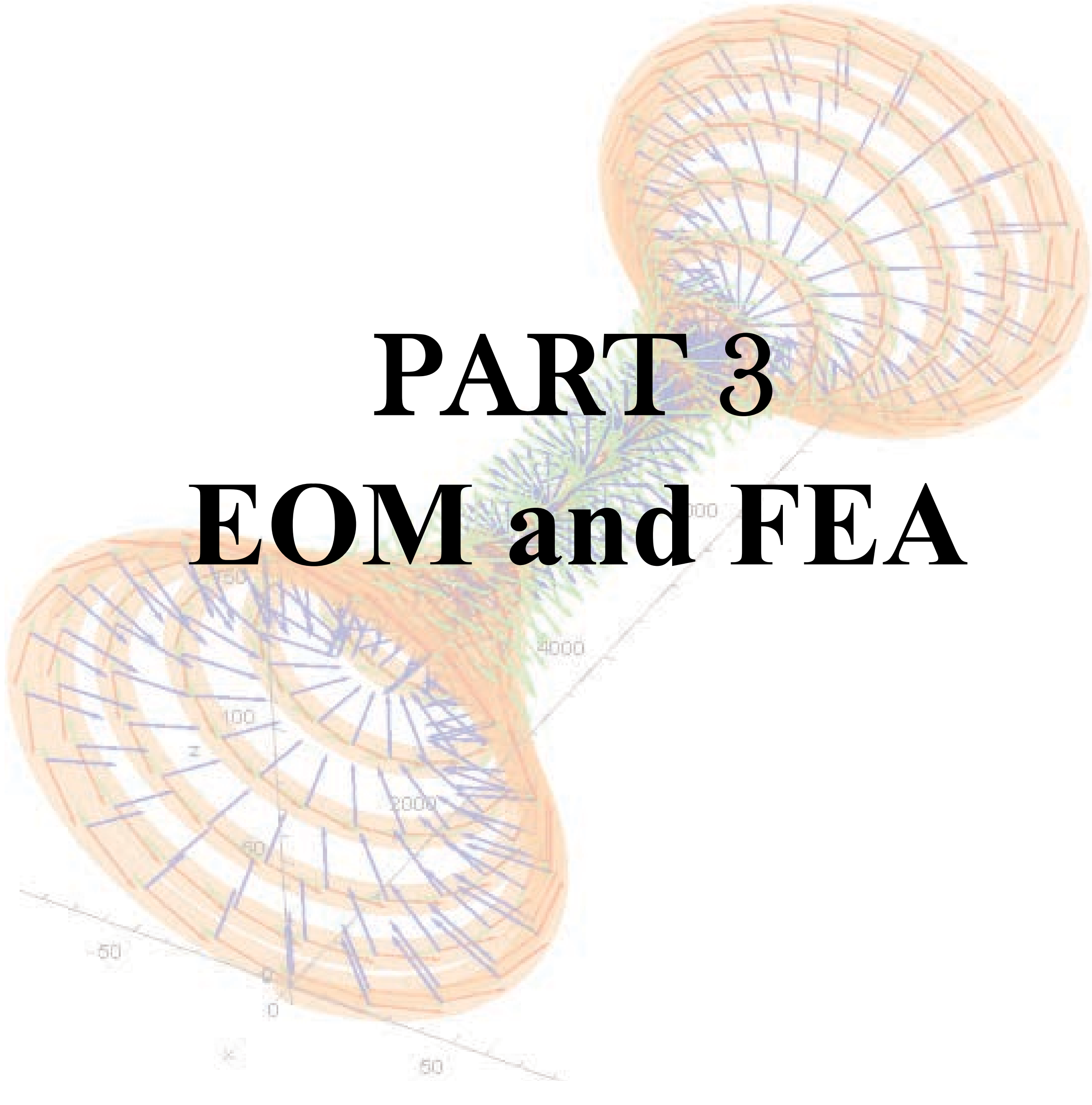
Floating DNA





Fluttering Leaf

PART 3
EOM and FEA



Assumed kinematics

Deformation map: $\mathbf{R} = \boldsymbol{\varphi} + \hat{\xi}_2 \mathbf{d}_2 + \hat{\xi}_3 \mathbf{d}_3 + \mathrm{W}(\xi_1, \xi_2, \xi_3)\mathbf{d}_1$;

- $\boldsymbol{\varphi}$ → Deformed Mid-curve
- $\hat{\xi}_2 \mathbf{d}_2 + \hat{\xi}_3 \mathbf{d}_3$ → Cross − sectional deformation due to Poisson's transformation
- $\mathrm{W}(\xi_1, \xi_2, \xi_3)\mathbf{d}_1$ → Cross − sectional deformation due to warping

Warping:

$$\mathrm{W}(\xi_1, \xi_2, \xi_3) = \mathrm{p}(\xi_1).\Psi_1(\xi_2, \xi_3) + \partial_{\xi_1}\bar{\kappa}_2(\xi_1).\Psi_2(\xi_2, \xi_3) + \partial_{\xi_1}\bar{\kappa}_3(\xi_1).\Psi_3(\xi_2, \xi_3)$$

- $\mathrm{p}(\xi_1).\Psi_1(\xi_2, \xi_3)$ → Torsional warping
- $\partial_{\xi_1}\bar{\kappa}_2(\xi_1).\Psi_2(\xi_2, \xi_3) + \partial_{\xi_1}\bar{\kappa}_3(\xi_1).\Psi_3(\xi_2, \xi_3)$ → Warping contributed by bending induced shear

Poisson′s transformation: $\hat{\xi}_i = (1 - \nu\varepsilon_l)\xi_i$; for $i = 2,3$.

where,

$$\varepsilon_l = \bar{\varepsilon}_1 + \xi_3\bar{\kappa}_2 - \xi_2\bar{\kappa}_3 + W_{,\xi_1}$$

- $\bar{\varepsilon}_1$ → Average mid − curve axial strain
- $\xi_3\bar{\kappa}_2 - \xi_2\bar{\kappa}_3$ → Axial strain due to bending across the cross-section
- $W_{,\xi_1}$ → Axial strain due to differential warping across the cross-section

Coupled Poisson′s and warping effect

Configuration and tangent space

For the assumed kinematics, the primary unknowns for the beam are: the midcurve position vector $\boldsymbol{\varphi}(\xi_1) \in \mathbb{R}^3$; the director triad field $\{\boldsymbol{d}_\mathrm{i}\} \equiv \boldsymbol{Q}(\xi_1) \in SO(3)$, and the warping amplitude field $\mathrm{p}(\xi_1) \in \mathbb{R}$. Therefore, the configuration space is:

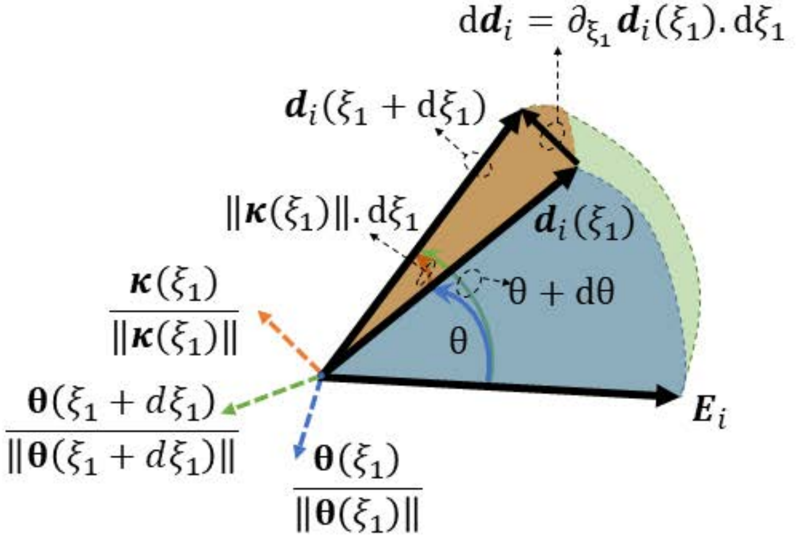
$$\mathbb{C} \coloneqq \{\boldsymbol{\Phi} = \big(\boldsymbol{\varphi}(\xi_1), \boldsymbol{Q}(\xi_1), \mathrm{p}(\xi_1)\big) \colon [0, \mathrm{L}] \rightarrow \mathbb{R}^3 \times SO(3) \times \mathbb{R}\}$$

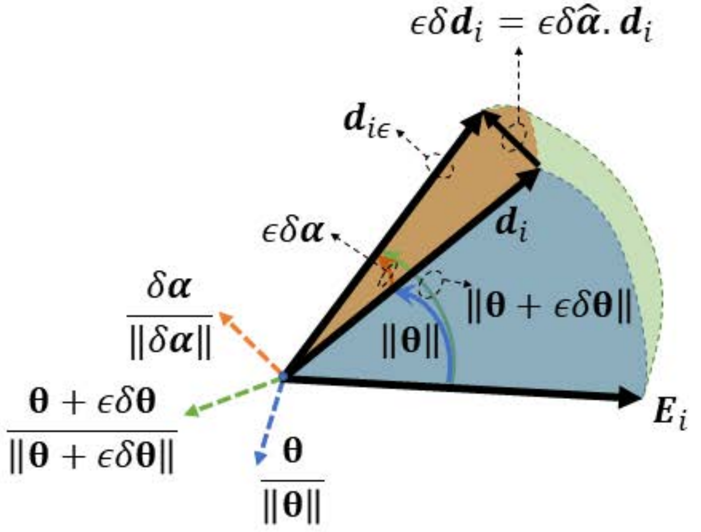
For any $\boldsymbol{\Phi} \in \mathbb{C}$, we define the tangent space $T_{\boldsymbol{\Phi}}\mathbb{C}$ as,

$$T_{\boldsymbol{\Phi}}\mathbb{C} \coloneqq \{\widetilde{\boldsymbol{\Phi}} = \big(\partial_{\xi_1}\boldsymbol{\varphi}, \partial_{\xi_1}\boldsymbol{Q}, \partial_{\xi_1}\mathrm{p}\big) \colon [0, \mathrm{L}] \rightarrow \mathbb{R}^3 \times T_{\boldsymbol{Q}}SO(3) \times \mathbb{R}\}$$

