Evolutionary Cooperative Games

Heinrich Harald Nax
University College
University of Oxford

a thesis submitted for the degree of
doctor of philosophy at the university of oxford.

michelmas term 2010

Grüße an Oma Käthi, Elisabeth Eiche und Wilhelm Busch.

Evolutionary Cooperative Games

HH Nax

University College, University of Oxford

Doctor of Philosophy in Economics

Michelmas Term 2010

Supervision:

H. Peyton Young (University of Oxford)

Françoise Forges (Université Paris-Dauphine)

Thesis outline

This thesis proposes a new approach to deriving cooperative solution concepts from dynamic interactive learning models. For different classes of cooperative games, the procedures implement the core. Within the core, tendencies towards equity are revealed and equitable outcomes are favoured in the long run.

Chapter 1. Motivation

Most of cooperative game theory is static, including solution concepts such as the core, Shapley value and the Nash bargaining solution. To support these solutions, dynamic implementation procedures have been proposed. In these models, multistage bargaining games or dynamic noncooperative games are played and the cooperative solution is implemented via equilibria of these related games. In contrast to existing approaches, we propose a class of fully dynamic models of learning based solely on each individual's assessment of how well he did in the past. These procedures reveal internal bias towards stable and equitable outcomes.

Chapter 2. Evolutionary cooperative bargaining

Many agents repeatedly bargain over a cooperative surplus. Agents feel their way to Pareto-efficient outcomes in a nonstrategic manner by incremental adjustments to their demand and aspiration levels. Transfers of surplus shares in between any two agents are always possible but the procedures reveal global tendencies away from substantial levels of inequity over time. This leads to the long-run implementation of equal splits of the surplus most of the time. In the intermediate run, however, the adjustment processes are considerably more complex. With growth in the possibility frontier, expected inequity may initially rise and only fall as the possibility frontier reaches a stable long-run position.

Chapter 3. Assignment games with trial-and-error

Many separate pairs of agents from two sides of a market may form. Partnerships are broken time and again, reshuffled and restored. Players experiment with various partnerships before settling on optimal partners. The dynamics reveal that, over time, matching becomes more and more stable. Most of the time, optimal partnerships form and share the gains in a pairwise stable way. Loss-averse partners share close to fifty-fifty with high probability, unless one or both of the partners have more profitable outside alternatives.

Chapter 4. Evolutionary coalitional games

Players learn to play coalitional games in characteristic function form. By way of incremental adjustments and varying coalitional commitments, individuals feel their way to cooperating. Over time, coalitions tend to grow and inner allocations of the cooperative gains become more equitable. For balanced games, the procedures lead to efficient outcomes that are robust against coalitional deviations; to the implementation of the core. For games with an empty core, inefficient subcoalitions persist. For large parts of the game, inherent equity biases are revealed, these tendencies are halted where subgroups have higher coalitional alternatives.

Chapter 5. Summary

We summarise our findings from the different classes of games regarding why and which coalitions form and how surpluses are shared. We pose open questions and discuss avenues that we aim to pursue in future research. In particular, we sketch a wider range of learning procedures and outline opportunities for software programming applications and experimental work.

Word count

The thesis contains 158 pages, its main body 151 pages. A typical page (p. 65) contains around 300 words, making the length of the thesis approximately 45,000 words (excluding thesis outline, acknowledgements and bibliography).

Acknowledgements

This D.Phil. thesis has been written under the joint supervision of Françoise Forges and Peyton Young. I thank both of them for their help and support.

Peyton Young, the main supervisor at the University of Oxford, has been very resourceful and helpful in developing my work. He is one of the few pioneers and specialists in both cooperative and evolutionary game theory. His view over the domains has been crucial to coordinate the task of bridging the two throughout the thesis, his guidance to complete it.

Françoise Forges, the external co-supervisor at Paris-Dauphine University, has been no less helpful and provided detailed comments throughout all stages of my research. Especially for understanding the cooperative underpinnings and for identifying the solutions that our learning procedures actually implement, her expertise in cooperative game theory was crucial to introduce and formulate the more general models we had in mind.

I am indebted to Steffen Issleib, with whom I went through a myriad of attempts for the proofs in Chapter Two. The technical results in Chapter Two are the result of joint efforts between Peyton Young, Steffen Issleib and myself. Steffen also introduced me to simulations with Visual C++, without which we would not know where our procedures really drift to.

Special thanks for helpful comments and discussions go to Jean-Paul Carvalho, Geoffroy de Clippel, Francis Dennig, Edoardo Gallo, Gérard Hamiache, Reza Saleh-Nejad, Tom Norman, Bary Pradelski, Bassel Tarbush, Oliver Walker and, last but not least, Chris Wallace.

I am also thankful for comments and discussions from participants at the *Gorman Workshop*, the *LSE Choice Group*, the *Discussion Group on Learning, Games and Networks* and the *UECE Lisbon Meetings 2010*. In particular, I would like to thank Matt Lloyd Elliott who presented at the *Discussion Group on Learning, Games and Networks* at Nuffield College. We discussed theory and experiments on matching, games that proved an ideal application and point of departure for extending our learning procedures to coalitional environments.

I am grateful to have been supported by the Economic and Social Research Council [grant number PTA-031-2005-00118] and for continuing support from University College Oxford.

All remaining errors are my own.

Contents

Chapter 1

Motivation

Ever since the beginning of modern game theory, cooperative theory has struggled to reconcile the axiomatic and strategic approaches.[1] On the one hand, many interesting solution concepts have been formulated using the axiomatic approach.[2] This approach, however, is, for the most part, static and lacks behavioural foundations to explain why and how particular outcomes are played. These issues are particularly apparent in complex repeated environments. The noncooperative approach, on the other hand, is built on Nash's [1950] definition of the equilibrium in strategies upon which no unilateral deviation can improve. In cooperative environments, it is difficult to extend the unilateral strategy space to incorporate binding coalitional agreements. Again, this problem arises primarily in complex environments where standard information and rationality assumptions are difficult to uphold.

In repeated noncooperative environments where binding collaborative agreements cannot be made, existing dynamic models show that agents learn to play Nash equilibria even if the interaction is complex.[3] Attempts have been made to adapt this learning theory from noncooperative to cooperative situa-

[1] These begin with von Neumann & Morgenstern [1947] and Nash [1950].

[2] Examples are the von Neumann & Morgenstern solution, the Nash bargaining solution, the Shapley value and the core.

[3] See Alexander [2009] for a review.

tions, in order to provide dynamic support for cooperative solutions.[4] These approaches, however, stumble on the problem of specifying the noncooperative strategies available to players who themselves, because individuals can collaborate, may be individuals and coalitions of individuals.

In this thesis, we posit a dynamic in which players do not have a strategic model at all and, thus, avoid many of the problems facing previous approaches. Each individual adapts his behaviour based solely on his assessment of how well he did in the past. The model has antecedents in early game theory on cooperative solutions with proto-dynamics.[5] The behaviour is based on psychological foundations from traditional behaviourial psychology and shares its nonstrategic heuristics with theories such as aspiration adaptation, reinforcement learning and fictitious play.[6] Over time, our agents learn to cooperate and play stable outcomes. Dynamics, furthermore, reveal tendencies towards equity, not because agents care about equity *per se* nor because social norms favour equitable outcomes in the long run, but because the dynamics lead to them naturally.

Real-world evidence proves that stable and equitable agreements are in fact frequently reached in complex and cooperative situations. Many common resources, for example, have been managed stably and shared equitably.[7] Similarly, persistent stability and growing equity are observed in wage-setting in the labour market.[8] Rich laboratory evidence also suggests that experimental subjects achieve cooperative outcomes with high levels of equity.[9] Two

[4]See Agastya [1997], [1999], Arnold & Schwalbe [2002], Rozen [2010], [2010a], Newton [2010] and Chapter Four of this thesis (pp. 103-104) for a discussion.

[5]Notably, see Zeuthen [1930] and Chapter Two of this thesis (pp. 38-39) for a discussion of his work and its relationship with our approach.

[6]See Sauermann & Selten [1962], Bush & Mosteller [1955] and Brown [1951].

[7]See Ostrom [1990], Ostrom, Gardner & Walker [1994] and Chapter Two of this thesis (pp. 16-19) for a discussion.

[8]See Bewley [1999].

[9]See Tietz, Weber, Vidmajer & Wentzel [1978], Selten [1987] and Charness, Corominas-Bosch & Fréchette [2007].

types of game-theoretic explanations for the equity phenomenon have been proposed.

The first explanation is based on social preferences. It assumes that individuals care about equity *per se* and are willing to sacrifice some of their own payoffs in return for increases in equity.[10] The second explanation is based on evolutionary success of equity-fostering strategies, that is, on the finding that "equity norms have an advantage over discriminatory norms in the very long run."[11]

We provide a third explanation for the emergence of equity without assuming inherent preferences for equity and without specifying a strategic learning model at all. In the models presented in the following chapters, players learn solely based on how well they did in the past. They change their behavior through incremental adaptations of aspirations that depend solely on each individual's assessment of success and failure in previous periods. Play is biased towards equitable solutions, not because of social preferences or norms, but because agents are naturally more likely to cooperate in equitable outcomes.

The remainder of this chapter is structured as follows. First, we will spell out the basic principles and relevant solution concepts of cooperative game theory which, for the most part, are static. Next, we discuss related noncooperative implementation mechanisms and evolutionary bargaining models. We then position our learning model amongst benchmark models in behavioural psychology and compare it to existing learning models in noncooperative theory. Finally, we outline the model in cooperative bargaining, assignment games and coalitional games. We find that players learn to play outcomes that are robust against coalitional deviations and equitable at the same time.

[10] See Fehr & Schmidt [1999].
[11] Axtell, Epstein & Young [2001], p. 206.

Cooperative game theory

Cooperative game theory begins with von Neumann & Morgenstern [1947] and is essentially a static representation of a game. Possible coalitions are associated with worths using simple characteristic functions. Allowing for coalitional agreements, cooperative theory aims to capture commitments and mechanisms (such as law) that enforce cooperative behaviour, which are a feature of many interactions in the real world.

This approach to games is attractive because two fundamental economic issues are addressed: cooperating to produce jointly and sharing surpluses. An outcome of a cooperative game is an agreement to form coalitions and to allocate gains. Von Neumann & Morgenstern propose a static solution concept but do not offer a dynamic theory that would explain why and how the solution is indeed played.[12] Without an explicit model for how the game is played, a cooperative game can be viewed as a summary of an interactive situation rather than as a game that can be solved.[13] Without additional structure, cooperative games are, therefore, somewhat under-specified for making clear predictions.

Because cooperative games cannot be solved using one standard procedure, cooperative solution concepts have traditionally been developed using an axiomatic and static approach, thus, providing solutions with explicit normative properties. Famous solution concepts of this type are the von Neumann-Morgenstern solution (von Neumann & Morgenstern [1947]), the Nash bargaining solution (Nash [1950]), the Shapley value (Shapley [1953]) and the core (Gillies [1959]).

[12] See von Neumann & Morgenstern [1947], p. 44: "A dynamic theory would unquestionably be more complete and therefore preferable."

[13] See Aumann & Maschler [1964], p. 443: "The basic difficulty in n-person [cooperative] game theory is due to the lack of a clear meaning as to what is the purpose of the game."

Different cooperative solution concepts capture different behavioural and strategic aspects of the situation. Some are derived from sets of axioms, while others have a dynamic interpretation. Zeuthen's [1930] bargaining point and Aumann & Maschler's [1964] bargaining set are well-known solutions of the latter kind. In Zeuthen's model, two parties start bargaining by initially demanding the whole surplus for themselves. Over the course of the bargaining, each side faces the decision of whether to accept the other side's proposal and to receive this payoff with certainty, or to make an alternative proposal at the risk of getting nothing, due to a breakdown in the bargaining process. Over time, the two parties make mutual concessions until a proposal is accepted. Zeuthen assumes that, at each stage, the side with the lower willingness to accept breakdown, (which is the party with the higher demand and, therefore, with a greater interest in accepting,) puts forward a new proposal demanding slightly less. This process continues until, eventually, a proposal is accepted and the final agreement is reached. Players with an equal willingness to accept a breakdown will end up sharing the surplus equally.

Aumann & Maschler [1964] propose the bargaining set, a solution that is the stable outcome of a sequence of proposals. Proposed allocations stir a series of objections, counterobjections, and alternative proposals. An objection can be made if an alternative agreement exists that is favoured and enforceable by some coalition. Any intermediate proposal is unstable if, for any possible objection by someone, there exists no counterobjection by someone else. An outcome is stable if either no objection exists or, for any objection, a counterobjection exists. The bargaining set contains all stable outcomes. Under standard assumptions, it is nonempty and contains the core.[14]

The core and the bargaining set capture different stability aspects of a solution,

[14] Depending on the strictness of stability relations, different (embedded) bargaining sets exist. For the class of convex games and assignment games, bargaining set and core coincide (Maschler, Peleg & Shapley [1972], Potters & Reijnierse [1995]).

consisting of allocations that are stable against different coalitional deviations. Solutions of this kind impose constraints on permissable payoffs and there may be many outcomes satisfying either notion of stability. The core may be empty.

The equal split of total surplus, Nash's and Zeuthen's bargaining solutions and the Shapley value, on the other hand, determine one unique outcome for any given cooperative game. The equity properties of these solutions can be assessed generally. A weakness of these solutions, however, is that they may be coalitionally unstable, that is, subcoalitions may exist that block them. In many applications, we therefore expect stable allocations to display persistent inequalities due to differences in coalitional opportunities.

Noncooperative implementation mechanisms

To make the strategic nature of any interaction explicit, Nash [1950a] introduces the equilibrium concept known as Nash equilibrium. Nash equilibria are outcomes upon which no unilateral deviation can improve. This concept is applicable to cooperative games where no collaboration across different positions of the game is permitted. If coalitions of players are able to collaborate, however, joint improvements upon the Nash equilibrium are possible.

The Nash [1950] bargaining solution and "Nash programs" have been studied in tandem, yielding outcomes that are derived from normative axioms and implemented via Nash equilibria of noncooperative bargaining mechanisms. Famous implementation mechanisms are Ståhl's [1972] and Rubinstein's [1982] sequential bargaining models with alternating offers and Raiffa's [1953] mutual concession model.

Mechanisms for noncooperative implementation of cooperative solutions to more general games where subcoalitions may form have also been proposed. Random room-entering stories, for example, motivate the Shapley value. To implement the core, Lagunoff [1994] develops a mechanism where players sequentially announce coalition plans that have to be ratified by the other players until an outcome that is ratified by all is played.

A general feature of noncooperative implementation models is that a strategic game is associated with the underlying cooperative game. The strategic structure regarding order of play, preferences, beliefs and actions is necessary to explain why a particular outcome is played.

Repeated cooperative games

Note that, even though many of the aforementioned bargaining models are multistage, the bargaining ends once a solution is reached. Bargaining takes time but cooperative gains are produced and consumed only once. Many cooperative situations in the real world, however, are repeated. Generalisations of the core, Shapley value and bargaining set have been proposed for this situation.[15] Konishi & Ray [2003] develop a model for playing repeated cooperative games dynamically. In their model, a population of perfectly rational agents with common knowledge and beliefs plays the same game repeatedly. Groups of players continue to be selected and are given the opportunity to reconfigure their coalitional agreements. Selected groups decide what to do based on their current inner allocation and on the present value of alternative configurations in the future. Coalitions reconfigure to alternative structures that promise higher present values. No reconfigurational deviation will occur when no reconfiguration promises improvement. For balanced games, core outcomes of the stage game are implemented if agents are sufficiently farsighted.

[15] See Oviedo [2000], Kranich, Perea & Peters [2005] and Hellman [2009].

