On the n^{th} doorkeeper of the pleasure garden in Fibonacci's Liber Abbaci — Proof techniques in Discrete Mathematics on a generalization of a problem possibly from 1202

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Abstract

This paper illustrates the need and sometimes beauty of an appropriate access to first of all find a formula and then later on to prove the statement — on a problem from one of our classics: Leonardo Pisano's Book of Calculation.
We realize that the insightful counting tool *recurrence relation* from Discrete Mathematics helps us to identify a formula — which might guide us in selecting a sensible order of presenting the various topics in school or university teaching.

1 Introduction

In [3], we may discover the following passage:

"*On Him Who Went into the Pleasure Garden to Collect Apples.*
A certain man entered a certain pleasure garden through 7 doors, and he took from there a number of apples; when he wished to leave he had to give the first doorkeeper half of all the apples and one more; to the second doorkeeper he gave half of the remaining apples and one more. He gave to the other 5 doorkeepers similarly, and there was one apple left for him. It is sought how many apples there were that he collected … ."

You find this exercise in a typical grammar school book ($9^{\texttt{th}}$ year). For university purposes, we generalize it to ask for the number of apples needed having an arbitrary number of doorkeepers:
$a_n :=$ # $a$pples for n doorkeepers.

Usually, the variety of students, if at all $\ddot{\smile}$, yield solutions which vary in ease of presentation. This article sheds some light on a didactic part of the area.

Section 2 states the brute-force approach which we may have in mind first when reading the exercise. Section 3 shows the way via a recurrence relation (*backward* as well as *forward substitution*) — thereby presenting a more elegant tool in finding a formula. Section 4 recapitulates and discusses the lessons learnt.

2 Brute force

Having a_n we hand over the following number of apples
to keeper 1: $a_n/2 + 1 = 1/2 \cdot (a_n + 2)$,
rest$_{-1}$: $a_n - (a_n/2 + 1) = a_n/2 - 1 = 1/2 \cdot (a_n - 2)$;
to keeper 2: $1/2 \cdot (a_n - 2) \,/\, 2 \; + 1 = 1/4 \cdot (a_n - 2 + 4) = 1/4 \cdot (a_n + 2)$,
rest$_{-2}$: $a_n - 1/2 \cdot (a_n + 2) - 1/4 \cdot (a_n + 2) = a_n - 3/4 \cdot (a_n + 2)$.
Idea: to keeper 3: $1/8 \cdot (a_n + 2)$; step by step:
$(a_n - 3/4 \cdot (a_n + 2)) \,/\, 2 \; + 1 = a_n/2 - 3/8 \cdot (a_n + 2) + 1 = 1/8 \cdot (4a_n - 3a_n - 6 + 8) =$
$1/8 \cdot (a_n + 2)$, according to the "Idea" above: $2^{(-3)} \cdot (a_n + 2)$, to doorkeeper 3.

Supposition: to *d*oorkeeper d: $2^{(-d)} \cdot (a_n + 2) \qquad \Rightarrow$

$a_n = 1 + \sum_{d:=1}^{n} 2^{(-d)} \cdot (a_n + 2) \Longleftrightarrow_{\texttt{[Auxiliary calculation, see below]}}$
$a_n = 1 + \left(1 - 2^{(-n)}\right) \cdot (a_n + 2) \qquad\qquad | - \left(1 - 2^{(-n)}\right) \cdot a_n$
$\left(1 - \left(1 - 2^{(-n)}\right)\right) \cdot a_n = 1 + 2 - 2^{(-n)} \cdot 2 \qquad | : 2^{(-n)} \; {\scriptstyle (>0)}$
$a_n = 3 \cdot 2^n - 2$.