Derivative and variation of rotation tensor:

$$\widehat{\boldsymbol{\kappa}} = \partial_{\xi_1}\boldsymbol{Q}.\boldsymbol{Q}^T$$
$$\delta\widehat{\boldsymbol{\alpha}} = \delta\boldsymbol{Q}.\boldsymbol{Q}^T$$





Governing equation of motion

Linear momentum conservation: $\partial_{\xi_1}\boldsymbol{n} + \boldsymbol{N}_{\boldsymbol{\varphi}} = \boldsymbol{\mathfrak{F}}_{\boldsymbol{\varphi}}$

Angular momentum conservation: $\partial_{\xi_1}\boldsymbol{m} + \partial_{\xi_1}\boldsymbol{\varphi}\times\boldsymbol{n} + \boldsymbol{N}_{\boldsymbol{\alpha}} = \boldsymbol{\mathfrak{F}}_{\boldsymbol{\varphi}}$

EOM due to warping: $\partial_{\xi_1}M_{\mathrm{f}} - N_{\mathrm{f}} + N_p = \mathfrak{F}_p$

Seven equations of motion

The reduced internal forces or moments are given by:

$$\boldsymbol{n} = \int_{\Omega_0} \mathbf{S}_1 \mathrm{d}\Omega_0 \,;\; \boldsymbol{m} = \int_{\Omega_0} \boldsymbol{r}\times\mathbf{S}_1 \mathrm{d}\Omega_0;\;\; M_{\mathrm{f}} = \int_{\Omega_0} \Psi_1\, \boldsymbol{d}_1.\mathbf{S}_1 \mathrm{d}\Omega_0\,.$$

The reduced inertial force vector can be written as:

$$\boldsymbol{\mathfrak{F}}_{\boldsymbol{\varphi}} = \int_{\Omega_0} \rho_0 \partial_t^2 \boldsymbol{R}\, \mathrm{d}\Omega_0 = \int_{\Omega_0} \rho_0 (\partial_t^2 \boldsymbol{\varphi} + \tilde{\partial}_t^2 \boldsymbol{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) + 2\boldsymbol{\omega}\times\tilde{\partial}_t \boldsymbol{r} + \partial_t \boldsymbol{\omega}\times\boldsymbol{r})\mathrm{d}\Omega_0$$

- $\partial_t^2 \boldsymbol{\varphi}$: Force corresponding to translational acceleration
- $\tilde{\partial}_t^2 \boldsymbol{r}$: Non − inertial force Due to cross-sectional deformation
- $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$: Centrifugral force
- $2\boldsymbol{\omega}\times\tilde{\partial}_t \boldsymbol{r}$: Coriolis force
- $\partial_t \boldsymbol{\omega}\times\boldsymbol{r}$: Euler force

Linearized form of equilibrium equation

$$G(\boldsymbol{\Phi}, \delta\boldsymbol{\Phi}) = (\delta U_{strain} - \delta W_{external}) + \delta W_{inertial} = 0$$

Linearizing at

$$\boldsymbol{\Phi}^{\#} = [\boldsymbol{\varphi}^{\#}, \mathbf{Q}^{\#}, p^{\#}]^T \in \mathbb{R}^3 \times SO(3) \times \mathbb{R}$$

Directed towards

$$\Delta\boldsymbol{\Phi} \equiv [\Delta\boldsymbol{\varphi}, \Delta\boldsymbol{\alpha}, \Delta p]^T$$

$$\mathfrak{D}G\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}] = \left(\mathfrak{D}\delta U_{strain}\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}] - \mathfrak{D}\delta W_{external}\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}]\right) + \mathfrak{D}\delta W_{inertial}\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}]$$

$$|||$$

$$\mathfrak{D}G\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}]$$

$$= \int_0^L \delta\boldsymbol{\Phi}^T [K_m + K_g - K_{nc} + K_c]_{\boldsymbol{\Phi}^{\#}} \Delta\boldsymbol{\Phi} \; d\xi_1 + \int_0^L \delta\boldsymbol{\Phi}^T \left\{ [M]_{\boldsymbol{\Phi}^{\#}} \begin{bmatrix} \Delta\ddot{\boldsymbol{\varphi}} \\ \Delta\bar{\boldsymbol{\psi}} \\ \Delta\ddot{p} \end{bmatrix} + [G_y]_{\boldsymbol{\Phi}^{\#}} \begin{bmatrix} \Delta\dot{\boldsymbol{\varphi}} \\ \Delta\bar{\boldsymbol{\omega}} \\ \Delta\dot{p} \end{bmatrix} \right\} d\xi_1$$

Increment in angular acceleration $\Delta\bar{\boldsymbol{\psi}}$ and angular velocity $\Delta\bar{\boldsymbol{\omega}}$ are in material coordinates. Thus time stepping schemes are obtained in Material form. The mass matrix is configuration dependent and in addition we have gyroscopic and centrifugal matrix.

Discretization of linearized EOM and FE formulation

$$\mathfrak{D}G\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}] = \sum_{e=1}^{N_{ele}} \mathfrak{D}G^{\mathrm{e}}\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}^{\mathrm{e}}]$$

Need minimum 4^{th} order polynomial

Galerkins approximation

$$\boldsymbol{\varphi}^{e} \approx \boldsymbol{\varphi}^{eh} = \sum_{\mathrm{I}=1}^{\mathrm{N_{nodes}}} \boldsymbol{N_I \varphi_I^e}$$

$$\Delta\boldsymbol{\alpha}^{e} \approx \Delta\boldsymbol{\alpha}^{eh} = \sum_{\mathrm{I}=1}^{\mathrm{N_{nodes}}} \boldsymbol{N_I \Delta\alpha_I^e}$$

Time updating on Non-Linear Lie Group $SO(3)$

$$\Delta\bar{\boldsymbol{\psi}} = \frac{1}{\Delta t^2.\beta}\mathbf{Q}^{\mathrm{T}}\mathbf{T}_{\alpha\theta}(\boldsymbol{\theta})\Delta\boldsymbol{\alpha}$$

$$\Delta\bar{\boldsymbol{\omega}} = \frac{\gamma}{\Delta t.\beta}\mathbf{Q}^{\mathrm{T}}\mathbf{T}_{\alpha\theta}(\boldsymbol{\theta})\Delta\boldsymbol{\alpha}$$

$$\left[\mathfrak{D}G\Big|_{\boldsymbol{\Phi}^{\#}}[\Delta\boldsymbol{\Phi}]\right]^h = \sum_{e=1}^{N_{ele}} \delta\boldsymbol{\Phi}_I^e.\left[K_{\mathrm{m}_{IJ}}^e + K_{\mathrm{g}\,IJ}^e - K_{\mathrm{nc}_{IJ}}^e + K_{\mathrm{c}_{IJ}}^e + M_{IJ}^e + G_{y_{IJ}}^e\right].\Delta\boldsymbol{\Phi}_J^e$$

$K_{\mathrm{dyn_{IJ}}}^e$

$$K_{\mathrm{dyn}} = \mathrm{Assemble}[K_{\mathrm{dyn_{IJ}}}^e]$$

$$K_{\mathrm{dyn}}.\Delta\boldsymbol{\Phi}^{\mathrm{g}} = 0$$

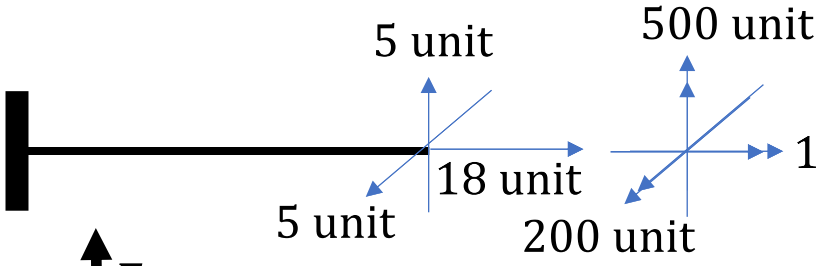
(at equilibrium)

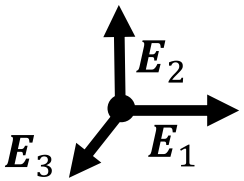
Note:

- Use uniform reduced integration for $K_{\mathrm{m}_{IJ}}^e, K_{\mathrm{g}\,IJ}^e$.
- Use full integration for $M_{IJ}^e, G_{y_{IJ}}^e, K_{\mathrm{c}_{IJ}}^e$ and $K_{\mathrm{nc}_{IJ}}^e$.