Evolutionary noncooperative bargaining

When the environment is sufficiently complex, the exact structure of repeated cooperative interactions may not be fully known to the players involved. In this event, the standard assumptions of common knowledge and perfect rationality are not particularly plausible. Evolutionary learning models have been proposed as a way to handle these situations. Less extreme informational (no common knowledge) and behavioural (rationality is bounded) assumptions suffice to provide evolutionary support for cooperative solutions. It turns out that social norms evolve over time that favour cooperative behaviour, leading to higher levels of equity than predicted by standard models.

The evolution of cooperation has received special attention in the evolutionary literature, often in contrast to the short-term lack of cooperation in one-shot games. This literature is divided into two strands.[16]

In the first strand of literature, long-run survival of cooperative strategies is studied in environments where cooperation is not a stage-game equilibrium. This literature deals with "The Evolution of Cooperation" (as in Axelrod & Hamilton [1981] and Axelrod [1984]) in the context of rival and noncooperative environments. Examples are prisoners' dilemmas, tournaments and public good provision, that is, games that cannot be modelled as cooperative games.

In the second strand of literature, long-run equity is studied in cooperative bargaining. It turns out that high levels of equity evolve through social norms. The first bargaining models of this kind analyse Nash demand games (Nash [1950a], Young [1993], Gale, Binmore & Samuelson [1995]). Players sample and hypothesise about the other players' behaviour and, given their current

[16]See Weibull [1995], Vega-Redondo [1996], Samuelson [1997] and Young [1998] for textbook treatments of that literature.

information, play as best replies as possible whenever called upon to play. Over time, players make errors and learn from them. As the game evolves, the players learn more and play settles closer to actual best replies. In Young's [1993] model, for example, pairs of agents are drawn at random from two populations. When the two populations are homogeneous and separate, populations evolve to play according to the Nash bargaining solution. When there is mixing between the two populations, the equal split is implemented. In contrast to noncooperative implementation mechanisms that stop once an outcome has been reached, evolutionary noncooperative procedures are fully dynamic and equitable outcomes remain stable over time as the game continues.

We contribute to the second strand of literature on evolutionary cooperative bargaining by developing new learning procedures that do not require a specification of the players' strategies. Cooperative and nonstrategic behaviour are the distinguishing features of our approach.

Our approach

In the following chapters, we develop a fully dynamic model for learning in cooperative games with many players. Rivalries amongst individuals exist over how to share joint gains and also over who forms coalitions with whom. The game is so large and the interaction so complex that information about the behaviour of others is limited and hard to obtain. We consider a new form of nonstrategic learning in such environments because we find it unreasonable to suppose that players would

> "perform complicated mathematical operations in an attempt to understand the strategic structure of the situation. [...] making cosmetic changes in the usual picture of Bayesian rationality is not a sufficient approach to the problem of bounded rationality. It would be better to look for theories that do not even mention such

constructs as subjective probabilities."[17]

In our model, agents feel their way to gains from cooperation in different coalitions, not via strategising but based solely on how well they did in the past. They change behavior through incremental adaptations of aspirations that depend solely on each individual's assessment of success and failure in previous periods. In the long run, core outcomes with high levels of equity are favoured. Players with similar coalitional possibilities receive almost the same most of the time, groups of individuals seldomly receive less than their worth as a coalition.

The proposed dynamics are such that coalitions continue to form and separate; no coalitional agreement lasts forever. Existing coalitions break, new coalitions form, and surplus shares are renegotiated and adjusted. We model two types of behavioural responses according to whether players are satisfied or not. Satisfied players, who partake in successful cooperative partnerships, occasionally experiment to see if they can improve their situation. Players are dissatisfied if they fail to enter a partnership, which leads them to reduce their demands in order to restore cooperation.

There are three basic elements to the process of aspiration adjustment. First, agents occasionally explore if alternative actions could lead to higher payoffs and abandon actions that result in worse positions quickly. Second, upward and downward adjustments are made locally and in small increments. Third, players are more likely to make restorative attempts and adjust aspirations downwards the higher the loss they experience during cooperative failure.

Over time, players learn to play stable outcomes that are robust against unilateral and coalitional deviations. Once stability is reached, the procedures reveal a tendency favouring equitable outcomes in the stable set.

[17] Selten [1987], pp. 42-43.

Completely uncoupled learning

The nonstrategic procedure we propose is a "completely uncoupled" learning rule. Completely uncoupled learning has recently been explored in noncooperative game theory and, indeed, Nash equilibrium behaviour can be learned in that way.[18] A learning procedure is completely uncoupled if solely dependent on the player's own history. When players do not understand the whole behavioural interplay, no attempt is made at playing best reply to the other players' possible actions. Instead, each player focusses on how well he did in the past and decides what to do without hypothesising about others. Such behaviour makes sense, especially when a large number of agents without common knowledge repeatedly interacts in complex environments with highly incomplete information.

There are precursors to this type of learning model in biology and psychology. In biology, Thorndike's [1898] animal experiments lead to the "Law of Effect" hypothesis that actions connected with positive or negative stimuli in the past are reinforced or avoided in current and future behaviour. An early psychological heuristic for adaptive learning for human interaction is reinforcement learning (Bush & Mosteller [1955]). Depending on the past success or failure of two alternative actions, players increase or decrease the probability of taking them respectively in the current period.

These heuristics are now also applied to economics. Karandikar, Mookherjee, Ray & Vega-Redondo [1998] show that, based on reinforcement learning, cooperation amongst satisficing players is implemented in the prisoners dilemma, and in two-by-two coordination games with common interest. The study of larger, more general noncooperative games using completely uncoupled learning procedures has been pioneered only recently. Based solely on the players'

[18] We use Young's [2009] terminology. Ordinary "uncoupled" learning rules (e.g. in Hart & Mas-Colell [2003], [2006]) depend on others' actions.

own payoff history, they experiment with new strategies in the hope of reaching better positions, not pursuing those that result in worse positions. For different classes of games, Nash equilibria are successfully implemented in Foster & Young's [2006] regret testing, in a simpler variant thereof in Marden, Young, Arslan & Shamma [2007] and in Young's [2009] interactive trial-and-error. Extending Young [2009], Pradelski & Young [2010] show that Nash equilibria with higher levels welfare are favoured.

The completely uncoupled nature of decision-making is the crucial feature for the setup of our evolutionary cooperative model because any coupled response dynamic would hinge upon strategic responses and, therefore, require an explicit noncooperative structure. Modelling the evolution of behaviour with a completely uncoupled procedure allows us to model the interactions dynamically without specifying the players' strategies in any detail. Behaviour is driven by incremental adaptations of aspirations and each individual's assessment of how well he did in the past. This opens the door for a new type of analysis of a wide range of cooperative situations, especially those with a large number of players.

The incremental adaptation approach has modelling antecedents in the early proto-dynamic bargaining models of Zeuthen [1930] and Raiffa [1953]. The earliest psychological theory to explain behaviour on the basis of satisfying momentary personal aspirations (*Anspruchsniveau*) instead of optimising behaviour is Hoppe [1931]. The economic models for dynamic adaptation of aspirations by Heckhausen [1955] and Sauermann & Selten [1962] have been tested in laboratory experiments of repeated bilateral bargaining. The studies of Tietz, Weber and coauthors analyse the adjustment processes displayed by the experimental subjects.[19] The results support our modelling assumptions

[19]See Tietz & Weber [1972], [1978], Tietz [1975], Weber [1976], Tietz, Weber, Vidmajer & Wentzel [1978], Tietz & Bartos [1983], Crössmann & Tietz [1983] and Tietz, Daus, Lautsch & Lotz [1988].

that "the basis of the aspiration levels changes according to the economic situation and is modified by success and failure in the previous negotiation."[20] Moreover, there is rich evidence in bilateral bargaining for our kind of adjustment procedures. Past success and failure determine whether aspirations are corrected upward or downward;

> "a subject lowers his aspiration level after a negative impulse. It is not lowered if the impulse is positive. After a neutral impulse the aspiration level is kept stable. [...] A subject raises his aspiration level after a positive impulse. It is not raised if the impulse is negative. After a neutral impulse the aspiration level is kept stable."[21]

In what follows, we develop models based on these learning heuristics for three classes of transferable utility cooperative games: cooperative bargaining, assignment games, and general coalitional games. The games differ crucially with regard to the coalitions that may form and with respect to the type of possible applications. More detailed motivations, discussions and references are given for each class of games and their applications in the respective chapters.

In all applications, the procedures lead to outcomes that are robust against coalitional deviations; to the implementation of the core. Somewhat unexpectedly, the dynamics in all games are furthermore biased towards equity, by which we mean that transitions to alternative coalitional arrangements are more likely if these feature more equitable shares amongst coalition partners. This equity bias is natural, neither due to social preferences nor caused by convergence of best reply at equitable social norms. As long as growing demands remain feasible, inequity may rise within existing coalitions. Once coalitional possibility frontiers are reached, players with higher aspirations reduce more quickly during cooperative failure, thus, tending towards more equitable outcomes within the coalitionally stable possibility frontiers.

[20] Tietz, Weber, Vidmajer & Wentzel [1978], p. 91.
[21] Tietz, Weber, Vidmajer & Wentzel [1978], p. 94.

Chapter 2

Evolutionary cooperative bargaining

Abstract

A class of learning procedures is introduced in which agents feel their way to Pareto-efficient outcomes in cooperative bargaining by incremental adjustments to their demands. The procedures reveal a global drift away from any substantial level of inequity and this drift leads to the long-run implementation of equitable solutions. In the intermediate run, however, the adjustment processes are considerably more complex. With growth in the possibility frontier, expected inequity may initially rise and then fall.

JEL classifications: C71, C73, C78, D83

Keywords: bargaining, cooperative game theory, equity, evolutionary game theory, (completely uncoupled) learning

2.1 Introduction

Dividing a pie is one of the most basic economic activities.[1] We address that question in an evolutionary setting for cooperative bargaining.

Synopsis

We propose a class of learning procedures applicable to cooperative bargaining where agents feel their way to Pareto-efficient outcomes by incremental adjustments to their demands. Each period, agents in a fixed population seek a cooperative agreement, individually demanding shares of the surplus. If no cooperative agreement is reached, cooperation breaks down and the players obtain their singleton payoffs until cooperation is restored. The central assumption concerning the individual's behaviour is that it is based solely on how well he did in the past. The behaviour differs according to whether the player receives a payoff that exceeds his aspirations (player is satisfied) or not (player is dissatisfied). Satisfied players in successful cooperative partnerships occasionally try to consume more. This experimentation comes at the risk of causing cooperative failure. Dissatisfied players reduce their demands during cooperative failure in order to restore cooperation. The stickiness of demand-reductions is decreasing in the loss of a player: the more he loses, the more likely he is to reduce.

Over time, the dynamics reveal that, as the joint possibility frontier is reached, transfers from relatively better-off players to relatively worse-off players become more likely than transfers the other way around. In the long run, the process is therefore close to equal splits of the surplus most of the time. In the intermediate run, however, the adjustment processes are considerably more

[1] Ellingsen [1997] (p. 581) even asks "Is there any economic activity more basic than two people dividing a pie?"

complex and reveal interesting interim phenomena. Growth in the possibility frontier, for example, may actually cause interim periods of rising inequity, whereas a contraction in the possibility frontier is, on average, equity-increasing.

Structure

The remainder of this introduction motivates the problem and discusses related literature. In the following two sections, we set up the evolutionary bargaining process and develop the behavioural procedures. Section Four discusses alternative behavioural specifications with proportional demand-revisions. Section Five looks at long- versus intermediate-run effects of shifts in the Pareto frontier and of changes in the individual outside options. Section Six concludes. An Appendix contains omitted technical details and mathematical proofs of the results.

Motivation

We consider an environment in which agents, without knowing the nature of their interaction, interact repeatedly in a situation that can be described as cooperative bargaining. Bargaining over the gains from cooperation "in other words, they are engaged in a game, but they do not know what the game is or who the other players are."[2]

Many common-pool resource problems share these features, being intrinsically cooperative situations where individuals bargain over the sharing of a complex joint resource. Gains from cooperation are spoilt if utilisation levels exceed

[2] Young [2009], p. 626.

certain thresholds. Information regarding the nature of the interaction is limited.

Groundwater used for irrigation and consumption in California's coastal locations, for example, is shared by many users.[3] Usage is rival and largely nonexcludable. Information regarding water availability is hard to obtain and there are many rivals whose identities and consumption levels are largely unknown. If extraction becomes too high and groundwater levels drop below sea level, however, saltwater may intrude and preclude utilisation of the groundwater basin altogether, until it has replenished itself naturally. That scarceness together with interest to consume more whenever available suggests tragedy of the commons. In reality, however, Californian groundwater is managed successfully and evenly shared most of the time.

The Geysers in northern California, the world's largest geothermal field, are another example.[4] Reservoirs filled with geothermally superheated water steam are tapped by different energy plants to generate electricity. The understanding of the reservoir is poor and the steam pressure has varied significantly over time. When extraction exceeds critical levels, that is, when "put simply, there are too many straws in the teapot" aiming to extract too much, The Geysers' reservoir "steams vigorously until it suddenly boils dry."[5] Periods of little or no energy generation ensue until the water and steam reservoirs replenish themselves. Stirred by rising competition amongst energy providers during the seventies and eighties, periods of rising extraction levels resulted in overexhaustion of the resource. Extended periods of pressure loss ensued in the nineties. Competitors withdrew and ownership is now highly consolidated with Calpine Corporation owning 19 of the 22 active plants and operating

[3] See Blomquist [1992] and Blomquist, Schlager & Heikkia [2004] for studies of the Californian water resources.

[4] See Kerr [1991] and www.geysers.com.

[5] Kerr [1991], pp. 134-135.

them cooperatively.

Other common-pool examples include different kinds of operating water-supply systems and the managing of forestry and fishery.[6] Other situations displaying similar characteristics are intra-firm wage-bargaining where wage schemes have to be agreed upon and large legal partnerships or similar entities where partners have to agree repeatedly on how to split realised profits and losses.

We model behaviour in such environments as completely uncoupled. The behaviour of each agent depends solely on his current frame of mind, which is based on his own assessment of how well he did previously. In particular, we model two types of behavioural responses according to whether players are satisfied or not, based on aspiration levels that are either fulfilled by current payoffs or not. Satisfied players who partake in successful cooperative partnerships occasionally experiment in order to improve their situation. Players are dissatisfied during cooperative failure and reduce their demands in order to restore cooperation.

A fisher, for example, revises his current target based on past catch. If his catch has matched or exceeded targets, he is satisfied and may increase his future targets. If his catch falls short of his targets, he is dissatisfied and may cut his target.

A particular feature of the behaviour is loss aversion: the more dissatisfied a player, that is, the higher his unfulfilled aspiration level, the more likely he is to revise. This assumption is based on adaptive aspiration heuristics from behavioural psychology and fares well in light of experimental bargaining evidence.[7]

[6] See Ostrom [1990] and Ostrom, Gardner & Walker [1994] for prominent treatments of these issues.

[7] Recall the discussion from Chapter One (pp. 11-13) or, for example, Tietz, Weber, Vidmajer & Wentzel [1978].

The procedures are applied to cooperative bargaining and lend new evolutionary support for equal splits of the surplus.

Equity in bargaining

Equitable bargaining outcomes are experimentally well supported and there is evidence for equity in complex sharing and bargaining situations in the real-world.[8] When the allocative task is to distribute a given cooperative surplus, other things equal, different theoretical bargaining solutions have been proposed that capture different aspects of equity.[9] The most prominent solution is Nash's [1950] bargaining solution (NBS). It is the unique allocation that satisfies the Nash axioms of invariance, Pareto optimality, independence of irrelevant alternatives and symmetry. When utility is transferable, the NBS splits the surplus equally: players get the same allocation if, and only if, their outside options are the same; higher outside options result in higher allocations. In this case, other solutions such as the Shapley value and the Nucleolus coincide with splitting the surplus in that way.[10] It remains to be explained why and how players share equitably or not.