Test: $n := 7$ (to get the answer to the question at the beginning of Section 1):

formula: $a_7 = 3 \cdot 2^7 - 2 = 3 \cdot 128 - 2 = 384 - 2 = 382$;

principle:
to keeper 1: $382/2 + 1 = 192$, rest$_{-1}$: 190,
to keeper 2: $95+1 = 96$, rest$_{-2}$: 94,
to keeper 3: $47+1 = 48$, rest$_{-3}$: 46,
to keeper 4: $23+1 = 24$, rest$_{-4}$: 22,
to keeper 5: $11+1 = 12$, rest$_{-5}$: 10,
to keeper 6: $5+1 = 6$, rest$_{-6}$: 4,
to keeper 7: $2+1 = 3$, rest$_{-7}$: 1 — ok.

Proof: induction on n

basis step: $n_0 := 0$
formula: $a_0 = 3 \cdot 2^0 - 2 = 3 - 2 = 1$
principle: a_0 (no doorkeeper) $= 1 \;\widehat{=}\;$ formula — ok
induction hypothesis: $a_{n-1} = 3 \cdot 2^{(n-1)} - 2$
induction step: $(0 \leq)\; n - 1 \to n \;(> 0)$
idea: $a_n = 3 \cdot 2^n - 2$
for 1 keeper: $(3 \cdot 2^n - 2) \,/\, 2 \;\; + \;\; 1 \;\; = \;\; 3 \cdot 2^{(n-1)} - 1 + 1 \; = \; 3 \cdot 2^{(n-1)}$
rest$_{-1}$: $3 \cdot 2^n - 2 - 3 \cdot 2^{(n-1)} \; = \; 3 \cdot 2^{(n-1)} \cdot (2 - 1) - 2 \; = \; 3 \cdot 2^{(n-1)} - 2$
$\widehat{=}$ induction hypothesis (backward construction, according to the task).

Auxiliary calculation: $\sum_{d:=1}^{n} 2^{(-d)} = 1 - 2^{(-n)}$.
Proof: induction on n

basis step: $n_0 := 0$
formula: $1 - 2^{(-0)} = 1 - 1 = 0$
principle: sum (with top index smaller than bottom index) $:= 0 \quad \hat{=} \quad$ formula
(alternatively: $n_0 := 1$: formula: $1 - 2^{-1} = 1 - 1/2 = 1/2 \;\hat{=}$ principle, sum: $\frac{1}{2}$)
induction hypothesis: $\sum_{d:=1}^{n-1} 2^{(-d)} = 1 - 2^{(-(n-1))}$
induction step: $(n_0 \leq)\ n - 1 \ \rightarrow\ n\ (> n_0)$
$\sum_{d:=1}^{n} 2^{(-d)} = \sum_{d:=1}^{n-1} 2^{(-d)} + 2^{(-n)} =^! 1 - 2^{(-(n-1))} + 2^{(-n)} = 1 - 2^{-n} \cdot (2-1) = 1 - 2^{-n}$.

3 Recurrence relation

a_0 (no doorkeeper) $:= 1$

Version a) "backward substitution"
$a_{n>0} \ = \ (2 \cdot a_{n-1} + 1) + 1 \ = \ (a_{n-1} + 1) \cdot 2 \ \left[=_{\texttt{produced later on, here as test:}} \ (a_{n-1} + 2) \cdot 2^1 - 2\right]$
$= \ (((a_{n-2} + 1) \cdot 2) + 1) \cdot 2 \ = \ (a_{n-2} + 1) \cdot 4 + 2 \ \left[=_{\texttt{produced later on, here as test:}} \ (a_{n-2} + 2) \cdot 2^2 - 2\right]$
$= \ (((a_{n-3} + 1) \cdot 2) + 1) \cdot 4 + 2 \ = \ (a_{n-3} + 1) \cdot 8 + 6 \ \left[=_{\texttt{produced later on, here as test:}} \ (a_{n-3} + 2) \cdot 2^3 - 2\right]$
$= \ (((a_{n-4} + 1) \cdot 2) + 1) \cdot 8 + 6 \ = \ (a_{n-4} + 1) \cdot 16 + 14 \ \left[=_{\texttt{produced later on, here as test:}} \ (a_{n-4} + 2) \cdot 2^4 - 2\right]$
$= \ (((a_{n-5} + 1) \cdot 2) + 1) \cdot 16 + 14 \ = \ (a_{n-5} + 1) \cdot 32 + 30 \ = \ (a_{n-5} + 1) \cdot 2^5 + 2^5 - 2$
$= \ (a_{n-5} + 2) \cdot 2^5 - 2$
$\vdots$
$=_? \ (a_{n-n} + 2) \cdot 2^n - 2 \ = \ (a_0 + 2) \cdot 2^n - 2 \ = \ (1 + 2) \cdot 2^n - 2 \ = \ 3 \cdot 2^n - 2.$