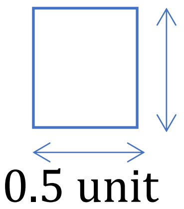
Numerical example 1: Static case

PART 3

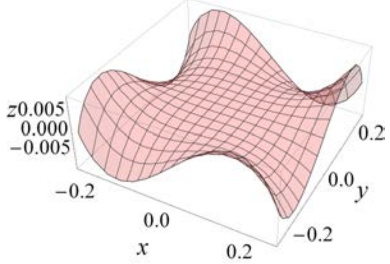




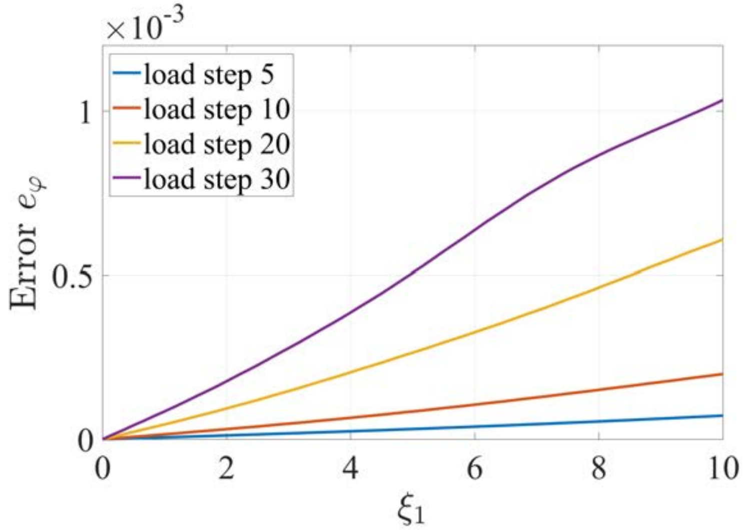
Conservative end load

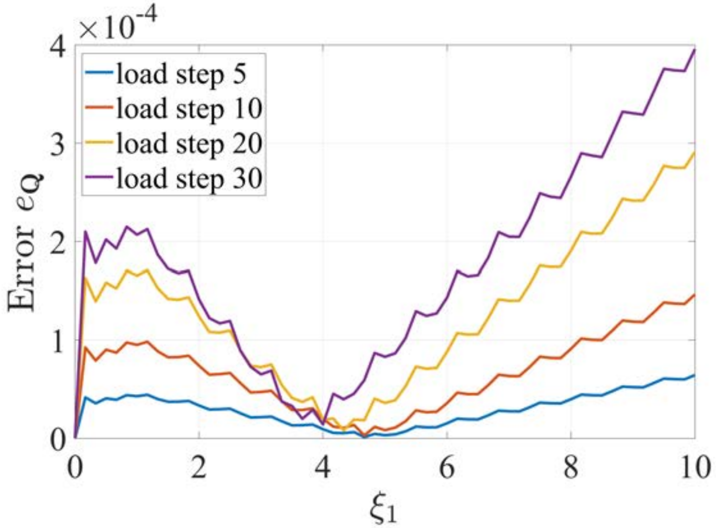


Square section

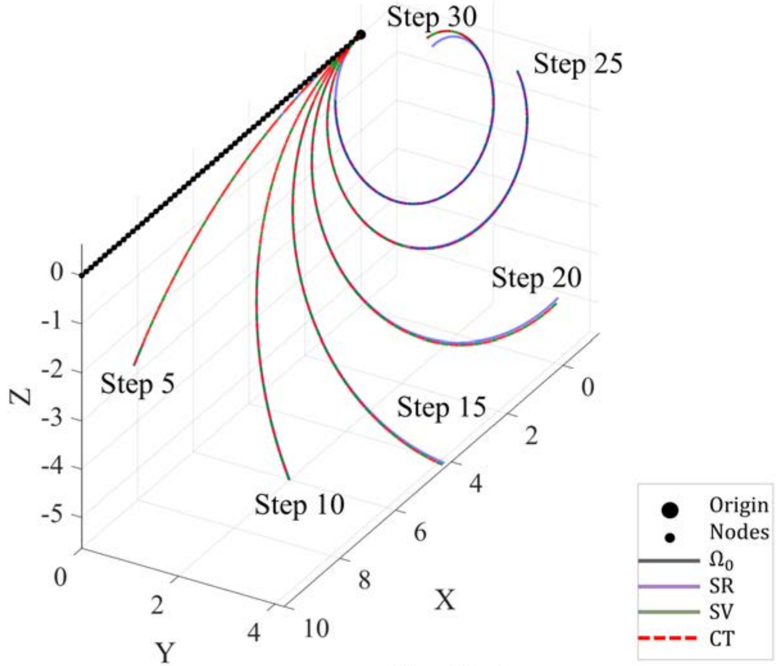


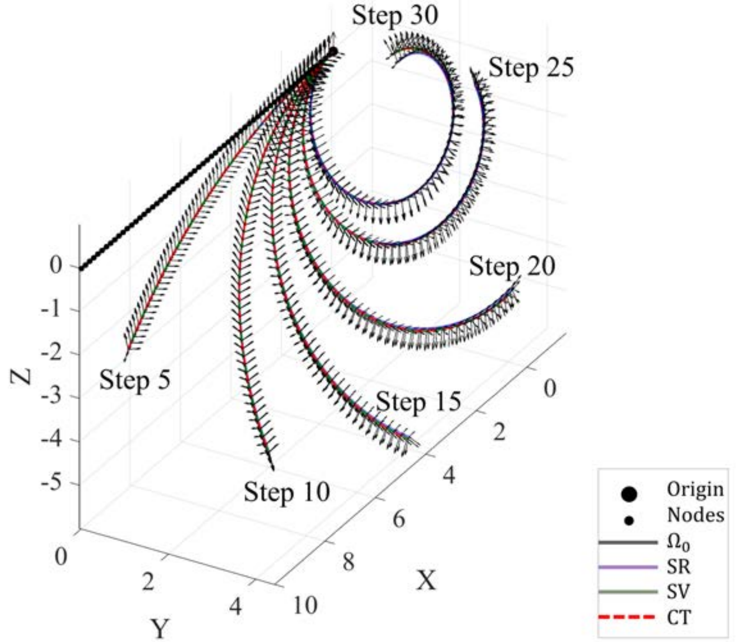
$\Psi_1(\xi_2, \xi_3)$





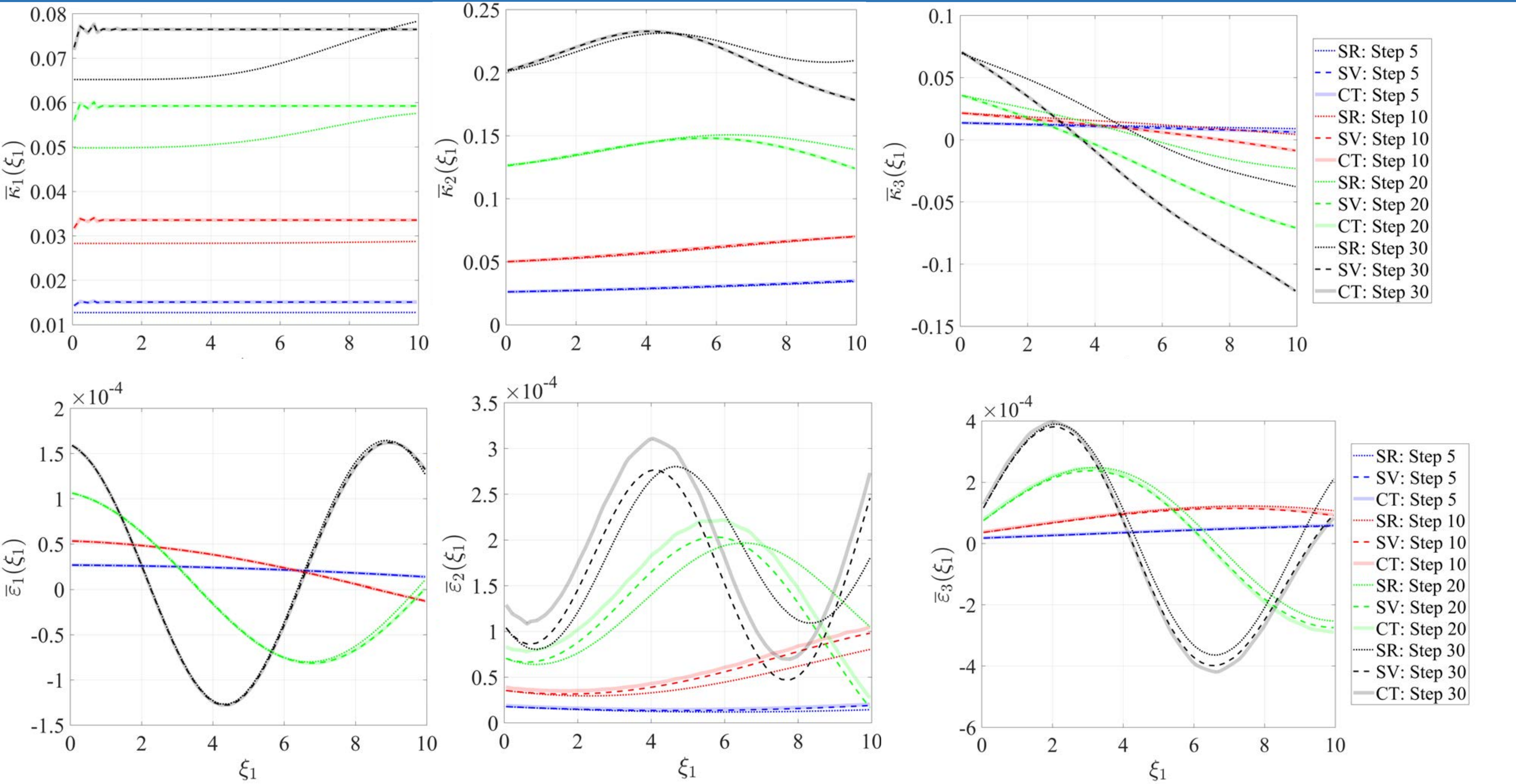
Error in Simo Vu − Quoc beam relative to the current beam model