Noncooperative models. Bargaining has been extensively studied combining cooperative and noncooperative tools, proposing noncooperative implementation mechanisms for cooperative solutions, explaining how and why the pie is split as some say it should be.[11]

The most famous bargaining model is Rubinstein's [1982] sequential bargaining, where two players make alternating offers as to how to split a pie until an

[8] See Roth [1995] and Young [1994] for a review of the classic behavioural experiments and surveys.

[9] Moulin [1988], [1995], [2003] discusses many of these notions of equity.

[10] Some bilateral bargaining solutions suggest that players share fifty-fifty in absolute terms unless one has an outside option above that (Binmore, Rubinstein & Wolinsky [1986], Binmore, Shaked & Sutton [1989], Dutta & Ray [1989]).

[11] Binmore [1994], [1998], [2005] discusses many of the contributions in this field.

offer is accepted. Rubinstein's model is an extension of Ståhl's [1972] model from the finite to the infinite horizon. With infinitely patient players, the unique subgame-perfect Nash equilibrium of the game implements the Nash bargaining solution. Equilibrium and solution coincide because of the way each player makes and evaluates the other's offer, which depends solely on outside options and on the rates at which future payoffs are discounted: as players become infinitely patient, the asymmetry of the first-mover advantage vanishes and the Nash bargaining solution is the outcome of reaching the equilibrium agreement in the first period of bargaining.

An early bargaining program is due to Raiffa [1953].[12] In Raiffa's model, bargaining starts at the disagreement point. Over time, players ratchet up towards a final allocation on the Pareto frontier along the "negotiation curve" of points half-way between each player's best possible allocation in the set of Pareto-efficient points above an interim position. The continuous approximation of the model with a linear negotiation curve leads to the same split as the Nash bargaining solution. Random offers in Ståhl's bargaining program (as opposed to the original model with alternating offers) lead to implementation of Raiffa's bargaining solution if the time interval between successive offers is taken to zero.[13]

Zeuthen's [1930] bargaining theory predates the notion of the Nash equilibrium in strategies.[14] In Zeuthen's model, two parties start bargaining by initially demanding the whole surplus for themselves. Over the course of the ensuing bargaining, each side faces the decision to accept the other side's proposal and to receive this payoff with certainty, or to make an own proposal at the risk of bargaining breakdown. If neither party accepts, the party with the

[12] In fact, Raiffa presents two models, the latter a continuous version of the other. See Raiffa [1953] and Luce & Raiffa [1957].

[13] See Sjöström [1991].

[14] We will discuss Zeuthen's model in more detail in this chapter (pp. 38-39) and interpret it in light of our results.

lower willingness to accept breakdown, (which is the party with the higher demand and, therefore, with a lower interest in accepting,) puts forward a new proposal demanding slightly less. Eventually, a proposal is accepted and the final agreement is reached. With symmetry, this will be at the equal split.

There are many more noncooperative bargaining stories and models. Most of them view bargaining as a game with a multi-stage pre-bargaining phase that precedes a single-period consumption. For symmetric transferable-utility cooperative bargaining, the equal split is the unique subgame-perfect Nash equilibrium of a number of such programs and also reached after a finite number of mechanistic and deterministic reductions in Zeuthen [1930]. Other bargaining models and real-world situations, in contrast, reveal substantial levels of inequity. Von Stackelberg [1934], for example, predicts inequity that results from inherent bargaining advantages in a one-shot game. An important research avenue has been to study the dynamics of bargaining, to explain how equity and inequity evolve when cooperation is repeated.

Evolutionary models. When the cooperative environment is large and complex, the standard rationality and information assumptions are less pervasive. The evolutionary branch of bargaining is concerned with the bargaining dynamics amongst boundedly rational agents. It turns out that agents learn to share equally in a range of situations. Young [1993] develops the first evolutionary model of noncooperative bargaining.[15] In the model, pairs of agents are drawn at random from two populations of bargainers to play the Nash demand game. In order to best-respond, each player randomly samples demands from previous bargaining encounters. When the two populations are homogeneous and separate, the population evolves to a population of agents who play according to the Nash bargaining solution. With mixing between the two populations, the equal split is implemented. Other important evolu-

[15]In Nash's [1950a] Ph.D. thesis, a less formal proto-evolutionary model was presented.

tionary models of bargaining are Gale, Binmore & Samuelson [1995], Ellingsen [1997], Alexander & Skyrms [1999], Saez-Marti & Weibull [1999] and Binmore, Samuelson & Young [2003]. The existing models show how (even simple) evolutionary dynamics implement (more complex) cooperative solutions because cooperative and equitable norms are successful long-run strategies.

Our approach. We model behaviour where no process of hypothesis-formation concerning the other participants' actions enters an individual's decision-making explicitly. Behaviour is driven solely by adaptations of the individual aspirations. Each individual makes incremental corrections to his own aspirations and demands depending on whether he experienced success or failure in the previous period.

Our bargaining process differs from existing models in several respects. First, our model is cooperative in the sense that the players, although limited in their understanding of the cooperative process, interact directly by demanding shares from the gains of their cooperation instead of playing a related noncooperative game. Second, it is based on nonstrategic revisions instead of traditional response dynamics. Third, we explicitly study the evolution of equity using dispersion measures and relax convergence to being "close most of the time," (the dispersion measure will be close to equity in the Euclidean metric most of the long-run time). In this framework, long- and intermediate-run predictions are possible and the effects of shifts and shocks can be analysed directly. With growth in the possibility frontier, Kuznets' [1955] hypothesis of rising inequity followed by equity is mirrored during the adjustment process that ensues after periods of growth. Finally, outside options play a different role. In our model, outside options are the payoffs that players receive in periods of cooperative failure.[16] In existing models, outside options are viewed

[16]This makes explicit the view held in Binmore, Rubinstein & Wolinsky [1986] that outside options matter only when they are short- and intermediate-run alternatives.

as threat points that determine agents' bargaining powers.[17]

2.2 Evolutionary cooperative bargaining

A fixed population of players, $N = \{1, ..., n\}$, repeatedly bargains over a common resource. Each period, each player holds a demand, d_i^t, for a share of the surplus and receives a payoff consisting of his singleton base payoff, o_i, and a share of the surplus, ϕ_i^t. Demands, surplus shares and base payoffs are taken from the set of nonnegative real numbers. If demands are jointly feasible, an agreement is reached and each player gets his base payoff plus his demand. When demands are infeasible, no agreement is reached and each player is left with his base payoff.

Cooperative bargaining. *Cooperative bargaining can be expressed by its superadditive-cover cooperative game, $G(v, N)$, with singleton worths $v(i) = o_i$ for all players, inessential subcoalitions $v(C) = \sum_{i \in C} o_i$ for all $C \subset N$, and cooperative worth $v(N) > \sum_{i \in N} o_i$. The surplus, $v(N) - \sum_{i \in N} o_i$, is normalised to one.*

The core of any cooperative bargaining game, $G(v, N)$, is nonempty, consisting of any allocation that distributes the surplus Pareto-optimally. A particular allocation in the core is the equal split of the surplus, when each player i gets his base payoff, o_i, plus an equal share of the surplus, $\frac{1}{n}$.

Surplus shares. In our process, a cooperative agreement pays all players their demands if, and only if, demands are jointly feasible, otherwise, no cooperation

[17] If the bargaining process is such that players always reach an agreement, Binmore, Shaked & Sutton [1989] argue that threat points below equal split are incredible (because they yield payoffs below the outcome of Nash bargaining without outside options) and should, therefore, have no effect on the equilibrium outcome.

takes place and individuals remain singletons: for all players $i \in N$ in any period t, $\phi_i^t = d_i^t$ if $\sum_{i \in N} d_i^t \leq 1$, or else $\phi_i^t = 0$.

Satisfaction. When cooperation takes place, each player receives a surplus share. If that share matches or exceeds his demand, he becomes satisfied: i is *satisfied* in period t if the surplus is generated and $\phi_i^{t-1} \geq d_i^{t-1}$. Otherwise, when no surplus is generated and he does not receive his surplus share, he becomes *dissatisfied*.

Demand transitions. Each period, nature gives some player i (with probability $\frac{1}{n}$) the opportunity to revise his demand, $d_j^t = d_j^{t-1}$ for all players $j \neq i$ who are not selected.

The central assumption concerning the individual's demand-making behaviour is that it is determined by his own experience alone and that it differs according to whether the player is satisfied or dissatisfied. When a satisfied player is selected, he demands more than d_i^{t-1} at a *rate of experimentation*, or else d_i^{t-1} again. When a dissatisfied player is selected, he demands less than d_i^{t-1} dependant on his *degree of stickiness*, or else d_i^{t-1} again.

States. $Z^t = (\phi^t, d^t)$ is the state in period t, specifying a pair of vectors of surplus shares, $\phi^t = (\phi_1^t, ..., \phi_n^t)$, and demands, $d^t = (d_1^t, ..., d_n^t)$, for players 1 to n. Since, for any state Z in the set of all states Ω, the transition probability between any two, $\mathbb{P}(Z^{t+1} = Z | Z^t, Z^{t-1}, ..., Z^0)$, is $\mathbb{P}(Z^{t+1} = Z | Z^t)$, the process is Markov.

Measures of dispersion

We want to study the evolution of allocative equity. But, instead of tracking the individual positions at every stage, we focus directly on the evolution of a function that measures inequity in terms of the dispersion of payoffs in the population at each stage. Of course, this measure is informative of the level of social welfare when equity and welfare are related. We consider the variance-to-mean ratio, a common measure in applied statistics, also known as index of dispersion, dispersion index or coefficient of dispersion. The variance-to-mean ratio measures randomness of observed phenomena, here, of the relative spatial dispersion of the players' positions in terms of demands and payoffs.

Variance-to-Mean Ratio (VMR).

The variance-to-mean ratio of a state Z *with a mean of surplus shares of* $\mu = \frac{1}{n}\sum_{i\in N}\phi_i$ *and variance* $\sigma^2 = \frac{1}{n}\sum_{i\in N}(\phi_i - \mu)^2$ *is* $VMR(Z) = \frac{\sigma^2}{\mu}$.

At zero, the minimum of the VMR, the equal split is reached. When not all players get the same, the VMR is positive. It is larger the greater the dispersion of the payoffs. When a transfer from a relatively better-off to a relatively worse-off takes place, the VMR goes down.[18] For positive levels of inequity, not only rich-to-poor transfers but also equal additions to all incomes diminish the variance-to-mean ratio.[19]

We will show that, over time, the dynamics of the process induce internal mechanisms that lead to diminishing of the VMR via sequences of transfers that tend to increase equity more often than not. In the long run, the VMR settles at negligible levels of inequity with high probability, resulting in very equitable splits of the surplus most of the time.

Other important measures of the inequity of allocations are range, range ratio, the coefficient of variation and the Gini [1912] coefficient. These measures also provide information about the (in)equity of an allocation. In our procedures,

[18]The small transfer principle is due to Pigou [1912] and Dalton [1920]: given any "two income-receivers, and a transfer of income takes place from the richer to the poorer, inequality is diminished" unless the transfers more than "reverse the relative positions of the two income-receivers." (Dalton [1920], p. 351.)

[19]Incremental changes of larger demands are relatively smaller than incremental changes of smaller demands. The same incremental change made to all demands, therefore, increases the variance by less than the incremental change in the mean; the VMR goes down. This is another property identified in Dalton [1920].

changes in the allocation occur over time via small local transfers. We are interested in the evolution of (in)equity as driven by our stochastic process. Of particular interest for tracking the evolution of equity in our procedure are, therefore, inequity measures that are responsive to such transfers in between any of the players' positions. Rankings and ratio measures, for example, unaffected by most transfers in our process that do not change order or positions of the best- and worst-off players, even though inequity may have substantially increased or decreased in large intermediate parts of the population. The VMR is sensitive even to small local transfers and can be used to study the effects of the dynamics at each stage. Whilst alternative dispersion measures (such as the Gini coefficient and the coefficient of variation) could also be used to evaluate the expected changes in (in)equity at each stage, the VMR is the most convenient instrument for our purpose. Using the VMR, we can derive convenient mathematical formulas summarising the procedure's directed tendency towards equity as a function of the current level of inequity. We will refer to these directed expected changes as "drifts."

Drifts

In evolutionary biology, "genetic drift" (also known as the Sewall-Wright-effect) refers to the kind of evolution in allele frequencies that is governed by pure randomness instead of natural selection. When changes in frequency of certain alleles do not improve the fitness of a species, evolution is not directed but subject to random genetic drift instead. Genetic drift between two alleles in a population is modelled by a pure random walk and will ultimately (and randomly!) result in fixation of one and extinction of the other.

Natural selection, on the other hand, statistically favours allele frequencies that improve fitness of the organism. Natural selection between two alleles that are relevant for fitness in a population is modelled by a random walk with a statistical drift term that increases the expected proportion (mean frequency) of the favourable allele over time.

The confusion over terminology dates back to Sewall Wright's own research, who used "drift" to refer to the stochastic drift present in a directed stochastic model of natural selection in his early work but then, and more famously so, developed a theory of random genetic drift for random selection. We use the term drift to mean directed stochastic drift, i.e. "drifting towards" rather than drifting in an undirected way.

VMR-Drift.

> *Let $VMR(Z)$ be the variance-to-mean ratio of state Z. The drift in the variance-to-mean ratio is* $\mathbb{E}[\Delta(VMR(Z))|Z^t = Z] = \mathbb{E}[VMR(Z^{t+1})|Z^t = Z] - VMR(Z)$.

If $\mathbb{E}[\Delta(VMR(Z))|Z^t = Z] < 0$, there is an equity-drift, that is, inequity diminishes in expectation.[20]

2.3 Loss-averse behaviour

We develop a model based on loss aversion, interpreted in the sense that players who suffer a greater loss during cooperative failure are more likely to adjust their demands. Downward, stickiness of demands is decreasing in the aspiration level. Demand revisions are incremental.[21] We will show that, given any state with substantial levels of inequity, the process drifts towards more equal

[20]These drifts are easier to compute for the VMR than for other measures such as the Gini coefficient or range.

[21]Later in this chapter (pp. 39-42), proportionality of demand revisions is shown to reinforce tendencies towards equity. The model with incremental changes is set-up against the equity result from proportional models.

splits of the surplus. In the long run, the process splits the surplus almost equally most of the time.

The first model of loss aversion is Kahneman & Tversky's [1979] prospect theory. This decision theory was developed because "much experimental evidence indicates that choice depends on the status quo or reference level. [...] The central assumption of the theory is that losses and disadvantages have greater impact on preferences than gains and advantages."[22] Prospect theory features reference dependence (gains/ losses in utility are evaluated relative to a reference level), loss aversion (losses decrease utility more than gains of the same size increase utility), diminishing sensitivity (the agents are diminishingly sensitive to outcomes further away from the reference point) and probability weighting (the decision-maker overweighs small probabilities and underweighs large probabilities). Prospect theory has been experimentally tested and widely applied, especially in studies of the housing market, in finance and insurance, and in labour economics.[23] The equity premium puzzle, for example, is explained through reasonable levels of loss aversion, compared to unreasonably high levels of risk aversion in the classic framework.[24]

Here, instead of analysing the implications of loss aversion for economic behavior via reformulations of the utility function, we directly build reference-dependence and loss aversion into the learning procedures that govern the economic behaviour of the agents. We interpret loss aversion to mean that players who suffer a greater loss are more likely to reduce their demands when dissatisfied. Instead of via the formulation of expectations, loss aversion directly translates into decision-making and likely demand reductions, stickiness

[22] Kahneman & Tversky [1991], p. 1039.

[23] See DellaVigna [2009] Section 2.2 for an overview of prospect theory, its applications and for further references.

[24] Returns on risky equity shares consistently and substantially outperform bond returns implying extraordinarily high levels of risk-aversion. Standard utility models therefore struggle to explain the observed investment behaviour.

increases with every reduction.

This assumption is experimentally supported, that is, observed behaviour in bargaining experiments displays the kind of revisions we model.[25] Indeed, "the share offered is a significant factor with regard to acceptance or rejection - higher proposed shares are more likely to be accepted."[26] The higher the demand level and the longer the period of continued bargaining failure, the more likely is acceptance of a proposed cut.