Version b) "forward substitution"
$a_1 \ = \ 1 + (2 \cdot a_0 + 1) = (a_0 + 1) \cdot 2 \ = \ a_0 \cdot 2 + 2 \ \left[= 4 =_{\texttt{produced later on, here as test:}} \ 3 \cdot 2^1 - 2\right]$
$a_2 \ = \ (a_1 + 1) \cdot 2 = (a_0 \cdot 2 + 2 + 1) \cdot 2 \ = \ a_0 \cdot 4 + 6 \ \left[= 10 =_{\texttt{produced later on, here as test:}} \ 3 \cdot 2^2 - 2\right]$
$a_3 \ = \ (a_2 + 1) \cdot 2 = (a_0 \cdot 4 + 6 + 1) \cdot 2 \ = \ a_0 \cdot 8 + 14 \ \left[= 22 =_{\texttt{produced later on, here as test:}} \ 3 \cdot 2^3 - 2\right]$
$a_4 \ = \ (a_3 + 1) \cdot 2 = (a_0 \cdot 8 + 14 + 1) \cdot 2 \ = \ a_0 \cdot 16 + 30 = a_0 \cdot 2^4 + 2 \cdot 2^4 - 2 = (1 + 2) \cdot 2^4 - 2$
$\quad = \ 3 \cdot 2^4 - 2$
$\vdots$
$a_n \ =_? \ 3 \cdot 2^n - 2.$

Proof: induction on n

basis step: $n_0 := 0$
formula: $a_0 = 3 \cdot 2^0 - 2 = 1$
principle: a_0 (no doorkeeper) $= 1 \ \hat{=}$ formula — ok
induction hypothesis: $a_{n-1} = 3 \cdot 2^{(n-1)} - 2$
induction step: $(0 \leq)\, n - 1 \ \rightarrow\ n\, (> 0)$
$a_n = 2 \cdot (a_{n-1} + 1) =^! 2 \cdot \left(3 \cdot 2^{(n-1)} - 2 + 1\right) = 3 \cdot 2^n - 2$
(forward construction, $n - 1 \ \rightarrow\ n$).

4 Discussion

A brute-force attack usually does not represent the most elegant solution; also here, this method (illustrated at the beginning) appears to be arduous. The (incremental[1]) *recurrence relation*, often used in the informatics area of algorithmics, eases the creative part of getting a counting formula. In teaching, an early introduction to this approach would be advantageous, e.g. in a first semester course of lectures in Discrete Mathematics — you may consult [1] $\ddot\smile$.

References

[1] Walter Hower. *Diskrete Mathematik — Grundlage der Informatik.* Oldenbourg Wissenschaftsverlag, München, 2009. ISBN: 978-3-486-58627-5.

[2] H. Lüneburg. L. Sigler — Fibonacci's Liber Abaci. *JB*, 105(4):29/30, 2003. Jahresbericht der Deutschen Mathematiker-Vereinigung, Teubner.

[3] Laurence E. Sigler. *Fibonacci's Liber Abaci — A Translation into Modern English of Leonardo Pisano's Book of Calculation.* Springer, pp. 397 f., 2003. (the original probably published in 1202 — however, cf. [2]).

[1] thereby reflecting the inductive step in an induction proof