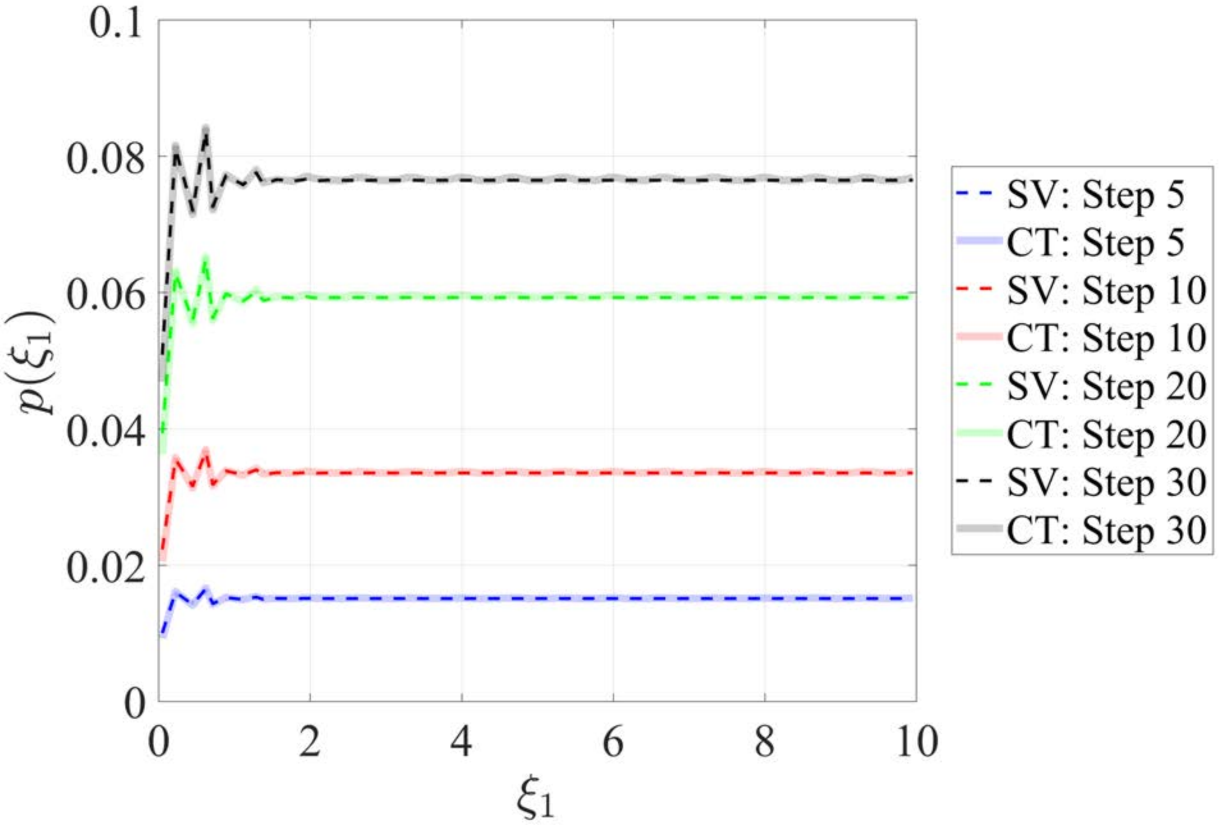
UC San Diego

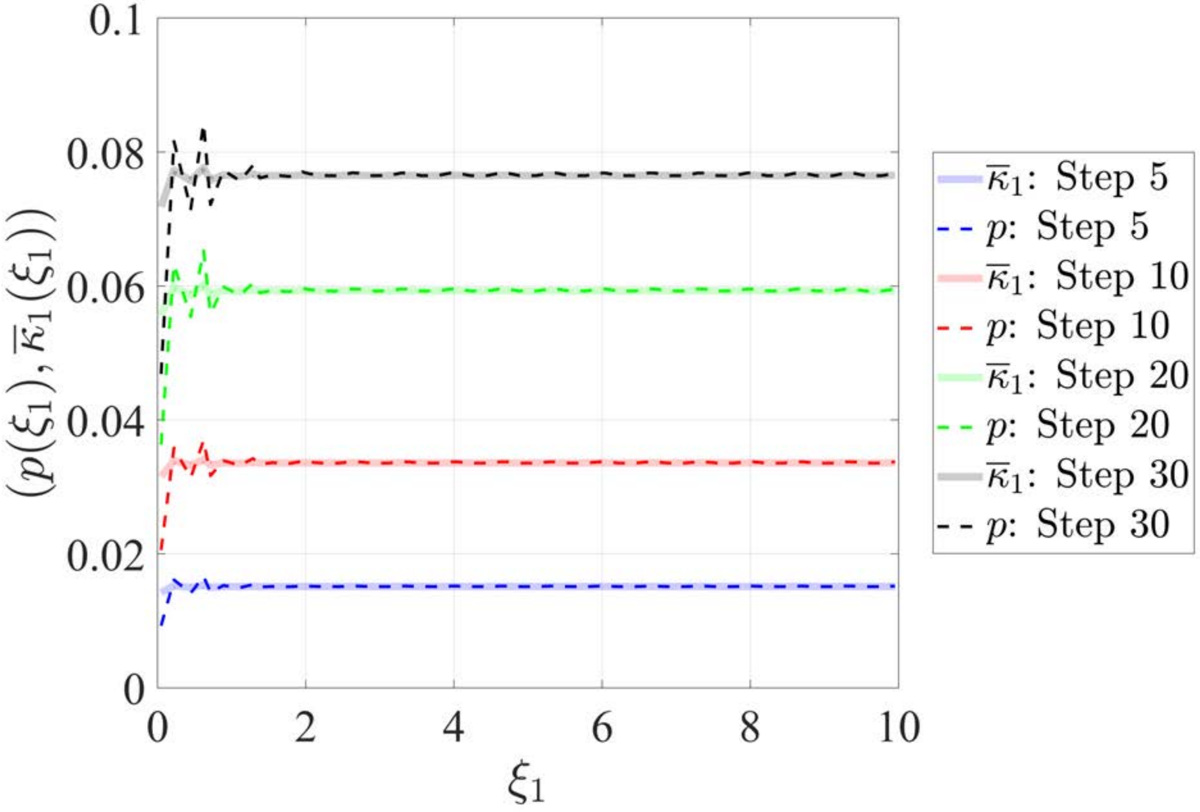
Numerical example 1: Curvatures and axial strains



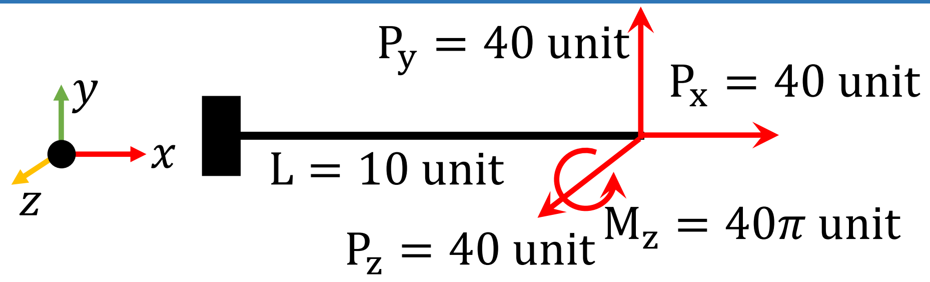
Numerical example 1: Curvatures and axial strains

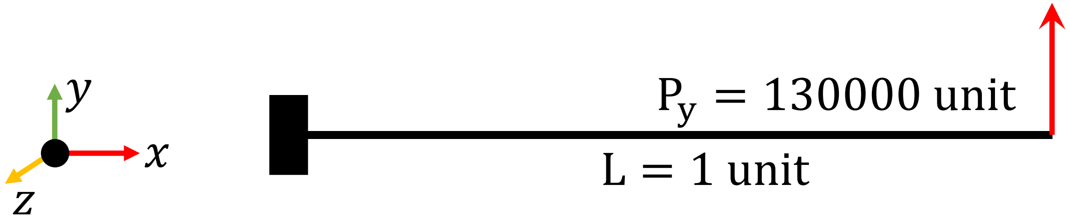
- Fixed end has non-zero torsional curvature but zero warping amplitude.
- The warping amplitude converges with the torsional curvature as we move away from the boundary.