Transitions

Demand transitions

Demands are adjusted by increments of size δ. When a player is invited to revise his demand (with probability $\frac{1}{n}$), his behaviour differs according to whether he is satisfied or not. r is the constant *rate of experimentation* with which a satisfied player demands a δ more. We assume r to be small. s_i^t is the *degree of stickiness* with which a dissatisfied player does not lower his demand when dissatisfied in period t. We assume s_i^t to be a constant function in time and to be the same for all players: $1 - s_i^t(\cdot) = f(\cdot)$ for all i and t with $f(0) = 0$, $f(\cdot) \in [0, 1]$ and $f(d) > f(d')$ if, and only if, $d > d'$. Typically, we assume stickiness not to be very small. The greater the loss relative to his singleton payoff, the more likely he is to reduce it by δ.

Interior and exterior states

State Z is *interior* when its demands are feasible, that is, when demands lie inside the Pareto frontier. Such states are of the form $((d_1, ..., d_n), (d_1, ..., d_n))$

[25]See the bilateral bargaining experiments by Tietz & Weber [1972], [1978], Tietz [1975], Weber [1976], Tietz, Weber, Vidmajer & Wentzel [1978], Tietz & Bartos [1983], Crössmann & Tietz [1983] and Tietz, Daus, Lautsch & Lotz [1988] as highlighted in Chapter One (pp. 11-13).

[26]Charness, Corominas-Bosch & Fréchette [2007], p. 46.

with $\sum_{i \in N} d_i \leq 1$. The set of interior states is Ω_I. The transitions from any $Z \in \Omega_I$ outwards to a state Z^i with demands $d^i = (d_1, ..., d_i + \delta, ..., d_n)$, where some player i makes a higher demand, has probability

$$\pi_{ZZ^i} = \begin{cases} \frac{r}{n} & \text{to any such } Z^i \text{ with } d^i = (d_1, ..., d_i + \delta, ..., d_n), \\ 1 - r & \text{when } Z^i = Z. \end{cases}$$

With probability $\frac{1}{n}$, a particular player i is selected, who then increases with probability r.

The states not in Ω_I are *exterior* and of the form $((0, ..., 0), (d_1, ..., d_n))$ with infeasible demands $\sum_{i \in N} d_i > 1$. The transitions from any $Z \notin \Omega_I$ inwards to a state Z^i with demands $d^i = (d_1, ..., d_j - \delta, ..., d_n)$, where some player j makes a lower demand, has probability

$$\pi_{ZZ^i} = \begin{cases} \frac{1}{n} f(d_i) & \text{to any such } Z^i \text{ with } d^i = (d_1, ..., d_j - \delta, ..., d_n), \\ 1 - \frac{1}{n} \sum_{i \in N} f(d_j) & \text{when } Z^i = Z. \end{cases}$$

With probability $\frac{1}{n}$, a particular player j is selected, who then reduces with probability $f(d_j)$.

Initial states, step sizes and transient states

Given any initial state Z^0 with demands $(d_1^0, ..., d_n^0)$, the future demands of any player lie on a δ-grid between zero and $\max\{d_i^0; 1 + \delta\}$ including the initial demand. For convenience, we choose a rational δ-grid of the form $\frac{1}{nk}$ for some $k \in \mathbb{N}$ and assume that all initial demands are taken as multiples of δ. These further assumptions ensure that all demands lie on the same δ-grid including all singleton payoffs, as well as zero and $\frac{1}{n}$. Because every demand vector implies a unique state, Ω is finite and we know a unique stationary equilibrium distribution exists.

Lemma 1. *Any state $Z \in \Omega$ with $\sum_{i \in N} d_i < 1$ or $> 1 + \delta$ is transient.*

Proof. of Lemma 1

From any interior state, the process exits with a positive probability in an outwards direction to larger demands, but not inwards to smaller demands. The direct neighbours of all interior states with $\sum_{i \in N} d_i < 1$ are also interior, whereas the interior states on the Pareto frontier with $\sum_{i \in N} d_i = 1$ have exterior neighbours with demands $\sum_{i \in N} d_i' = 1 + \delta$.

From any exterior state, the process exits with a positive probability in an inwards direction to smaller demands, but not outwards to larger demands. The direct neighbours of all exterior states with $\sum_{i \in N} d_i > 1 + \delta$ are also exterior, whereas an exterior state with $\sum_{i \in N} d_i' = 1 + \delta$ is the neighbour of interior states on the Pareto frontier with $\sum_{i \in N} d_i' = 1$.

This implies that all interior states with $\sum_{i \in N} d_i < 1$ (exterior states with $\sum_{i \in N} d_i' > 1 + \delta$) are transient because the process exits these states with a positive probability in an outward (inward) direction but, once left, they are never again reached. The states that are not transient are the interior states on the Pareto frontier with $\sum_{i \in N} d_i = 1$ and the exterior states with $\sum_{i \in N} d_i' = 1 + \delta$. $\square$

Drifting towards splitting the surplus

Adding the same amount to all demands reduces the VMR. Subtracting the same amount from all demands increases the VMR. When there is randomness regarding which player increases or decreases, the expected change in the VMR (drift) depends on the respective probabilities. Equiprobable increases cause negative drifts, equiprobable decreases cause positive drifts. When a transfer takes place, drifts are measured in two transitions (increase and decrease). In the embedded chain of states on the Pareto frontier, any two neighbours differ by a single transfer from one agent to another.

States on the Pareto frontier

Consider the chain of recurrent states on the Pareto frontier. Each such state Z with demands $(d_1, ..., d_n)$ is interior with a constant sum of demands $\sum_{i \in N} d_i = 1$. The neighbours of Z in the chain are states Z_{ij} with demands of the form $(d_1, ..., d_i + \delta, ..., d_j - \delta, ..., d_n)$: single transfers take place between neighbours in the chain; first some player i ups his demand to $d_i + \delta$ (causing demand infeasibility), then some player $j \neq i$ reduces his demand to $d_j - \delta$ (restoring feasibility), all other demands remain at their previous levels.

We know that the set of recurrent states on the Pareto frontier and their neighbours just outside is finite. Any split of the surplus on the Pareto frontier can be reached via a series of transfers from another and all are reached with positive probability. For some small rate of experimentation, r, we spend an expected long time, $\frac{1}{r}$, in each interior state. In between two neighbours in that chain, some time (negligible compared to the time spent in interior states) is spent in an exterior state. The two-period probability of transition between any two neighbours, Z and Z_{ij}, in that chain is

$$\pi_{ZZ_{ij}} = \frac{1}{n} r \times \frac{1}{n} f(d_j). \tag{2.1}$$

We will use these probabilities and treat transitions in the embedded chain that take at least two periods in the original chain as taking one single period. We indicate these times with hats ($\widehat{\ }$).

In the embedded chain, the drift in the VMR is

$$\mathbb{E}[\Delta(VMR(Z))|\widehat{Z^t} = Z] = 2r\delta\frac{1}{n}\sum_{i\in N} f(d_i)\{\delta\frac{n-1}{n} - [\frac{\sum_{i\in N} f(d_i)d_i}{\sum_{i\in N} f(d_i)} - \frac{1}{n}]\}. \tag{2.2}$$

If we consider $f(\cdot)$ to be a linear function of the form $f(d_i) = ad_i$ with some $a \in [0, 1]$ and we substitute $f(d_i) = ad_i$ for all i in Equation 2.2, we get

$$\mathbb{E}[\Delta(VMR(Z))|\widehat{Z^t} = Z] = 2ar\delta\frac{1}{n}[\delta\frac{n-1}{n} - VMR(Z)], \tag{2.3}$$

which is negative if, and only if,

$$VMR(Z) > \delta\frac{n-1}{n}. \tag{2.4}$$

When $VMR(Z) < \delta\frac{n-1}{n}$, the maximal change in the VMR in the embedded chain is $\delta^2\frac{n-1}{n}$. This change occurs when $VMR(Z) = 0$ initially, then someone increases and another decreases.

We, therefore, establish the truths of the following two facts:

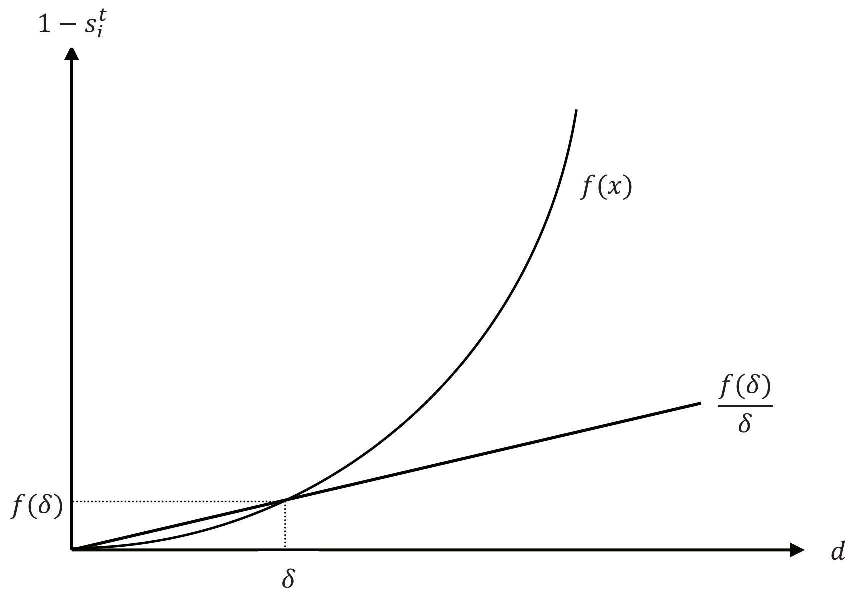
1.) **Negative drifts.** $VMR(Z) > \delta \Rightarrow \mathbb{E}[\Delta(VMR(Z))|\widehat{Z^t} = Z] < 0$

2.) **Small adverse tendencies.** $VMR(Z) \leq \delta \Rightarrow \Delta(VMR(Z)) < \delta$

The Appendix contains the calculations behind Equations 2.2 and 2.3.

We can view the linear function as an approximation for a more general function or as a lower bound for a function that first-order dominates the linear bound (e.g. a more convex function or a step function). Using ad_i with $a = \frac{f(\delta)}{\delta}$ for any convex function $f(\cdot)$ with $f(0) = 0$, $f'(x) > 0$ and $f''(x) \geq 0$ for all $x > 0$, for example, would only understate the real equity drifts but not alter the results.

Linear stickiness: The following graph illustrates linear stickiness derived from a convex function.



For a range of functional forms of $f(\cdot)$, no equity bias exists. With a constant function $f(d_i) = 1 - s$ for some small s, for example, increases for all satisfied players are equiprobable and decreases for all dissatisfied players are equiprobable. Away from zero, rich-to-poor and poor-to-rich transfers are then equiprobable and the process will not drift towards the equal split. Worst still, if $f(\cdot)$ is such that $f(d) < f(d')$ if, and only if, $d > d'$ for all positive levels of d, d', that is, if stickiness is increasing in aspirations (no loss aversion), an inequity bias is revealed. Poor-to-rich transfers become more likely than rich-to-poor transfers and some individual may ultimately end up with the whole surplus. The experimental evidence, however, suggests that loss-averse behaviour and diminishing stickiness are the behavioural norm.[27]

Results

For any cooperative bargaining game, our procedures lead to outcomes with very low levels of inequity (as measured by the VMR) most of the long-run time. This implies that we will be close to the equal split in the Euclidean metric with high probability in the long run. The result follows from the two above facts. First, we have negative drifts for VMRs greater than δ. Second, adverse tendencies in VMRs smaller than δ don't leave the 2δ-neighbourhood. In expectation, the VMR will, therefore, decrease and settle close to zero with high probability. The formal statement of the convergence is divided into the following arguments:

Theorem 2. *In the embedded chain of exterior states, for any game with step size δ and the same linear stickiness, relative inequity, as measured by the VMR, will in expectation be less than 2δ after a finite time.*

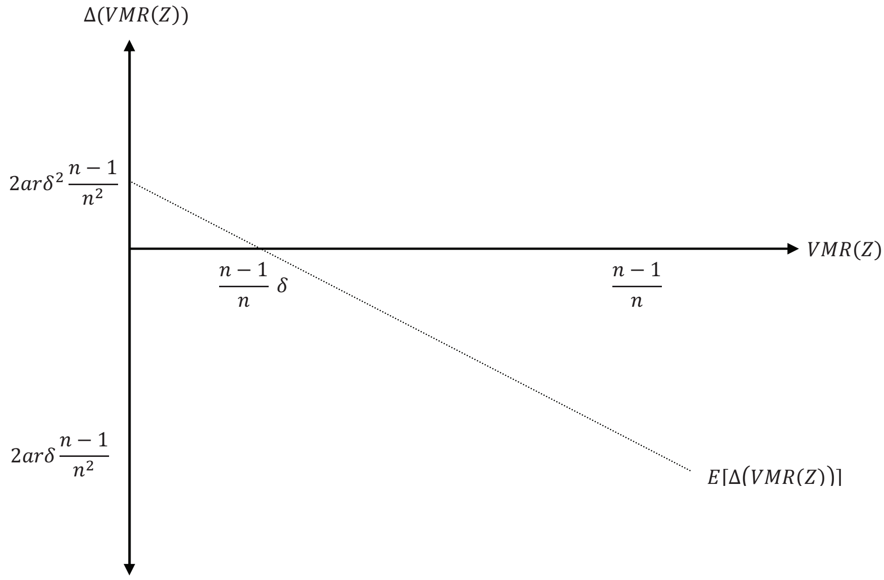
Corollary 3. *For any small level of relative inequity β and for any large probability $1 - \gamma$, a class of games with small enough step sizes exists, for which the probability of being more inequitable than β is less than γ.*

Proposition 4. *The expected hitting times of the original chain necessary to reach the Pareto frontier are derived.*

The Appendix contains the proofs.

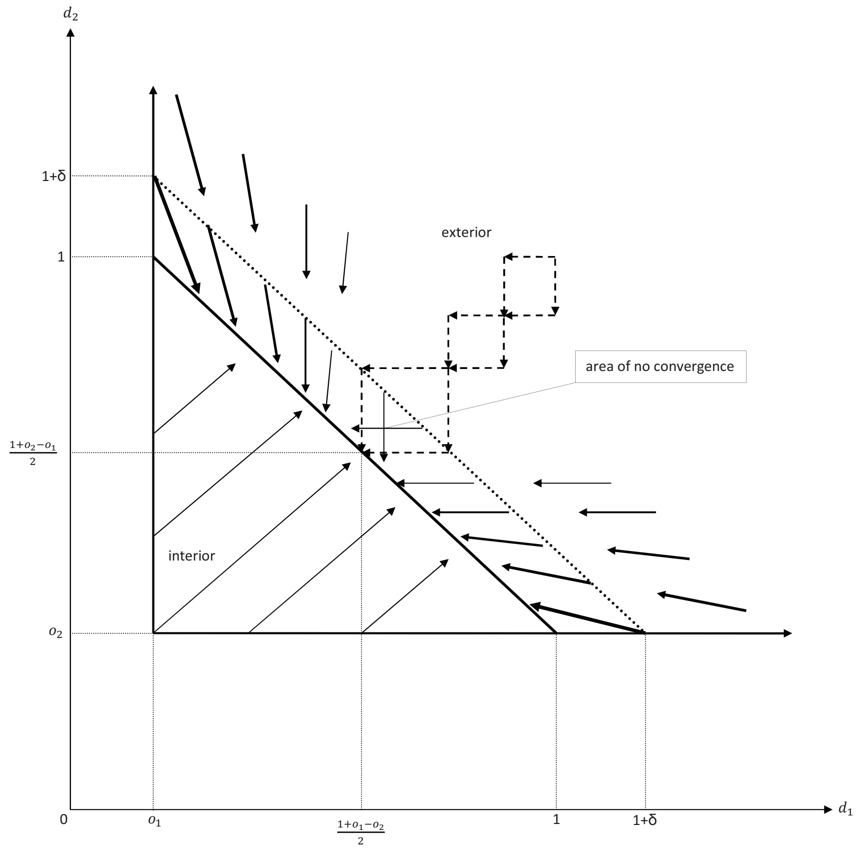
[27] See the discussion in Chapter One (pp. 11-13).

Drift graphic (1): The following graph illustrates drifts when $f(d_i) = ad_i$ for all players.



As seen in the graph, the drifts are positive, if, and only if, the VMR is very small (less than $\frac{n-1}{n}\delta$). Otherwise, they are negative. The drifts in the area where they are positive are small compared to the large negative drifts that exist for larger values of the VMR.

Drift graphic (2): For two players, the following graph illustrates the process.



The process moves above the singleton payoffs. Interior states below the Pareto frontier are transient, expected movement is outwards along 45-degree rays towards the Pareto frontier (like Raiffa's linear negotiation curves). In the external region, the process tends inwards and towards an equal split of the surplus (the zigzags illustrate Zeuthen's paths). In the long run, the process moves between interior states with sums of demands equal to one (fat diagonal) and exterior states with sums of demands equal to $1+\delta$ (dashed diagonal). Mass concentrates around the equal split of the surplus.

Recovering Zeuthen's bilateral bargaining

Zeuthen [1930] models bargaining as a sequence of mutual concessions. The process starts with each player demanding the whole surplus, proposing to leave nothing to the other. As bargaining continues, players make mutual concessions based on their relative willingness to risk conflict. At any moment in time, the party with the lower willingness to accept breakdown, which is the party with the higher demand in the symmetric two-player bargaining game, adjusts his demand to a slightly smaller demand with probability one.

In any arbitrary period t, the demands of players 1 and 2 are d_1^t and d_2^t, their respective probabilistic willingness to risk conflict n_1^t and n_2^t.[28] In the model, any one i of the players accepts the other's proposal d_{-i}^t in that period if, and only if, $1 - d_{-i}^t \geq n_i^t d_i^t$. i is indifferent between accepting and rejecting when $n_i^t = \frac{1-d_{-i}^t}{d_i^t}$. If both demands are so high that $1 - d_2^t < n_1^t d_1^t$ and $1 - d_1^t < n_2^t d_2^t$, nobody accepts the other's offer and we move to the next period. In period $t+1$, the party with the lower willingness to accept breakdown (the player with the lower willingness to risk conflict), adjusts his demand by decreasing his demand by some small amount δ. This continues until a proposal is accepted.

Consider a symmetric bargaining situation and suppose $n_1^{t+1} > n_2^{t+1}$ in period $t+1$. In period $t+1$, player 1 reduces his demand to $d_1^{t+1} = d_1^t - \delta$, resulting in $n_1^{t+1} = \frac{1-d_2^t}{d_1^t - \delta} > n_1^t$ and $n_2^{t+1} = \frac{1-d_1^t+\delta}{d_2^t} < n_2^t$. 2 accepts 1's proposal d_1^{t+1} if, and only if, $1 - d_1^{t+1} \geq n_2^{t+1} d_2^{t+1}$. If 2 does not accept the offer, the process moves to period $t+2$. Concessions continue and alternate until, in some period T, $n_1^T = n_2^T$, at which point the players split the pie equally and bargaining ends.

Note that, in evaluating whether to accept or reject a proposal, each player

[28] The demands are public and the respective willingnesses to risk conflict are basic ingredients of the model.

follows a decision rule without forming a formal expectation of what is to happen in future periods. Otherwise, unless that period was the last period, the conceived payoff from rejecting would have included another term.[29] Zeuthen formulates a mechanistic revision-protocol instead of a standard optimisation model.

Furthermore, note that, for symmetric bargaining games, the assumption that the side with the higher demand reduces with probability one coincides with a limiting case of our process. In our model, we assume that, during bargaining breakdown, the player with the higher demand (the higher loss) reduces with a larger probability than the other player. If we set that probability to one, we would, as in Zeuthen, move to the equal split from where both players demand everything. Instead of stopping once demands are feasible for the first time, however, our process continues and implements equal split of the surplus in a zonal rather than pinpoint way: continuing its "drifting towards [...] area around the middle in which no party is substantially more eager to secure an agreement than the other. Establishing the existence of such centripetal forces - powerful around the edges of the bargaining area but weaker towards the middle."[30]

2.4 Proportional demand-revisions

For many situations, when the size of change is positively related to a current level, proportionality of demand-revisions may be a better assumption than constant incremental changes. Throttling the extraction of water or of water steam, for example, may result in proportional rather than incremental cuts. The depletion of fishing grounds too may result in proportional rather than

[29]See Brems [1968], pp. 232-233 for a discussion of Zeuthen's original model and Brems' strategic reinterpretation of the model.

[30]Brems [1976], p. 404

incremental decreases in catch. In intrafirm wage-bargaining, wage-changes are also best modelled with proportionality: hourly wages of labourers are renegotiated by smaller amounts than managerial salaries.[31]

We will illustrate the effects of proportionality in bilateral bargaining with outside options normalised to zero. We find that, when increases and reductions are modelled as uniformly proportional changes, the procedures bear stronger inherent drifts towards equity than before, reinforcing convergence. In fact, the drifts due to proportional demand-revisions alone are strong enough, that, even in the absence of loss aversion, outcomes close to the equal split are implemented most of the time.

Proportional demand-revisions. A satisfied player occasionally demands $(1+\delta)$ times his previous demand when selected, a dissatisfied player is likely to lower his demand to $(1-\delta)$ times his previous demand.

We measure inequity between a pair of demands x and y (w.l.o.g. say $x \geq y$) by their coefficient of variation (CV).[32]

Coefficient of Variation (CV).

> *Between any two demands x and y in any state Z (where $x \geq y$), the coefficient of variation is $CV(Z) = \frac{x-y}{x+y}$.*[33]

Note that a uniform proportional change in both positions would leave the CV unchanged. In absolute terms, however, the gap between x and y widens with

[31]Uniform proportionality is consistent with observed wage structures, as compiled in Bewley's [1999] extensive study. Bewley finds that general pay raises are usually proportional. (See p. 154 from section 10.1. "Type and Size of Raises.") Furthermore, smaller wages of salaried employees are usually downward rigid, whereas managerial wages are usually not downward rigid. (See p. 72 from section 6.1. "Types of Internal Pay Structure.") The assumption of uniformly proportional raises and cuts, therefore, underestimates the upward-movement and downward-rigidity in pay of the worse-off, thus, working against the obtained results regarding equity from the analyst's point of view.

[32]The coefficient of variation (also known as relative standard deviation, unitised risk or variation coefficient) is another common (dimensionless) dispersion measure.

[33]For an arbitrary number of players n, $CV(Z) = \frac{\sigma}{\mu}$. For $n = 2$, $\frac{\sigma}{\mu} = \frac{\sqrt{\frac{1}{2}[(x-\frac{x+y}{2})^2+(y-\frac{x+y}{2})^2]}}{\frac{x+y}{2}} = \sqrt{\frac{1}{2}}\frac{\sqrt{(\frac{x-y}{2})^2}}{\frac{x+y}{2}} = \sqrt{\frac{1}{2}}\frac{x-y}{x+y}$. We suppress the $\sqrt{\frac{1}{2}}$ for simplicity.

increases because the absolute change in the larger demand is larger than the change in the smaller demand. When there is randomness and the revisions by all players are equiprobable, we find a negative drift in CV in any (interior or exterior) state, unless the changes reverse their relative positions.

Drifts

For any state $Z \in \Omega_I$ with demands $x + y \leq 1$, both players are satisfied and receive their demands. When a satisfied player is selected, he will, with a constant positive probability, demand proportionally more. When the increase in the smaller demand does not reverse the players' relative position ($x > (1+\delta)y$), $CV(Z) > \frac{\delta}{2+\delta}$. For any $Z \in \Omega_I$ with $CV(Z) > \frac{\delta}{2+\delta}$,

$$\mathbb{E}[\Delta(CV(Z^t))|Z^t = Z] < -\frac{r}{2}\,\frac{\delta^2}{1+(1+\delta)^2}\,CV(Z) < 0. \tag{2.5}$$

For any state $Z \notin \Omega_I$ with demands $x + y > 1$, both players are dissatisfied. When a dissatisfied player is selected, the probability of demand-reduction is increasing in loss (as before). When the reduction in the larger demand does not reverse the players' relative position ($(1-\delta)x > y$), $CV(Z) > \frac{\delta}{2-\delta}$. For any $Z \notin \Omega_I$ with $CV(Z) > \frac{\delta}{2-\delta}$,

$$\mathbb{E}[\Delta(CV(Z^t))|Z^t = Z] < -\frac{f(x)+f(y)}{4}\,\frac{\delta^2}{1+(1-\delta)^2}\,CV(Z) < 0. \tag{2.6}$$

We note that both drifts are negative when $(1-\delta)x > y$. For any state Z (interior or exterior) with $CV(Z) > \frac{\delta}{2-\delta}$, therefore, $\mathbb{E}[\Delta(CV(Z^t))|Z^t = Z] < 0$ and stronger the higher current $CV(Z)$.

When x and y are closer together and the CV is too small, drifts turns positive. Both interior and exterior states with demands as close together as $(1-\delta)x < y \leq x$ may have adverse positive drifts. A single-period positive change never exceeds

$$\Delta(CV(Z)) \leq \frac{\delta}{2-\delta}. \tag{2.7}$$

As before, we therefore have the following two facts:

1.) **Negative drifts.** $VMR(Z) > \delta \Rightarrow \mathbb{E}[\Delta(VMR(Z))|\widehat{Z^t} = Z] < 0$

2.) **Small adverse tendencies.** $VMR(Z) \leq \delta \Rightarrow \Delta(VMR(Z)) < \delta$

Even without the effects of loss aversion, when demand increases and reductions are made with constant probability, the above holds.[34] Proportionality alone suffices to create drifts strong enough to implement the equal split.

Aside: See the Appendix for the drift calculations and for additional arguments.

2.5 Shifts and shocks

We now turn to the long- and intermediate-run effects on equity of shifts in the Pareto frontier and of shocks to outside options. At the beginning of period t, before the shift or shock takes place, state Z^t is such that $\sum_{i\in N} d_i^t = 1 = v(N) - \sum_{i\in} o_i$ and all players are satisfied receiving $\phi_i^t = d_i^t$.

Growth of Pareto frontier

The possibility frontier shifts from $v(N)$ to $v(N) + s$ or $v(N) - s$.[35]

In the long run, the surplus will still be split equally and each player's long-run allocation will either increase or decrease by $\frac{s}{n}$, depending on the direction of the shift. Long-run equity in in splitting the surplus will remain unaffected.

[34] $\frac{f(x)+f(y)}{4}$ in Equation 2.6 becomes $\frac{1-s}{2}$ when $f(\cdot) = 1 - s$ constantly.

[35] In order for this shift to affect the process and to preserve the grid on which me move, we assume s to be a multiple of $n\delta$.

In the intermediate-run, however, the effects are considerably more complex and depend on the direction of shift and on the underlying type of demand-revisions.

Let us consider the effects of a positive shift first. When demand-revisions are proportional, inequity in absolute terms is likely to increase: agents with higher demands would demand larger additional shares, which are now feasible due to the outward shift of the possibility frontier. Absolute inequity therefore increases in expectation until the new Pareto frontier is reached and the corrective drifts towards equity are again effective. This offers a new interpretation of Kuznets' [1955] hypothesis that, as income grows, inequity first rises and then falls. Here, the reason for that phenomenon is the superior ability of better-off players to add more onto their (already better) positions in the intermediate-run periods of growth. This is reversed as the Pareto frontier is again reached and bargaining becomes a purely redistributive task. Then, rich-to-poor transfers become more likely than poor-to-rich transfers. In relative terms, expected inequity with proportional demand revisions remains unaffected. When demand-revisions are incremental, on the other hand, the outward movements are equiprobable but relatively smaller for larger demands which leads to no expected change in absolute inequity but a decrease in expected relative inequity.

A negative shift renders current demands infeasible. Loss aversion means that players with higher demands are more likely to reduce. Hence, absolute and relative inequity are expected to decrease. When demand-revisions are proportional, absolute inequity would decrease even in the absence of loss aversion, relative inequity would remain unaffected.

Shock to outside option

An individual outside option changes from o_i to $o_i + s$ or $o_i - s$.

In the long run, two effects take place. On the one hand, player i's base payoff changes by s, which, on the other hand, causes the same change in surplus available to the public. Let us suppose that the surplus was split equally before the shock took place. In the long run after the shock, the available surplus will be split equally again so that the change in available surplus is split equally amongst everybody too, whereas the change in base payoff affects i alone. Hence, i either enjoys an increase or suffers a loss of $s - \frac{s}{n}$ as a result of change in his outside option. All others' surplus shares change by $\frac{s}{n}$. Depending on whether i has a relatively high or low outside option, this may be viewed to represent a rise or fall in equity overall.

In the intermediate run, the effects of changes in players' outside options depend on their current position. Note that the long-run effects on all other players but i are the same whether due to a change in the Pareto frontier or in i's outside option. A change in i's outside option, however, leaves the location of the possibility frontier unaffected. Player i, whose outside option has changed, however, will now behave differently during cooperative failure: he will be more likely to reduce his demand than before if his outside option fell, less likely to reduce his demand if his outside option increased. For now, the others behave as before. Since the changes in loss aversion are marginal and effective only if i is called upon to revise (with probability $\frac{1}{n}$), the adjustment processes resulting from shocks to outside options are much slower than those resulting from growth in the possibility frontier.

2.6 Concluding remarks

A number of interesting features of the procedures should be noted. First, instead of analysing the positions of all players in every state, we use simple summary statistics to measure inequity in terms of dispersion. Negative drifts in these statistics are revealed whenever current inequity is substantial. Second, instead of establishing convergence by taking the probability of error or of a shock variable down to zero and then analysing the welfare properties of the implemented allocations, we track the evolution of dispersion measures directly. Long- and intermediate-run trends concerning equity are explicitly studied. Third, convergence of the dispersion measure is into a small equity-neighbourhood around the equal surplus split. That convergence is close most of the time rather than pinpoint: in equilibrium, the probability of being outside that neighbourhood is small but remains positive. Fourth, the equity results are robust for different procedures featuring loss aversion, incremental and proportional demand-revisions.

A forte of the proof techniques employed in this chapter lies in their ability to give good bounds for waiting times that are easy to interpret. Experimentation and stickiness jointly determine waiting times and closeness of the implementation. Both small rate of experimentation and large stickiness result in longer waiting times. The rate of experimentation is more important for waiting times because more time is spent in interior states. The effects of larger rates of experimentation dominate those of smaller degrees of stickiness out of equilibrium, so that a larger rate of experimentation is more important for fast movement into a neighbourhood (then obviously larger) around the equal split. Very small rates of experimentation and very large degrees of stickiness together can make the convergence to equal split arbitrarily close but waiting times very long. One natural avenue is to explore the interplay and tradeoff between rate of experimentation and stickiness in more detail and to estimate waiting times for parameter values consistent with data.

In the presence of shifts of the Pareto frontier and shocks to outside options, we noted differences in relative and absolute convergence. Temporarily conflicting tendencies concerning the evolution of equity may result in the intermediate run, depending on the underlying types of demand-revisions. It seems a promising avenue to pursue this distinction in more detail and to evaluate empirical trends in light of the models' implications. In particular, the consequences of growth can be tested depending on the types of adjustment: on average, proportional revisions should cause a temporary increase in equity, incremental revisions should cause a temporary decrease in equity.

A natural extension of the framework is to allow for the formation of coalitions other than the grand coalition, for cooperation to continue or fail, globally and locally.

Appendix

Mathematical results under loss aversion

Drift calculations

Equation 2.2.