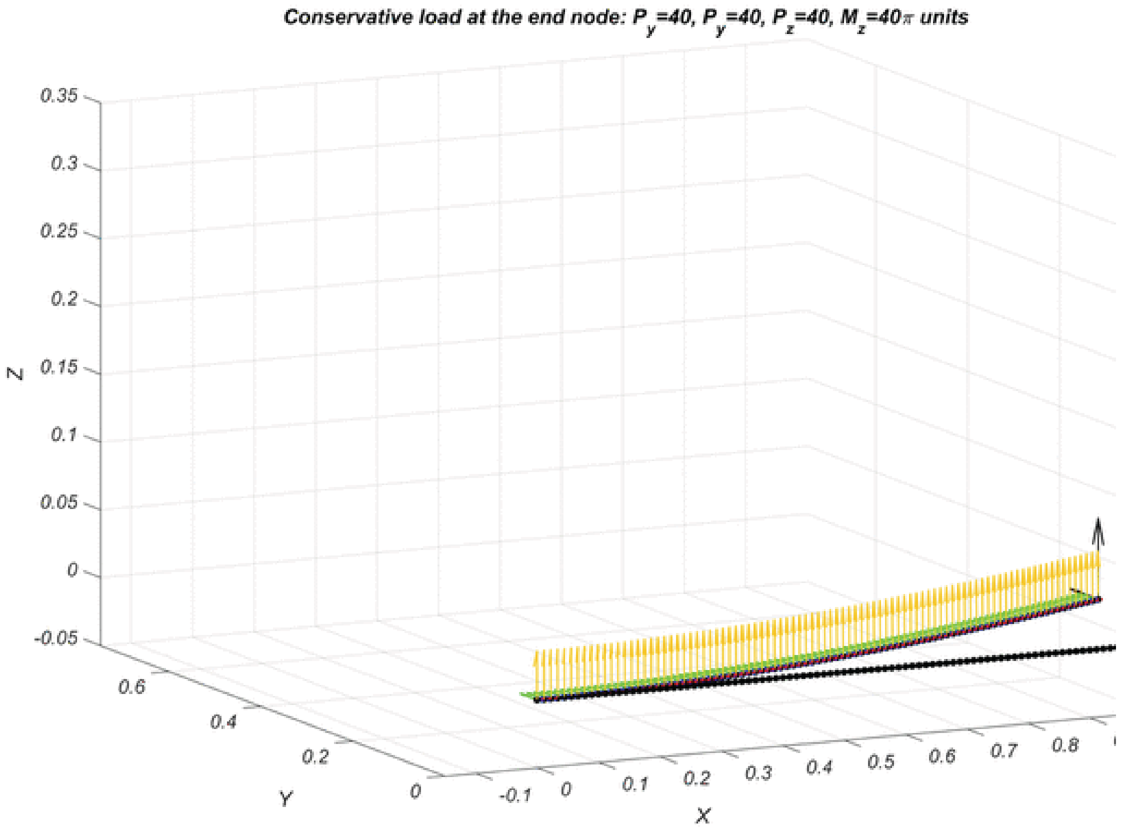


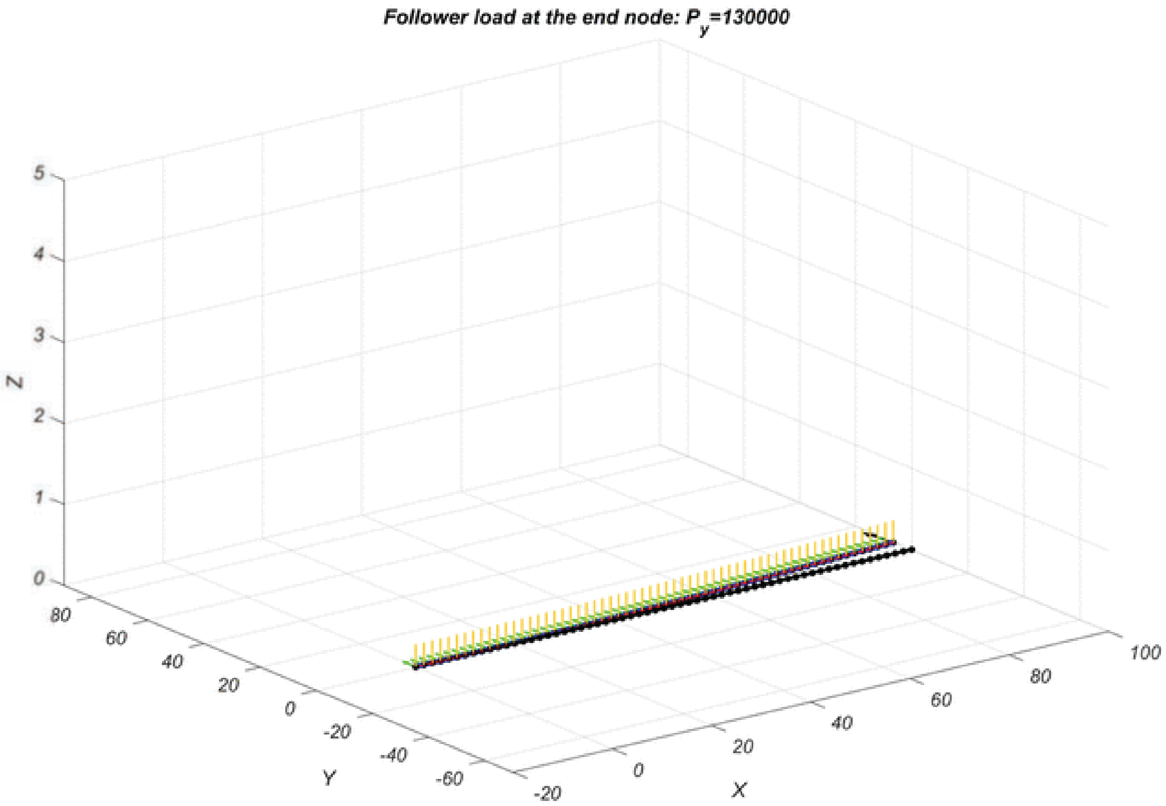


Some other examples

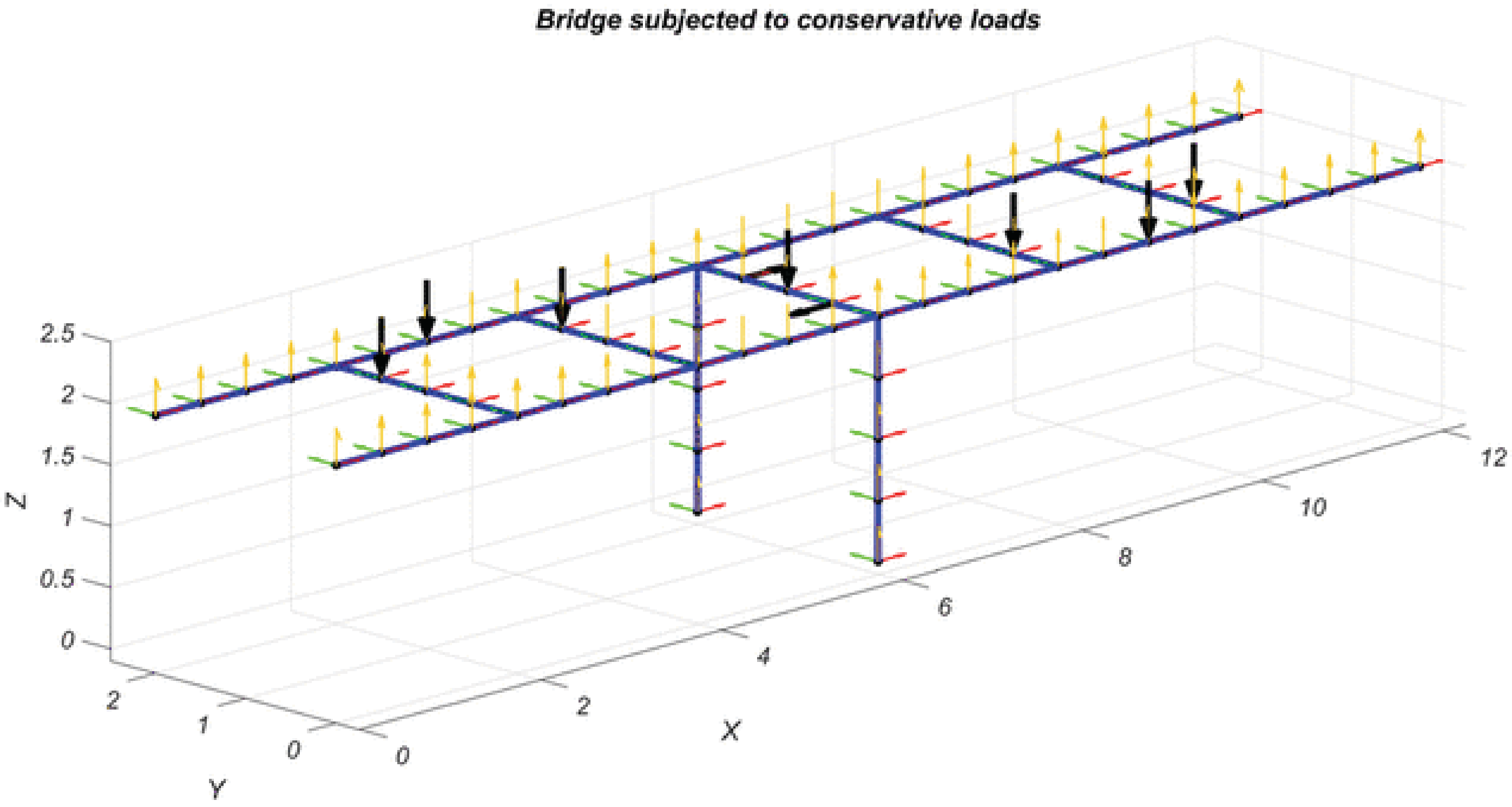


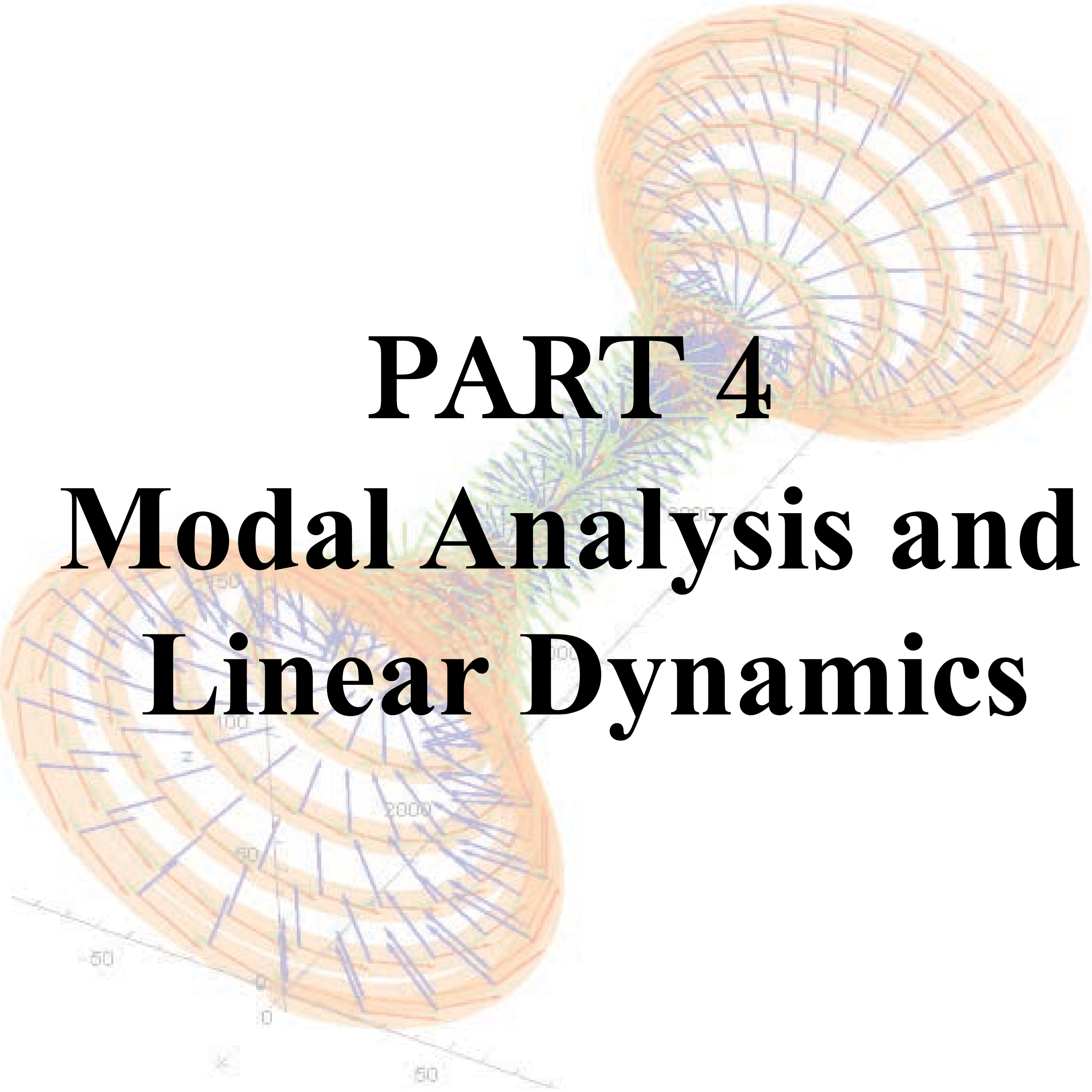






Example: Frame





PART 4
Modal Analysis and Linear Dynamics

Deformation map for linear case

We find that the contribution to Poisson's effect due to bending and warping deformations are negligible as compared to axial strains for linear case. Therefore, we assume $\varepsilon_l = \partial_{\xi_1} u$. The simplified deformation map $\boldsymbol{R} = \boldsymbol{\varphi} + \boldsymbol{r} = R_i \boldsymbol{E}_i$, for linear case is given by:

$$R_1 = u - \left(1 - \nu.\partial_{\xi_1} u\right)\xi_2\theta_3 + \left(1 - \nu.\partial_{\xi_1} u\right)\xi_3\theta_2 + p\Psi_1 + \partial_{\xi_1}^2\theta_2.\Psi_2 + \partial_{\xi_1}^2\theta_3.\Psi_3$$

$$R_2 = v - \left(1 - \nu.\partial_{\xi_1} u\right)\xi_3\theta_1 - \nu.\partial_{\xi_1} u\xi_2$$

$$R_3 = w - \left(1 - \nu.\partial_{\xi_1} u\right)\xi_2\theta_1 - \nu.\partial_{\xi_1} u\xi_3$$

Here: $\boldsymbol{\varphi} = (u, v, w)$; and $\kappa_i = \bar{\kappa}_i = \partial_{\xi_1}\theta_i$.