From any recurrent interior state Z with $\sum_{i\in N} d_i = 1$, we move to some other Z_{ij} or come back to the starting Z with probability $rf(d_j)\frac{1}{n^2}$. With probability $1-\sum_{j\neq i} rf(d_j)\frac{1}{n^2}$, we stay in Z. The next expected sum of squares of demands in the embedded chain is therefore $\mathbb{E}[\sum_{i\in N}(\widehat{d_i^{t+1}})^2|\widehat{Z^t} = Z] =$

$\frac{r}{n^2}\sum_i\{\sum_{j\neq i} f(d_j)([d_i+\delta]^2+[d_j-\delta]^2+\sum_{k\neq i,j} d_k^2)\}+(1-\sum_{j\neq i} rf(d_j)\frac{1}{n^2})\sum_i d_i^2.$

Expanding the squares, this becomes

$$\frac{r}{n^2}\sum_i\{\sum_{j\neq i} f(d_j)(\sum_k d_k^2 2\delta^2+2\delta[d_i-d_j]\}+(1-\sum_{j\neq i} rf(d_j)\frac{1}{n^2})\sum_i d_i^2,$$

which is

$$\sum_i d_i^2+2\delta r\frac{1}{n}\sum_i f(d_i)[\frac{n-1}{n}\delta-(\frac{\sum_i f(d_i)d_i}{\sum_i f(d_i)}-\frac{\sum_i d_i}{n})].$$

Equation 2.3.

Substituting $f(d_i)=ad_i$ in Equation 2.2, the drift in the sum of squares of demands is $\mathbb{E}[\Delta(\sum_{i\in N}(\widehat{d_i^{t+1}})^2)|Z^{\hat{t}}=Z]=$

$$2\frac{ar\delta}{n}\sum_i d_i[\delta\frac{n-1}{n}-(\frac{\sum_i d_i^2}{\sum_i d_i}-\frac{\sum_i d_i}{n})]=2\frac{ar\delta}{n}\sum_i d_i[\delta\frac{n-1}{n}-VMR(Z)],$$

which is also the drift in the VMR because $\Delta(\sum_{i\in N} d_i^2)=\Delta(VMR(Z))$ in the embedded chain as

$$VMR(Z)=\frac{\sum_{i\in N}(d_i-\frac{1}{n})^2}{1}=\sum_{i\in N} d_i^2-\frac{1}{n}$$

when $\sum_{i\in N} d_i=1$.

Convergence results

Theorem 2. *For any game with step size $\delta>0$ and the same linear stickiness, there exists a time $\widehat{T}$ such that, for every $\hat{t}>\widehat{T}$ and for all Z^0 with $\sum_{i\in N} d_i^0=1$,*

$$\mathbb{E}[VMR(Z^{\hat{t}})|Z^0]\leq 2\delta. \tag{2.8}$$

Proof. Note that we start and move in the embedded chain of interior states on the Pareto frontier with times $\hat{t}$. We prove this theorem in two steps. First, we prove that, from any state Z in the embedded chain with $VMR(Z)\leq 2\delta$, all expected future values of the coefficient of variation are less than 2δ, and that, from any state with $VMR(Z) > 2\delta$, all expected future values of the coefficient of variation are less than $VMR(Z)$. Second, we prove that, for any initial state Z^0 on the Pareto frontier, it takes at most time $\widehat{T}$ for the expected VMR to be less than 2δ. Jointly, these two facts imply that, in the embedded chain of exterior chain, for any initial $Z^0 \notin \Omega_I$, $\mathbb{E}[VMR(Z^{\widehat{t}})|Z^0] \leq 2\delta$ after time $\widehat{T}$.

Step 1. Expression 2.3 is negative for all states $Z^{\widehat{T}} = Z$ with $VMR(Z) > \delta\frac{n-1}{n}$. If $Z^{\widehat{T}} = Z$ is such that $VMR(Z) \leq \delta\frac{n-1}{n} < \delta$, a maximum $\Delta(VMR(Z)) = \delta^2\frac{n-1}{n} < \delta$ may occur in one period and, thus, result in a $VMR(Z^{\widehat{T+1}})$ no larger than 2δ.

Hence, for any state Z with $VMR(Z) > 2\delta$, it is true that, for all $\widehat{t'} > \widehat{t}$,

$$\mathbb{E}[VMR(Z^{\widehat{t'}})|Z^{\widehat{t}} = Z] < VMR(Z). \tag{2.9}$$

For any state Z with $VMR(Z) \leq 2\delta$, it is true that, for all $\widehat{t'} > \widehat{t}$,

$$\mathbb{E}[VMR(Z^{\widehat{t'}})|Z^{\widehat{t}} = Z] < 2\delta. \tag{2.10}$$

Step 2. We now prove that $\mathbb{E}[VMR(Z^{\widehat{T}})|Z^0] \leq 2\delta$ indeed holds for all $\widehat{t} > \widehat{T}$ and for any Z^0 on the Pareto frontier. Note that $\mathbb{E}[\Delta(VMR(Z))|Z^{\widehat{t}} = Z] < 0$ when $VMR(Z) = 2\delta$. Hence, for any state Z with $\frac{VMR(Z)}{2\delta} > 1$, $\mathbb{E}[\Delta(VMR(Z))|Z^{\widehat{t}} = Z] = 2\frac{ar\delta}{n}[\delta\frac{n-1}{n} - VMR(Z)] < -ar\delta\frac{n+1}{n^2}$. Writing $c \equiv ar\delta\frac{n+1}{n^2}$, we obtain the expression

$$\mathbb{E}[VMR(Z^{\widehat{t+1}})|Z^{\widehat{t}} = Z] \leq VMR(Z) - c, \tag{2.11}$$

for any Z in the embedded chain on the frontier with $VMR(Z) > 2\delta$. Substitution in Equation 2.11 yields

$$\begin{aligned}\mathbb{E}[VMR(Z^{\widehat{t}})|Z^0] &= \mathbb{E}[\mathbb{E}[VMR(Z^{\widehat{t}})|Z^{\widehat{t-1}}]|Z_0] \\ &\leq \max\{\mathbb{E}[VMR(Z^{\widehat{t-1}}) - c|Z^0], 2\delta\}.\end{aligned} \tag{2.12}$$

As long as $\mathbb{E}[VMR(Z^{\widehat{t-1}}) - c|Z^0] > 2\delta$, we iterate Expression 2.12 repeatedly forward to obtain

$$\mathbb{E}[VMR(Z^{\widehat{t}})|Z^0] \leq \max\{VMR(Z^0) - c\widehat{t}; 2\delta\}, \tag{2.13}$$

which is less than or equal to 2δ for every $\widehat{t} > \widehat{T}$ when $\widehat{T} \geq \frac{1}{c}(1-2\delta) \geq \frac{1}{c}(VMR(Z^0) - 2\delta)$. □

Note that the size of a, which measures the degree of stickiness, determines the speed of convergence but leaves its precision unaffected.

Corollary 3. *Given any $\beta, \gamma > 0$, for any game with step size $\delta \leq \frac{\beta\gamma}{2}$, the above time $\widehat{T}$ is such that, for every $\widehat{t} > \widehat{T}$,*

$$\mathbb{P}(VMR(Z^{\widehat{t}}) < \beta) > 1 - \gamma. \tag{2.14}$$

Proof. For any $\beta \in (0,1]$ and Z^0 with $\sum_{i\in N} d_i^0 = 1$, Theorem 2 implies that $\mathbb{P}([VMR(Z^{\widehat{t}}|Z^0] \geq \beta) \times \beta + \mathbb{P}([VMR(Z^{\widehat{t}})|Z^0] < \beta) \times 0 \leq \mathbb{E}[VMR(Z^{\widehat{t}})|Z^0]$ for any $\widehat{t} > \widehat{T}$. Rearranged, it follows that, for any Z^0 on the frontier,

$$\mathbb{P}([VMR(Z^{\widehat{t}})|Z^0] \geq \beta) \leq \frac{\mathbb{E}[VMR(Z^{\widehat{t}})|Z^0]}{\beta} \leq \frac{2\delta}{\beta} \leq \gamma, \tag{2.15}$$

which holds for any $\beta > 0$ and $\gamma > 0$ by appropriate choices of $\delta \leq \frac{\beta\gamma}{2}$ and

$\widehat{T} \geq \frac{1}{c}(1-2\delta)$.[36] $\square$

For the embedded chain of exterior states, Corollary 3 establishes the proportion with which the demands have small variance-over-mean ratios after time $\widehat{T}$. It remains to give the time it may take to reach the Pareto frontier. We do this in Proposition 4.

Proposition 4. *For any game with step size $\delta > 0$, there is a time T' of the form $\frac{1}{r\delta}$ such that, for every $t > T'$ and for all $Z^0 \in \Omega_I$,*

$$\mathbb{E}[\sum_{i \in N} d_i^t | Z^0] \geq 1. \tag{2.16}$$

Similarly, for every $t > T''$ and for all $Z^0 \notin \Omega_I$,

$$\mathbb{E}[\sum_{i \in N} d_i^t | Z^0] \leq 1. \tag{2.17}$$

Proof. From Lemma 1, we know that all states with $\sum_{i \in N} d_i > 1 + \delta$ and < 1 are transient. For any state $Z \in \Omega_I$, $\mathbb{E}[\sum_{i \in N} d_i^{t+1} | Z^t = Z] = \sum_{i \in N} d_i^t + r\delta$. Starting at $Z^0 \in \Omega_I$, therefore, $\mathbb{E}[\sum_{i \in N} d_i^t] \geq \min\{1; \sum_{i \in N} d_i^0 + tr\delta\}$, which exceeds 1 after $t > T' = \frac{1}{r\delta}$.

For any state $Z \notin \Omega_I$ with $\sum_{i \in N} d_i > 1+\delta$, $\mathbb{E}[\sum_{i \in N} d_i^{t+1} | Z^t = Z] = \sum_{i \in N} d_i^t - a\delta \sum_{i \in N} \frac{1}{n} d_i^t$. Starting at $Z^0 \notin \Omega_I$, therefore, $\mathbb{E}[\sum_{i \in N} d_i^t] \leq \max\{1+\delta; \sum_{i \in N} d_i^0 - t\frac{a\delta}{n}\}$, which falls below $1 + \delta$ after $t > T'' = \frac{n}{a\delta}$. $\square$

[36] Note Equation 2.15 is the Markov inequality.

Mathematical results with proportional demand-revisions

Drift calculations

Equation 2.5.

When $Z \in \Omega_I$ and $x > (1+\delta)y$, $\frac{x-y}{x+y} > \frac{\delta}{2+\delta}$ and $\mathbb{E}[\Delta(CV(Z^t))|Z^t = Z] =$

$$\frac{r}{2}\frac{(1+\delta)x-y}{(1+\delta)x+y} + \frac{r}{2}\frac{x-(1+\delta)y}{x+(1+\delta)y} - r\frac{x-y}{x+y} = -r\frac{xy\delta^2}{((x+(1+\delta)y)(x(1+\delta)+y))}\frac{x-y}{x+y}.$$

When $Z \in \Omega_I$ and $x = (1+\delta)y$, $\frac{x-y}{x+y} = \frac{\delta}{2+\delta}$ and $\mathbb{E}[\Delta(CV(Z^t))|Z^t = Z] =$

$$\frac{r}{2}\frac{(1+\delta)x-y}{(1+\delta)x+y} - r\frac{x-y}{x+y} = -\frac{r}{2}\frac{\delta^2}{1+(1+\delta)^2}\frac{\delta}{2+\delta}.$$

Equation 2.6.

When $Z \notin \Omega_I$ and $(1-\delta)x > y$, $\frac{x-y}{x+y} > \frac{\delta}{2-\delta}$ and $\mathbb{E}[\Delta(CV(Z^t))|Z^t = Z] =$

$$\frac{1}{2}f(x)\frac{(1-\delta)x-y}{(1-\delta)x+y}+\frac{1}{2}f(y)\frac{x-(1-\delta)y}{x+(1-\delta)y}-\frac{f(x)+f(y)}{2}\frac{x-y}{x+y} < -\frac{f(x)+f(y)}{2}\frac{xy\delta^2}{((x+(1-\delta)y)(x(1-\delta)+y))}\frac{x-y}{x+y}$$

because $f(x) > f(y)$ and $\frac{(1-\delta)x-y}{(1-\delta)x+y} < \frac{x-(1-\delta)y}{x+(1-\delta)y}$.

When $Z \notin \Omega_I$ and $(1-\delta)x = y$ (which implies $x > (1+\delta)y$), $\frac{x-y}{x+y} = \frac{\delta}{2-\delta}$ and

$$\mathbb{E}[\Delta(CV(Z^t))|Z^t = Z] = \frac{1}{2}f(y)\frac{x-(1-\delta)y}{x+(1-\delta)y} - \frac{f(x)+f(y)}{2}\frac{x-y}{x+y} < -\frac{f(x)+f(y)}{2}\frac{\delta^2}{(1-\delta)^2+1}\frac{\delta}{2-\delta}.$$

Equation 2.7.

When $Z \in \Omega_I$ and $x = y$, $\Delta(CV(Z)) \leq \frac{(1+\delta)x-x}{(1+\delta)x+x} = \frac{\delta}{2+\delta}$.

When $Z \notin \Omega_I$ and $x = y$, $\Delta(CV(Z)) \leq \frac{x-(1-\delta)x}{x+(1-\delta)x} = \frac{\delta}{2-\delta}$.

Convergence results

Say the process starts at some Z^0 with strictly positive demands $(d_1^0, ..., d_n^0)$. Any future demand d_i^t of a player will be of the form $(1+\delta)^q(1-\delta)^r d_i^0$ for some positive integers q and r, counting the player's past demand increases in periods of satisfaction and the demand reductions in periods of dissatisfaction. Any such d_i^t will be a positive real number, smaller than or equal to $\max\{d_i^0, 1+\delta\}$. Because every demand vector implies a unique state, Ω is a countable set.

Lemma 5. *All exterior states $Z \notin \Omega_I$ such that*

$$y > \max\{(1+\delta) - (1+\delta)x, 1 - \frac{x}{1+\delta}\}$$

and all interior states $Z \in \Omega_I$ with

$$y \leq \min\{(1-\delta) - (1-\delta)x, 1 - \frac{x}{1-\delta}\}.$$

are transient states.

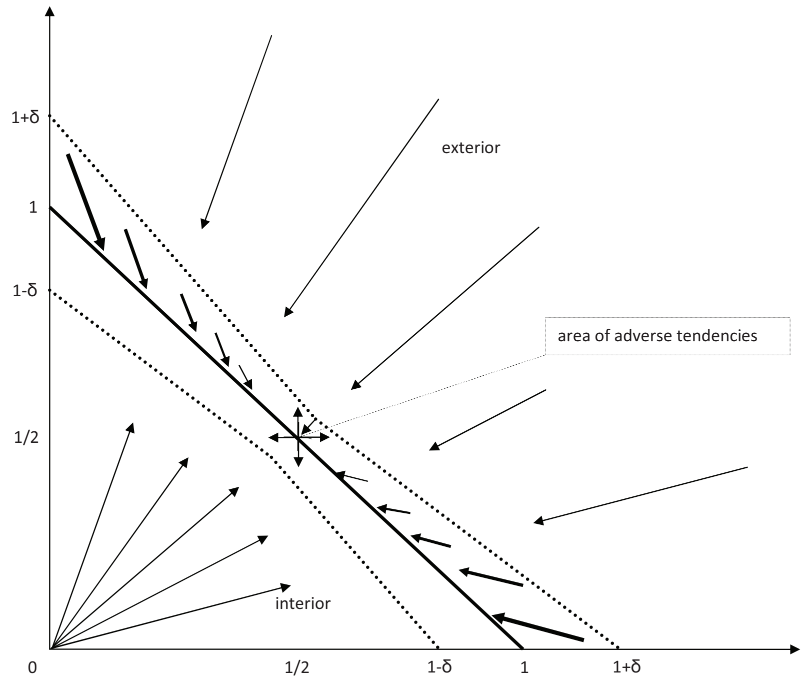
Proof. From any interior state, the process exits with a positive probability in an outwards direction to larger demands, but not inwards to smaller demands. From any exterior state, the process exits with a positive probability in an inwards direction to smaller demands, but not outwards to larger demands. Hence, only interior states that are reachable from exterior states and exterior states that are reachable from interior states are recurrent.