We also realize that the effect of Poisson's deformation on to bending and torsions are negligible for linear case.

$$R_1 = u - \xi_2\theta_3 + \xi_3\theta_2 + p\Psi_1 + \partial_{\xi_1}^2\theta_2.\Psi_2 + \partial_{\xi_1}^2\theta_3.\Psi_3$$

$$R_2 = v - \xi_3\theta_1 - \nu\partial_{\xi_1} u.\xi_2$$

$$R_3 = w - \xi_2\theta_1 - \nu\partial_{\xi_1} u.\xi_3$$

Strain energy, kinetic energy and external work done

$$V = \frac{1}{2}\int \sigma_{ij}\varepsilon_{ij} d\Omega$$

Strain energy due Poisson's non-uniform shear warping

$$= \int_0^L \left(\frac{1}{2}EA.\partial_{\xi_1}u^2 + \frac{1}{2}GA.\left(\partial_{\xi_1}w + \theta_2\right)^2 + \frac{1}{2}GA.\left(\partial_{\xi_1}v - \theta_3\right)^2 + \frac{1}{2}EI_{11}\left(\partial_{\xi_1}\theta_1\right)^2 + \frac{1}{2}EI_{22}\left(\partial_{\xi_1}\theta_2\right)^2 + \frac{1}{2}EI_{33}\left(\partial_{\xi_1}\theta_3\right)^2 \right.$$

$$+ \frac{1}{2}GI_k p^2 + \frac{1}{2}E\Xi\left(\partial_{\xi_1}p\right)^2 - GI_k p.\partial_{\xi_1}\theta_1 + \frac{1}{2}\left(\frac{GI_{11}\nu^2}{1+\nu}\right)\left(\partial_{\xi_1}^2 u\right)^2 + \frac{1}{2}C_a\left(\partial_{\xi_1}^3\theta_2\right)^2 + \frac{1}{2}C_b\left(\partial_{\xi_1}^3\theta_3\right)^2 + \frac{1}{2}C_d\left(\partial_{\xi_1}^2\theta_3\right)^2$$

$$\left. + \frac{1}{2}C_e\left(\partial_{\xi_1}^2\theta_3\right)^2 \right) d\xi_1$$

Ignore these terms because the section constants C_a, C_b, C_d and C_e are small compared to other section constants

$$T = \int_0^L \left(\frac{1}{2}A\rho(\partial_t u^2 + \partial_t v^2 + \partial_t w^2) + \frac{1}{2}\rho I_{33}(\partial_t\theta_3)^2 + \frac{1}{2}\rho I_{22}(\partial_t\theta_2)^2 + \frac{1}{2}\rho I_{11}(\partial_t\theta_1)^2 + \frac{1}{2}\nu^2 I_{11}\rho\left(\partial_t\partial_{\xi_1}u\right)^2 \right.$$

$$\left. + \frac{1}{2}\rho\Xi\partial_t p^2 \right) d\xi_1$$

K.E due to cross-sectional deformation

$$W_{\text{ext}} = \int_0^L \left(N_1 u + N_2 v + N_3 w + M_1\theta_1 + M_2\theta_2 + M_3\theta_3 + N_p p\right) d\xi_1$$

$$\delta A = \delta \int \mathfrak{L} d\xi_1 = \delta \int (T - V + W_{\text{ext}}) d\xi_1 = 0$$

EOM

$$\rho\Xi\partial_t^2 p - E\Xi\partial_{\xi_1}^2 p + GI_k(p - \partial_{\xi_1}\theta_1) = N_p; \qquad [E\Xi\partial_{\xi_1}p.\delta p]_0^L = 0$$

$$\rho I_{11}\partial_t^2\theta_1 - \partial_{\xi_1}(GI_{11}\partial_{\xi_1}\theta_1 - GI_k p) = M_1; \qquad [(GI_{11}\partial_{\xi_1}\theta_1 - GI_k p).\delta\theta_1]_0^L = 0$$

Coupled in θ_1 and p

$$\rho A\partial_t^2 w - GA.\partial_{\xi_1}(\theta_2 + \partial_{\xi_1}w) = N_3; \qquad \left[\left(GA(\theta_2 + \partial_{\xi_1}w)\right)\delta w\right]_0^L = 0$$

$$\rho I_{22}\partial_t^2\theta_2 + GA(\theta_2 + \partial_{\xi_1}w) - EI_{22}\partial_{\xi_1}^2\theta_2 = M_2; \qquad [EI_{22}\partial_{\xi_1}\theta_2.\delta\theta_2]_0^L = 0$$

Coupled in θ_2 and w

$$\rho A\partial_t^2 v + GA.\partial_{\xi_1}(\theta_3 - \partial_{\xi_1}v) = N_2; \qquad \left[\left(GA(\theta_3 - \partial_{\xi_1}v)\right)\delta v\right]_0^L = 0$$

$$\rho I_{33}\partial_t^2\theta_3 + GA(\theta_3 - \partial_{\xi_1}v) - EI_{33}\partial_{\xi_1}^2\theta_3 = M_3; \qquad [EI_{33}\partial_{\xi_1}\theta_3.\delta\theta_3]_0^L = 0$$

Coupled in θ_3 and v

$$\left(\frac{G\nu^2 I_{11}}{1+\nu}\right)\partial_{\xi_1}^4 u - (EA)\partial_{\xi_1}^2 u - (\rho I_{11}\nu^2)\partial_t^2\partial_{\xi_1}^2 u + (\rho A)\partial_t^2 u = N_1;$$

$$\left[\left(-EA\partial_{\xi_1}u - (\rho I_{11}\nu^2)\partial_t^2\partial_{\xi_1}u + \left(\frac{G\nu^2 I_{11}}{1+\nu}\right)\partial_{\xi_1}^3 u\right)\delta u\right]_0^L = 0$$

$$\left[\left(\left(\frac{G\nu^2 I_{11}}{1+\nu}\right)\partial_{\xi_1}^2 u\right).\delta\partial_{\xi_1}u\right]_0^L = 0$$

Uncoupled EOM for u

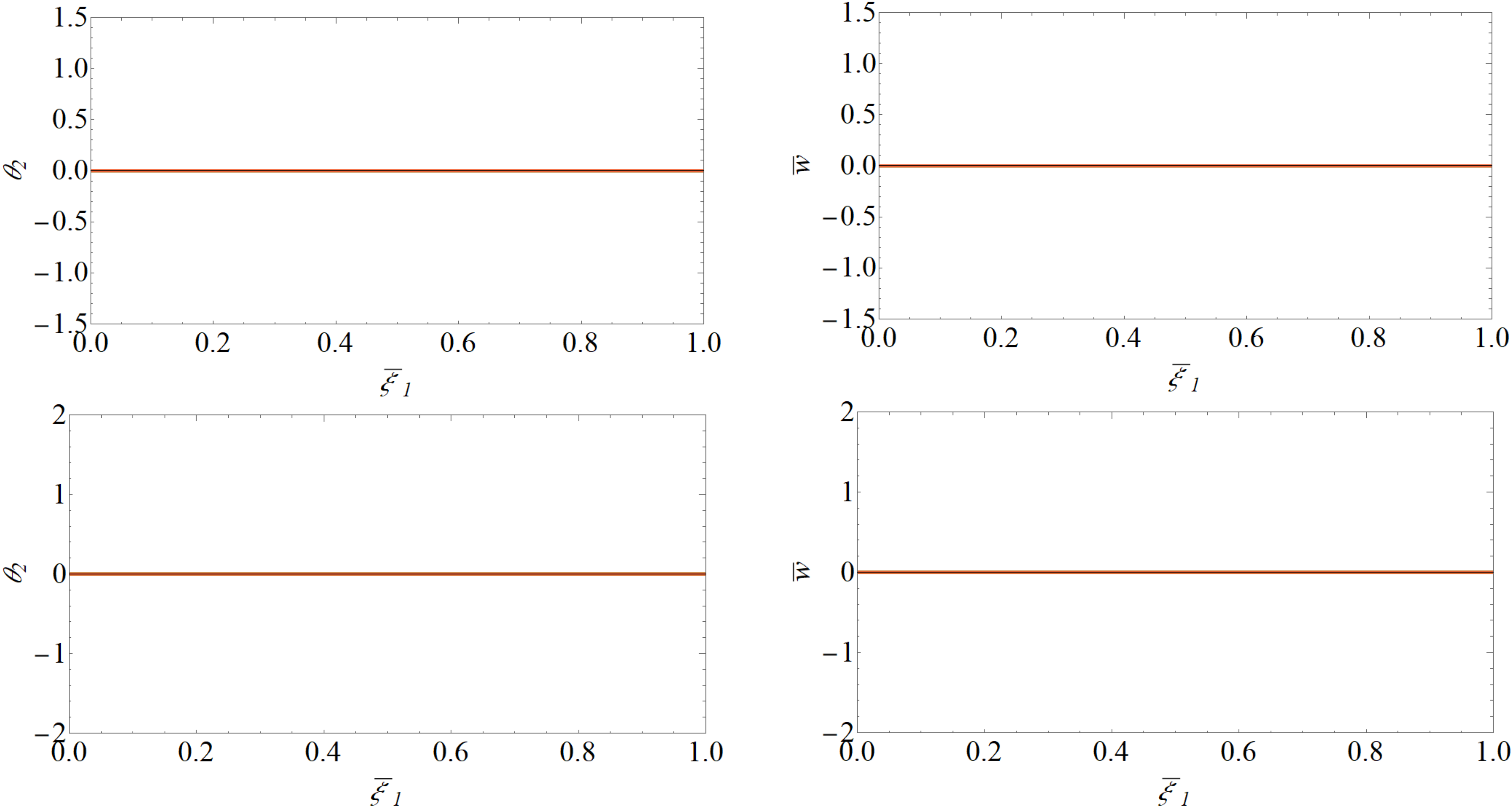
These EOM can be non dimensionalized and uncoupled to solve for the mode shapes.