How large can demands of exterior states be that are reached from an interior state? An interior state Z_I on the Pareto frontier with $x + y = 1$ has two exterior neighbours of the form $[(0,0),((1+\delta)x, y)]$ and $[(0,0),(x,(1+\delta)y)]$. Hence, for any recurrent exterior state, the sum of demands exceeds 1 but cannot exceed $(1+\delta)(1-y)+y$ if $x > y$, or $x + (1+\delta)(1-x)$ if $x < y$. All exterior states, with demands such that $y > \max\{(1+\delta) - (1+\delta)x, 1 - \frac{x}{1+\delta}\}$ are transient.

How small can demands of interior states be that are reachable from an exterior state? When $x + y > 1$ but arbitrarily close to 1, its neighbouring states are both interior and of the form $[((1-\delta)x, y), ((1-\delta)x, y)]$ or $[(x, (1-\delta)y), (x, (1-\delta)y)]$. Hence, for any interior state reached from an exterior state, the sum of demands $x+y$ exceeds $(1-\delta)(1-y)+y$ if $x > y$, or $x+(1-\delta)(1-x)$ if $x < y$. Any interior state that is reachable from some exterior state, therefore, must have demands that lie between $y \leq 1-x$ and $y > \min\{(1-\delta)-(1-\delta)x, 1-\frac{x}{1-\delta}\}$. All other states with $y \leq \min\{(1-\delta) - (1-\delta)x, 1 - \frac{x}{1-\delta}\}$ are transient. $\square$

In the long-run, the mass of the process accumulates in a small region around the equal split because the drifts are negative whenever $VMR(Z) \geq \delta$ and adverse tendencies in areas with $VMR(Z) < \delta$ are small ($< \delta$).

Drift graphic (3): The graph illustrates bilateral bargaining with proportional revisions.



From any interior state, the expected movement is outwards along rays through the origin. Interior states with demands so small that they cannot be reached from exterior states are transient. Conversely, any exterior state with demands so large that they cannot be reached from interior states is transient and the expected movement is inwards along a ray through the origin. In the long run, the process moves between interior and exterior states close to the frontier (between dotted lines), mass mostly around the equal split.

Chapter 3

Assignment games with trial-and-error

Abstract

Agents from two sides of a market (firms and workers) seek one another to form partnerships and bargain over their shares. Playing repeated assignment games with dynamic matching, players feel their way to stable partnerships over time. Instead of associating bargaining powers with either side of the market, the matching procedure is based on individual demand increases and concessions made locally by agents on both sides. Matched agents occasionally probe higher demands, unmatched agents shop at reduced rates if they remain unmatched for extended periods of time. Partnerships are broken time and again, reshuffled and restored. In the long run, optimal partnerships form often and are robust against pairwise deviations most of the time. Partners share close to fifty-fifty most of the time unless one or both of them have more profitable alternatives.

JEL classifications: C71, C73, C78, D83

Keywords: assignment games, cooperative game theory, equity, evolutionary game theory, one-to-one matching, learning, (pairwise) stability

3.1 Introduction

Our learning procedures are applied to assignment games, the simplest transferable utility model of matching amongst partners from two sides of a market. In contrast to cooperative bargaining as in the previous chapter, feasibility in assignment games is not a global condition and coalitioning is not restricted to one large coalition involving everybody (the grand coalition). Instead, a large number of separate pairs may form. Our learning procedures, suitably enriched to model search for and match amongst partners, reveal dynamics tending towards optimal assignments and stable matchings. Within optimal partnerships, equity is favoured. In the long run, optimal partners share close to fifty-fifty most of the time unless one or both of the partners have profitable outside alternatives. Conclusions and modelling assumptions fare well in light of experimental evidence.

Two-sided matching. Two-sided matching models are a class of interesting cooperative games, rich for their real-world applicability and theoretical tractability.[1] A situation is describable as two-sided matching when players or coalitions of players on two sides of a market seek to form partnerships or coalitions with players on the other side. Examples are employers and employees matching in labour markets, universities and students in university admission, and men and women in marriage markets. A preference order for each player ranks the different possible matchings. When utility is transferable, each matching is associated with a value that is allocable amongst the partners.

One distinguishes between many-to-many, many-to-one and one-to-one matching models.

[1] Roth & Sotomayor [1990] is a classic study of these models and discusses evidence, applications and early results.

Many-to-many matching is studied in Roth & Sotomayor [1990], Sotomayor [1999], [2004], Baïou & Balinski [2000], Echenique & Oviedo [2006] and Konishi & Ünver [2006]. Applications include university admissions and labour markets. Different stability concepts and stable sets are proposed. Not surprisingly, as many agents are able to match with and be matched with many others, in coalitions of arbitrary sizes, the results from standard cooperative theory are mirrored: as with the core in general games, standard stable sets may be empty.

One significant simplification is to restrict coalitions to many matching only one.[2] The many-to-one model and stability criteria are introduced in Gale & Shapley [1962] who study student admission schemes: one university is matched with many students but each student is matched with only one university. Many labour markets also share this feature. Theory and applications of many-to-one matching have been studied, providing different definitions of and sufficient conditions for stability in Roth & Sotomayor [1990], Balinksi [2001], Baïou & Balinski [2003], Echenique & Oviedo [2004], Hatfield & Milgrom [2005] and Pycia [2007]. Stability conditions equivalent to the one-to-one matching case are recovered (Roth & Sotomayor [1990]).

In one-to-one matching as in marriage (Gale & Shapley [1962]) and assignment games (Shapley & Shubik [1970]), the most important stability concept is pairwise stability: a match is pairwise stable if no alternative partnerships exist that could improve upon the current match by deviating jointly.[3]

Assignment games. Utility in assignment games is transferable and matching involves two tasks: forming a bilateral partnership and sharing its gains. A stable solution specifies which partnerships form and how their surpluses

[2]Baïou & Balinski [2000] show that restrictions on preferences may produce equivalent simplifications in many-to-many matching as the restriction to many-to-one.

[3]The first part of Roth & Sotomayor's [1990] textbook is an excellent review of these models.

are shared in a way that is robust against deviations.

Stability in assignment games has convenient properties and is easy to interpret in terms of pairwise stability. Famously, Shapley & Shubik [1972] show that the core of any assignment game is always nonempty and coincides with the set of optimal assignments with pairwise stable allocations. Balinsky & Gale [1987] compute the maximal number of extreme points of the core dependent on market size and Quint [1991] characterises the structure of core allocations within the core. Sotomayor [2003] links the number of optimal assignments to the size of the core. Allowing for incomplete information, Forges [2004] finds that the *ex ante* incentive-compatible core of the assignment game is always nonempty.

Noncooperative implementation mechanisms for the core of assignment games have been proposed by Demange & Gale [1985], Demange, Gale & Sotomayor [1986], Kamecke [1989] and Perez-Castrillo [2003]. Quite different from one another, these mechanisms share two important features. First, they associate different levels of bargaining power with players depending on which side of the market they belong to (one side of the market being essentially more powerful than the other). Second, information is generally assumed to be complete.

Our approach. We are interested in complex environments with large markets involving many participants on each side. The information available to the individual concerning past and current behaviour of others is limited and hard to obtain, especially from the other side of the market.

We propose a model where each individual's decision is based solely on his own assessment of how well he did in the past. We do not assume that either of the two markets consists of intrinsically different players or possesses structurally more power. Instead, we treat all agents in the same way and propose one learning procedure for all. As in the previous chapter, the players' behavioural dynamics are rooted in small and local adjustments that vary according to whether they are currently satisfied or dissatisfied. The model we propose is fully dynamic, which permits a complete analysis of the long-run behaviour of the process. To our knowledge, no fully dynamic procedure has previously been proposed for assignment games.

In large assignment games, each player has many different candidate partners and the success of arrangements depends crucially on which partnerships form. Searching for partners plays a major role in the individual's behaviour, how he searches depends on his current position and satisfaction level. A satisfied player will occasionally break his current partnerships in order to search for more elsewhere. A dissatisfied player who finds no partner will decrease his aspirations and demands in the hope of finding a feasible partner at the reduced rate in the future.

Partnerships are broken time and again, reshuffled and restored. No partnership will last forever as individuals continue to explore alternative partnerships. Over time, however, more worthy partnerships form more often and partnerships become more stable as fewer alternative partnerships remain feasible. A large proportion of the long-run time, optimal partnerships form and share payoffs in a pairwise stable way. Loss-averse partners share close to fifty-fifty most of the time unless one or both of them have more profitable alternatives.

The dynamics we propose are aimed at explaining large, repeated and complex assignment environments. We consider large, disconnected markets such as firm-worker, consumer-to-consumer, business-to-business or business-to-consumer matching. In such markets, agents from two sides of a market anonymously and repeatedly are matched by a decentralised market maker. Instead of more classic firm-worker matching, the matching process is inherently based on varying and loose partnerships that may be anonymous and are being reconfigured time and again, reshuffled and restored.

Consider, for example, electronic market makers where neither sellers nor buyers are informed about the other side or are able to bargain directly with them, the decentralised electronic market maker matching the two anonymously and bindingly before the match is disclosed.[4] Interesting programming applications are software agents who participate in repeated electronic matching representing real human beings. In these applications, standard strategic and coupled evolutionary learning models have been abandoned in many of these programs because they seem to hard to implement.[5] Our completely uncoupled dynamics provide promising alternatives because learning can be done off-line.

The completely uncoupled learning heuristics are particularly interesting in these environments for two main reasons.

First, the structure of the assignment game features two basic elements: partnerships that form across two sides of a market, and bilateral bargaining that occurs within each partnership. When markets are large and complex and agents are largely disconnected, a two-sided market structure may restrict informational flows from players on one side of the market to the other. Agents may be informed about their own partners and about players in their own market but no reliable information about the other players in the other market may be available, especially with regards to their past behaviour. Instead of assuming that individuals perform complicated calculations in attempts to outguess the other market, we consider the case where individuals learn in

[4] An example of such a market maker is *Priceline.com*'s "Name-Your-Own-Price" (NYOP): buyers submit a price they are willing to pay for a particular good or service and are matched with a seller who is willing to supply at that price, *Priceline.com* keeping the difference between sell and buy prices. (See www.priceline.com.)

[5] See Fatima, Wooldridge & Jennings [2005].

a completely uncoupled way, deciding what to do solely based on their own assessment of how well they did in the past.

Second, our modelling assumptions are supported by evidence. Studies of the labour market, for example, find that wage-adjustments and wage-rigidities observed in reality are well-modelled by small incremental adjustments with downward wage-rigidity (Bewley [1999]).[6] Within partnerships, experiments supports these modelling assumptions in bargaining and provides evidence for our long-run predictions.[7] Bilateral bargaining experiments show that, based on trial and error, agents adapt their aspirations gradually and end up sharing equitably because higher shares are more often reduced. Similarly, repeated one-to-one matching with more than two players reveals trial-and-error adjustments leading to optimal assignments and stable matchings with high levels of equity.[8]

Related literature & evidence. Convergence to pairwise stable matchings is studied in Roth & Vande Vate [1990], Chang [2000], Kojima & Ünver [2006] and Klaus & Klijn [2007]. In these models, subsets of agents are temporarily closed to outside partners and bargain amongst themselves until internally stable. Once internal stability is achieved, some new partners are allowed in and bargaining recommences, again closed to all others outside until internally stable. The set of bargainers continues to expand after internal stability of subsets is reached until, finally, the whole market matching is stable; an optimal assignment has formed.

As opposed to a mechanism where subsets of agents remain temporarily closed

[6]See Sections 10.1. "Type and Size of Raises" and 6.1. "Types of Internal Pay Structure" and Chapter One (p. 40).

[7]See Tietz & Weber [1972], [1978], Tietz [1975], Weber [1976], Tietz, Weber, Vidmajer & Wentzel [1978], Tietz & Bartos [1983], Crössmann & Tietz [1983] and Tietz, Daus, Lautsch & Lotz [1988].

[8]See Charness, Corominas-Bosch & Fréchette [2007] and p. 62 of this chapter for a discussion.

to others until internally stable, all agents continue to interact with each other at all times in our process. All partnerships continue to be broken, are reshuffled and restored. Over time, they become more and more stable. Within stable matchings, we identify a bias for equity that is constrained by outside alternatives. Optimal players will share close to fifty-fifty most of the time unless one or both of the partners have more profitable outside alternatives. This finding ties in well with experimental evidence from Charness, Corominas-Bosch & Fréchette [2007] who test the strategic bargaining procedure by Corominas-Bosch [2004]. In repeated matching, it is revealed that equity is an important phenomenon and that stability and equity increase over time: "point predictions generally fail" in terms of the strategic bargaining theory, which "seems to fare poorly when it predicts a very uneven split."[9] Furthermore, the individual adjustment behaviour displayed is consistent with our modelling of loss-aversion: "the share offered is a significant factor with regard to acceptance or rejection - higher proposed shares are more likely to be accepted."[10]

Equitable matchings in the core of assignment games have been studied and characterised. Thompson [1981] proposes a solution that gives equal power to each side of the market: the equal division point lies halfway between the core allocations that are either optimal for all players on one or the other side of a market. Rochford [1984], on the other hand, looks at optimal pairs individually and gives equal power to the players in those. The solution he proposes shares the gains from each optimal partnership according to the Nash bargaining solution with outside options implied by each agent's best alternative partners who also share according to the Nash bargaining solution with their optimal partners. Hougaard, Thorlund-Petersen & Peleg [2001] identify

[9] Charness, Corominas-Bosch & Fréchette [2007], p. 45.
[10] Charness, Corominas-Bosch & Fréchette [2007], p. 46.

Lorenz-optimal matchings in the core, applied to kidney exchange problems in Roth, Sönmez & Ünver [2005].

In the next two sections, we introduce assignment games formally and develop dynamic procedures for adjustment and search. Section Four discusses pairwise stability. In Sections Five and Six, we prove convergence into the core and convergence to equity with complete search. Section Seven studies incomplete search and obtains bounds for instability of the core. The Appendix proposes a probabilistic interpretation of stability.

3.2 Assignment games

Consider the game described by $[F, W, \alpha, A]$:

- $F = \{f_1, ..., f_m\}$ is a set of m firms (or men or sellers).
- $W = \{w_1, ..., w_n\}$ is a set of n workers (or women or buyers).
- $\alpha = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \alpha_{ij} & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}$ is the matrix of partnership worths.

α specifies, for every possible one-to-one partnership between a firm i and a worker j, $(i, j) \in F \times W$, a worth of $\alpha_{ij} \geq 0$. α_{ij} represents the total that can be split between i and j when they are matched.

- $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$ is the assignment.

A specifies whether a pair forms or not. For any $(i, j) \in F \times W$, $a_{ij} = 1$ if (i, j) forms and $a_{ij} = 0$ if (i, j) does not form. For all firms i, $\sum_{j \in W} a_{ij} \leq 1$. For all workers j, $\sum_{i \in F} a_{ij} \leq 1$. If $a_{ij} = 1$, $a_{ij'} = 0$ for all alternative partners, $j' \neq j$, of i and $a_{i'j} = 0$ for all alternative partners of j, $i' \neq i$.