UC San Diego

Example of mode-shapes θ_2 and w

Pinned-Pinned

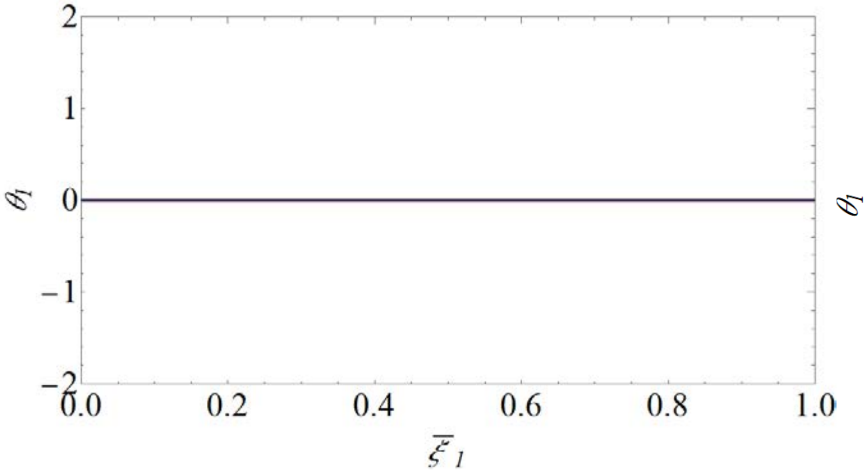
Fixed-Free

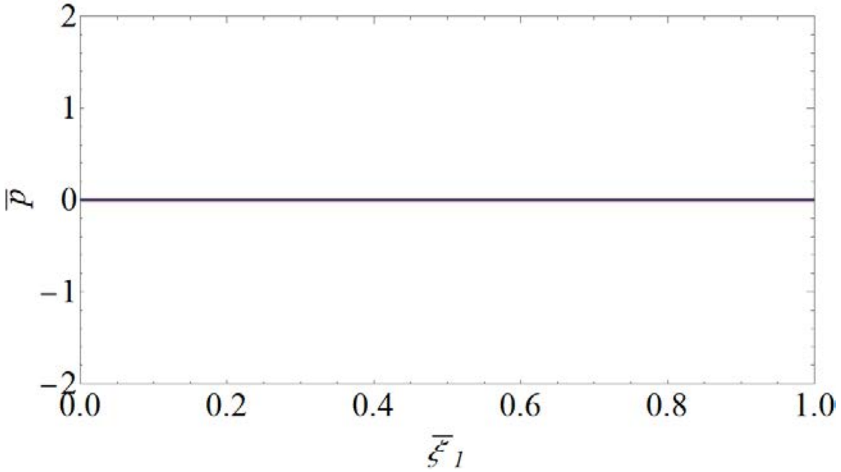


Example of mode-shapes θ_1 and p

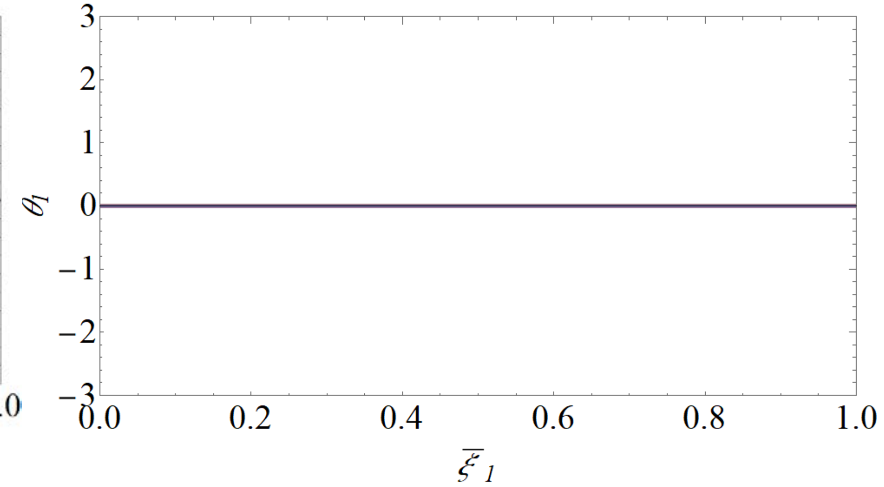
Fixed-Fixed	Fixed-Free	Free-Free

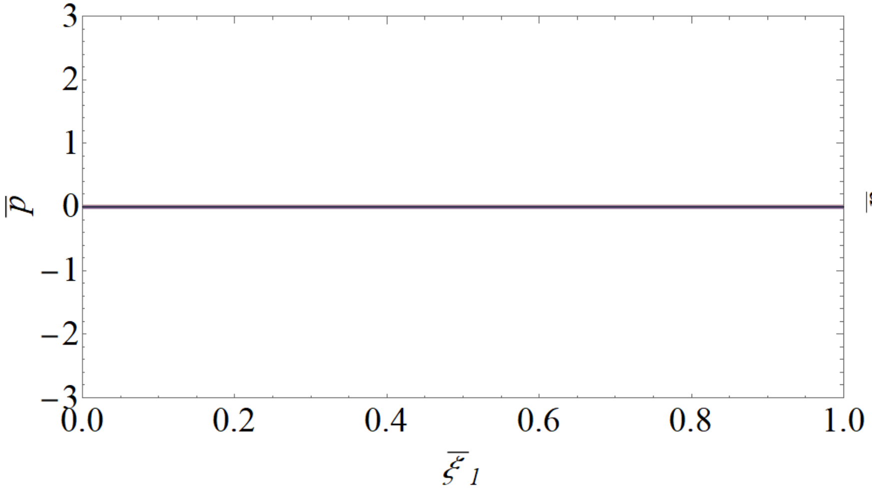
Fixed-Fixed



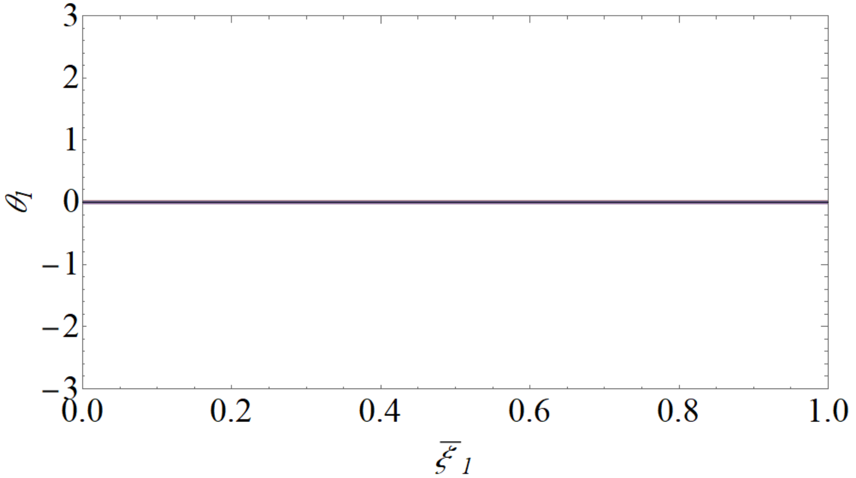


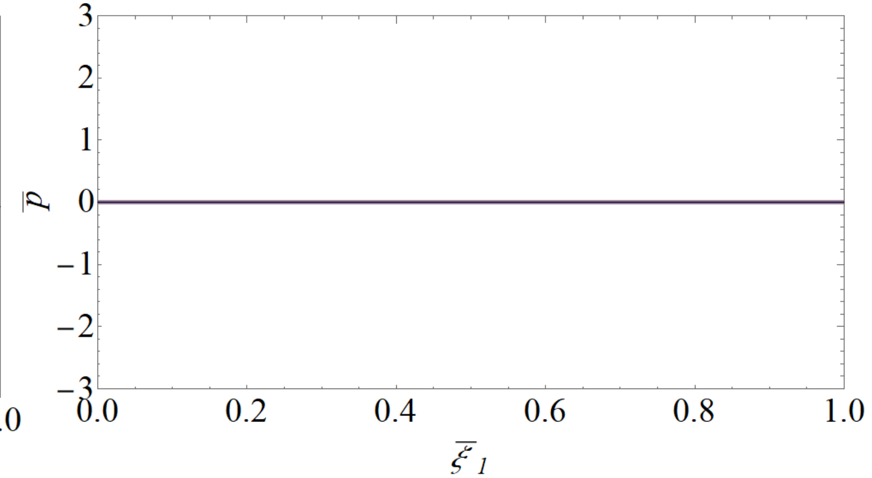
Fixed-Free





Free-Free

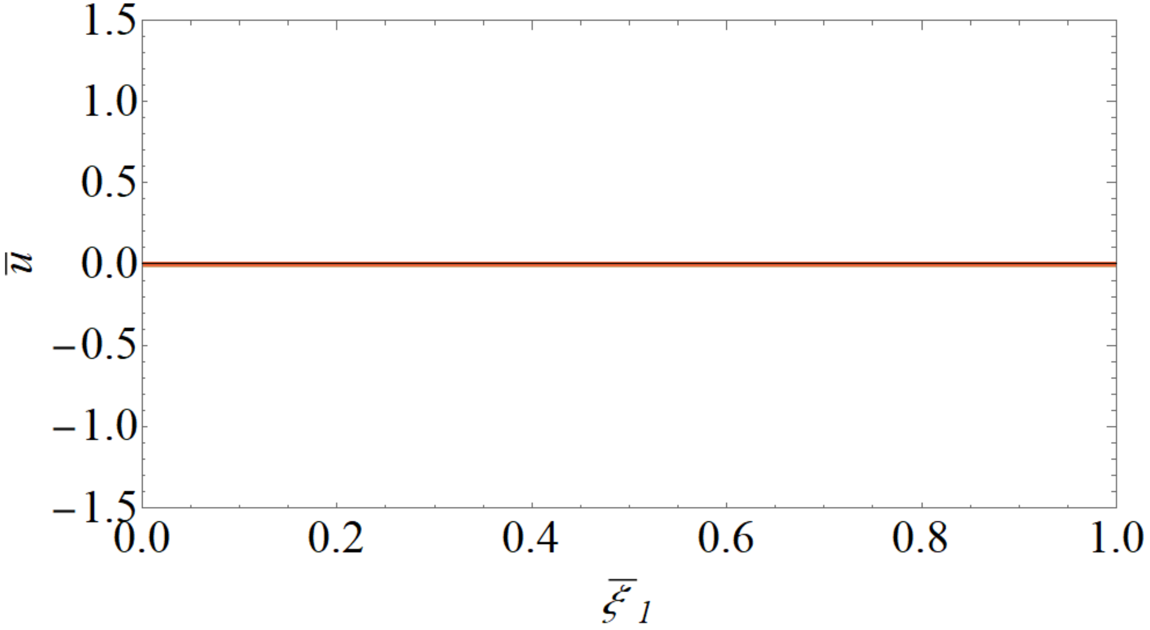




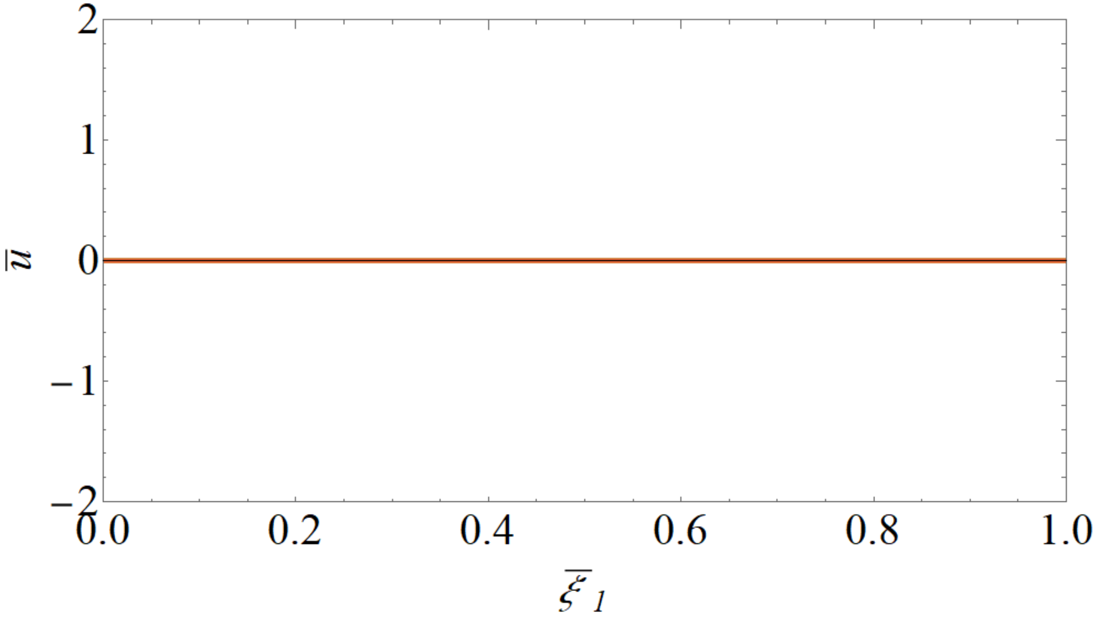
mode-shapes for u

PART 4

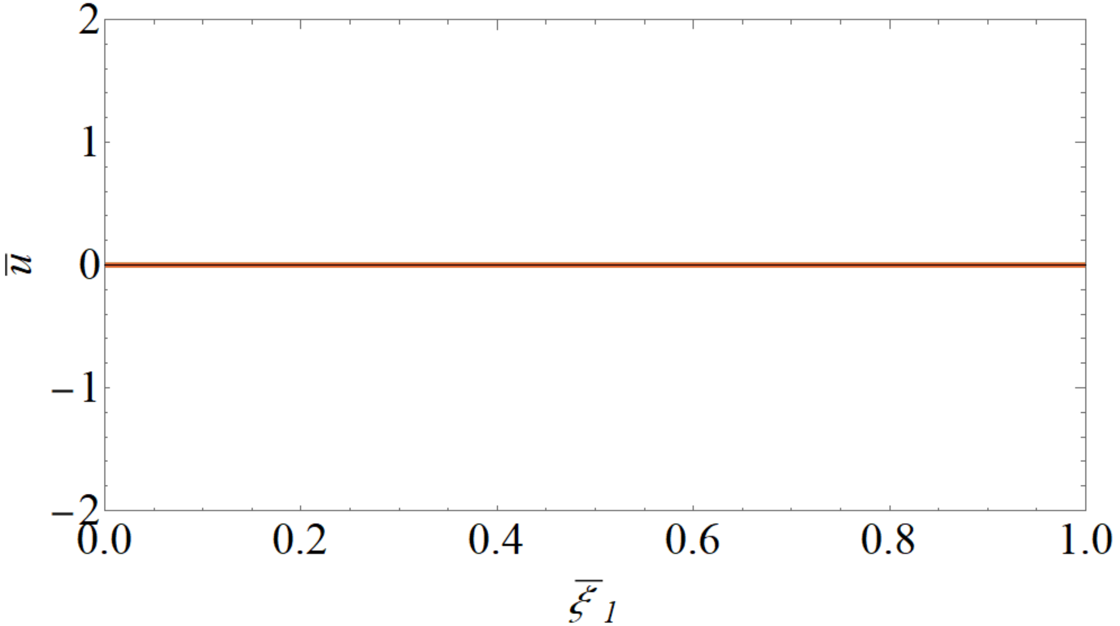
Pinned-Pinned



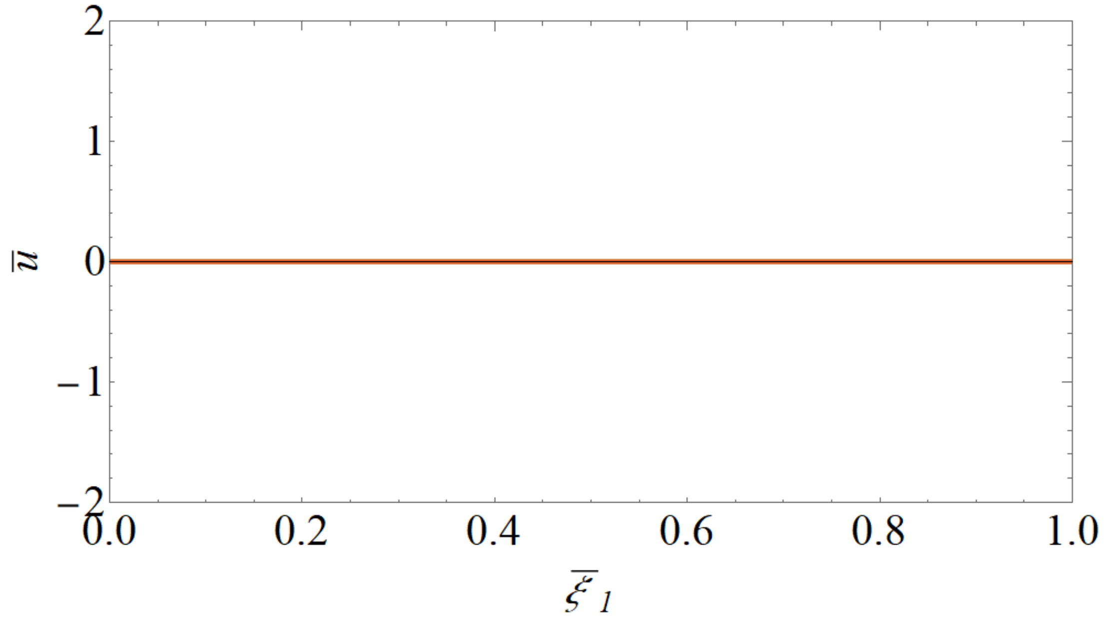
Fixed-Fixed



Fixed-Free



Free-Free



Conclusion

- A comprehensive kinematics of geometrically-exact beam taking into account- multiple curvatures, uniform and non-uniform shear, axial strains and a fully coupled Poisson's and warping effect, maintaining single-manifold character of the beam, is presented.
- The understanding of framed space curves by means of *Relatively Parallel Adapted Frames (RPAF)* and *Material Frames (MF)* is used to reconstruct shape of slender structure, and generate computer graphics.
- Governing equations of motion and finite element formulation of the beam considering general kinematics is presented. The beam kinematics not only depend on the axial strain and curvatures but also on their derivatives, making it *higher-order* beam theory.
- The coupled non-linear equations can be written as 7 uncoupled non-dimensionalized linear equations to perform modal analysis.
- We aim at improvising our work to include non-linear material model and perform non-linear dynamics to investigate stability issues like bifurcation etc.

Relevant publications

- Chadha, M., Todd, M. D., 2017 "A generalized approach to reconstructing the three-dimensional shape of slender structures including the effect of curvature, shear, torsion, and elongation." *American Society of Mechanical Engineering* (ASME), *Journal of Applied Mechanics*

- Chadha, M., Todd, M. D., 2017- "An introductory treatise on reduced balance laws of Cosserat beams", *International Journal of Solids and Structures*

- Chadha, M., Todd, M. D., 2018- "A comprehensive kinematic model of single-manifold Cosserat beam structures with applications to a finite strain measurement model for strain gauges", *IJSS*

- Chadha, M., Todd, M. D., 2019- "On the material and material-adapted approach to curve framing and its application to path estimation, shape sensing and computer graphics", *Computers and Structures*

- Chadha, M., Todd, M. D., 2019- "On the derivatives of curvature of framed space curve and their time-updating scheme", *Applied Mathematics Letters, Elsevier*

- Chadha, M., Todd, M. D., 2019 (expected)- "Mathematical theory of higher-order geometrically-exact single-manifold Cosserat beam with deforming cross-section" (under progress)