Cooperative assignment game. *From $[F, W, \alpha, A]$, we derive the cooperative assignment game $G(v, N)$, where $N = F \cup W$ and $v : S \subseteq N \to \mathbb{R}$ with*

- $v(i) = v(\varnothing) = 0$ *for all singletons* $i \in N$*,*
- $v(S) = \alpha_{ij}$ *for all* $S = (i, j)$ *such that* $i \in F$ *and* $j \in W$*,*
- $v(S) = \max\{v(i_1, j_1) + ... + v(i_k, j_k)\}$ *for any arbitrary* $S \subseteq N$*,*

where $\max\{v(i_1, j_1) + ... + v(i_k, j_k)\}$ *is taken over all sets* $\{(i_1, j_1), ..., (i_k, j_k)\}$ *containing at most* $k \leq \min\{|S_F|, |S_W|\}$ *pairs that may be formed from matching the respective firms and workers contained in* S*.* $v(N)$*, in particular, specifies the worth of any optimal assignment.*

An outcome of the assignment game specifies an assignment A and a payoff allocation ϕ of the gains generated by A.

To simplify notation, we add dummies (with whom all partnerships are worth zero) to the smaller side of the market so that there is an equal number (n) of firms and workers in the market. This means we have two equal-sized sides of the market: $|N| = 2n = |F| + |W|$.

3.3 Evolving play

A fixed population of players, $N = F \cup W$, repeatedly plays the assignment game $G(v, N)$. Each period, each player i makes a demand d_i^t for a share of a partnership and receives a payoff ϕ_i^t, both nonnegative real numbers. If two demands are jointly feasible in a partnership, the two may be matched and split their partnership worth in some way. When two demands are jointly infeasible in a partnership, no agreement can be reached that matches those two. A partner who remains unassigned receives zero.

Some partnerships are better than others, meaning that they generate higher payoffs. Players always seek higher payoffs and aim to achieve them by searching for the best possible partners. There was no searching in Chapter Two because the only relevant coalition in cooperative bargaining is the grand coalition. Apart from the grand coalition, no subcoalitions form. In assignment games, many pairs may form and searching for a good partner is, therefore, an important part of the matching process. We propose different search mechanisms, modelling different attitudes of the searcher.

We assume that both a player's demand revisions and his search behaviour depend solely on his own assessment of how well he did in the past. Players do not hypothesise about the others' actions and do not strategise in order to play best replies. Such nonstrategic behaviour is most realistic in complex and large markets where limited information is available about other players' behaviour (especially concerning past behaviour and about the other side of the market). Over time, partnerships change but optimal partnerships tend to form more often as the game continues. Within optimal partnerships, bilateral bargaining is an ongoing process of individual demand increases and of mutual concessions.

States. Z^t is the state in period t of the form $[A^t, \phi^t, d^t]$ with assignment A^t and vectors ϕ^t and d^t of payoffs, $\{\phi_1^t, ..., \phi_{2n}^t\}$, and demands, $\{d_1^t, ..., d_{2n}^t\}$, held by each player. The set of all states is Ω.

To simplify notation, we choose all initial demands and all partnerships' worths, α_{ij}, as multiples of the rate of demand-adjustment, δ, and restrict α_{ij} in between zero and one. This ensures that a finite number of demand-pairs (d_i, d_j) constitutes the Pareto frontier for any partnership (i, j), all of which with $d_i + d_j = \alpha_{ij} \leq 1$.

Payoffs and satisfaction. In period t, $\phi_i^t = d_i^t$ if i is assigned, 0 otherwise.

If $\phi_i^t \geq d_i^t$, he becomes satisfied in period $t+1$. Otherwise, if he is unassigned in period t and receives 0, he becomes dissatisfied in period $t+1$.

In other words, a player is satisfied in any period $t+1$ if, and only if, he was assigned in period t or demanded nothing. If he is assigned in period $t+1$, his payoff will match his demand. Otherwise, he will receive zero.

Demand transitions. Each period, one player is selected (with probability $\frac{1}{2n}$) and given the opportunity to revise his demand and search for a new partner.[11] The selected player may increase or decrease his demand, dependent on whether he is satisfied or not.

When a satisfied player is selected, he breaks his current partnership with some small probability, r, and demands a small increment, δ, more than previously. With this higher demand, he then searches for a better match. With high probability $1-r$, he remains inactive, his current partnership and demands unchanged.

When a dissatisfied player is selected, he immediately searches. If he finds no match at the end of his search and remains a singleton, he demands δ less than previously with a high probability, $1-s_i^t$. With probability s_i^t, he does not decrease his demand.

We assume that the probability r, with which a satisfied player breaks an agreement in order to demand more, is constant and small.

Loss aversion. The probability $1-s_i^t$, with which a dissatisfied player reduces his demand after an unsuccessful search is large. The greater a player's loss, the

[11] In our nonstrategic model, the assumption of selecting only one player at a time is not as restrictive as in a strategic model where hypotheses about others' actions determine one's own best reply. Since our process is completely uncoupled, the activation of only one player at a time conventiently sequences actions that otherwise happen at the same time if several players are activated at once. Activation of only one player is technically convenient but comes at little conceptual cost because the actions depend on satisfaction levels and not on the others' actions.

more likely he is to reduce: i is likely to reduce his demand to $d_i^t = d_i^{t-1} - \delta$ in period t after an unsuccessful search. We assume the probability of reduction to be ad_i^{t-1}.[12]

The assumption of decreasing stickiness is experimentally supported by experiments.[13] Players are more likely to accept higher shares and less likely to accept lower shares and become more likely to accept over time if no agreement is reached.

Search. The player who is selected may search, both when satisfied and when dissatisfied. A satisfied player searches after breaking his previous partnership in order to try and get more. A dissatisfied player searches when he is currently unassigned in order to find a partner.

Search has the following meaning: a player who is selected and currently unassigned (either because he had already previously been unassigned or because he broke his partnership), submits his willingness to form partnerships to potential partners together with a demand. A candidate who is addressed by the searcher may become his partner if, jointly with his own current demand, their partnership is feasible. Partnerships amongst players with jointly infeasible demands do not form. When several candidate partnerships may form, one is randomly selected.

Completeness of search. Completeness of search, p, measures the probability of addressing someone on the other side of the market. The probability of addressing any set of players, $\mathcal{S}$, is identically and independently distributed with $p \in (0, 1]$ for addressing each one. When $p = 0$, there is no search and the probability of addressing any partner is zero. For $p \in (0, 1]$, the probability of addressing some subset $\mathcal{S}$ of the other market is $p^{|\mathcal{S}|}(1-p)^{n-|\mathcal{S}|}$ for all $\mathcal{S}$ with

[12] See Chapter Two (pp. 27-35) for more detail on loss aversion and alternative demand-revisions with proportionality (pp. 39-42).

[13] See, for example, Weber [1976], Tietz, Weber, Vidmajer & Wentzel [1978] and Charness, Corominas-Bosch & Fréchette [2007].

$|\mathcal{S}| \leq n$. In particular, the probability that everybody is addressed is p^n, the probability that nobody is addressed is $(1-p)^n$.

Complete search.

> *When $p = 1$, the search mechanism is complete and the searcher addresses everybody at the same time with probability one so that all feasible partnerships may form.*

Incomplete search.

> *When $p \in (0, 1)$, the search mechanism is incomplete and the searcher addresses any one player in particular with probability p, resulting in a random sample $\mathcal{S}$ from the other side of the market of size $|\mathcal{S}|$ with probability $p^{|\mathcal{S}|}(1-p)^{n-|\mathcal{S}|}$.*

Complete search is realistic when many alternative partners can be contacted at the same time and switching partnerships can be achieved at negligible costs (as in computerised matching applications). More incomplete searches are realistic if search is costly and takes time.

Indeed, in the presence of search and switching costs, agents may consider the possible profitability of alternative behaviour and decide to experiment based on their assessment. In that case, our trial-and-error dynamics may need to be complemented by some explicit formation of expectations. Our simple model with trial and error without explicit expectations is better suited to explain behaviour when search and switching costs are negligible and expectations are not explicitly formed. In that case, the complete search case is the benchmark and our model reveals that it provides us with support for efficient and equitable market outcomes in the long run.

Assignment transitions. If no player searches in period $t+1$, $Z^{t+1} = Z^t$ and $A^{t+1} = A^t$. When some player i searches in period $t+1$, the assignments may change. Depending on whether i is satisfied or dissatisfied, he either searches with $d_i^{t+1} = d_i^t + \delta$ or $d_i^{t+1} = d_i^t$. All other demands remain $d_j^{t+1} = d_j^t$ as in period t for all $j \neq i$.

Suppose i is satisfied and was previously assigned to j. With probability r, he searches with $d_i^{t+1} = d_i^t + \delta$. The search by i reaches a set of $\mathcal{S}$ players, none to all of which may be feasible partners; denote the feasible set by $\mathcal{F} \subseteq \mathcal{S}$ with $d_i^{t+1} + d_{j'}^t \leq \alpha_{ij'}$ for all $j' \in \mathcal{F}$ and $d_i^{t+1} + d_{j''}^t > \alpha_{ij''}$ for all $j'' \in \mathcal{S} \setminus \mathcal{F}$. Let $|\mathcal{F}| = f \geq 0$. If $\mathcal{F}$ is empty, both i and j will be left as singletons and all other partnerships remain unchanged. If $\mathcal{F}$ is nonempty, some partnership with a member of $\mathcal{F}$ forms.[14] If j, the old partner with whom i was assigned in period t, is no longer the partner in $t+1$, j is left as a singleton in $t+1$. If the new partner j' was previously assigned to some other $i' \neq i$, that i' is also left as a singleton. If the new partner j' was previously unassigned, the other partnerships remain unchanged. When i and j are immediately rematched, all other pairs remain unchanged.

Suppose i is dissatisfied and not assigned in period t. With probability one, he searches with $d_i^{t+1} = d_i^t$. The search by i reaches a set of $\mathcal{S}$ players, $\mathcal{F} \subseteq \mathcal{S}$ of which feasible. If $\mathcal{F}$ is empty, all existing partnerships remain unchanged and i remains a singleton. If $\mathcal{F}$ is nonempty, some partnership of i with some $j \in \mathcal{F}$ forms. If j was previously assigned to some other $i' \neq i$, that i' is left as a singleton. If j was previously also unassigned, all previous partnerships remain intact.

Aside: Selecting a coalition in $\mathcal{F}$ equiprobably is just one possible specification. Alternatively, we could, for example, select the most efficient or the largest feasible coalition. It will become apparent in the later proof sections that indeed any selection mechanism leading to successful search ultimately leads

[14]We can consider the case when each partnership forms randomly with the same probability, $\frac{1}{|\mathcal{F}|}$, or many other selection mechanisms as most profitable, closest, etc. What matters is that the searcher forms a partnership with someone in $\mathcal{F}$, no matter with whom.

to the same results.

3.4 Pairwise stability

We now have a complete description of the dynamic process and turn to observations regarding pairwise stability of states in our procedure. Let $[A, \phi]$ specify an assignment and payoffs.

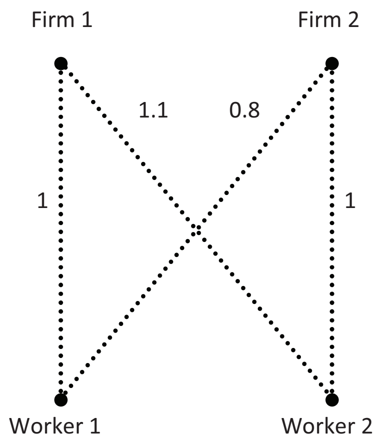
Pairwise stability.

> *Given any $[A, \phi]$, the pair (i, j) with $a_{ij} = 1$ is stable if $\phi_i + \phi_j = \alpha_{ij}$ and $\phi_{i'} + \phi_j \geq \alpha_{i'j}$ for any alternative firm i' and $\phi_i + \phi_{j'} \geq \alpha_{ij'}$ for any alternative worker j'.*
>
> *$[A, \phi]$ is pairwise stable if, and only if, all assigned partnerships are stable.*

If $[A, \phi]$ is not pairwise stable, some pairwise deviation is profitable: two players exist who have a common incentive to deviate and form a partnership. In particular, when $\phi_i + \phi_j < \alpha_{ij}$ for some (i, j) with $a_{ij} = 1$, there is room for Pareto improvement within that existing pair. When $\phi_i + \phi_j < \alpha_{ij}$ and $a_{ij} = 0$, that pair prefers to form instead of remaining in its existing arrangement.

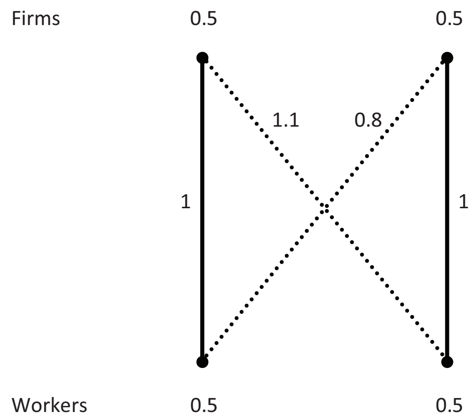
Consider the following illustration of transitions in an example when search is complete.

Example 3.1. *Two firms and two workers play the assignment game with $v(f_1, w_1) = 1$, $v(f_2, w_2) = 1$, $v(f_1 w_2) = 1.1$, $v(f_2 w_1) = 0.8$ and $v(i) = 0$ for all other $i \in N$.*

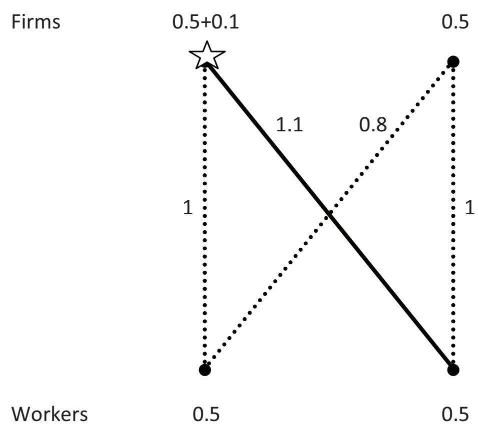


Complete search: moving from instability to stability

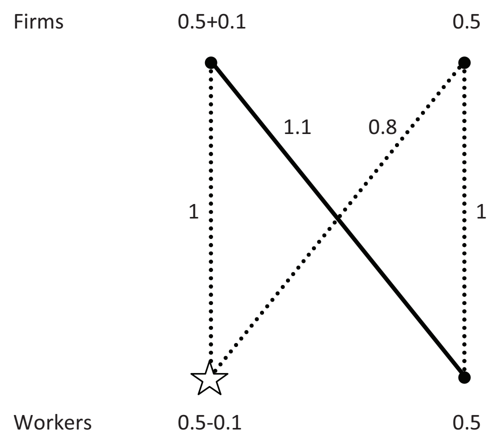
Suppose a situation as illustrated below is the state in *period t*. Note the state is not pairwise stable: Firm One and Worker Two, currently receiving 0.5 each in their respective pairs, would prefer to form a pair (worth 1.1) and distribute the extra 0.1 amongst them. We will illustrate how the existing unstable arrangement can transit into a pairwise stable arrangement in a few periods.



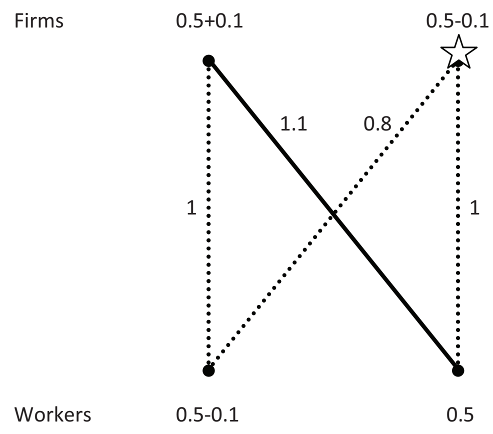
In period $t + 1$, suppose that Firm One is selected (moving player indicated by the star) who breaks the partnership with Worker One and increases its demand by $\delta = 0.1$, finding a partner in Worker Two. Worker One and Firm Two are left without partner.



In period $t+2$, suppose that Worker One is selected who searches but finds no partner, resulting in reduction of demand by 0.1.

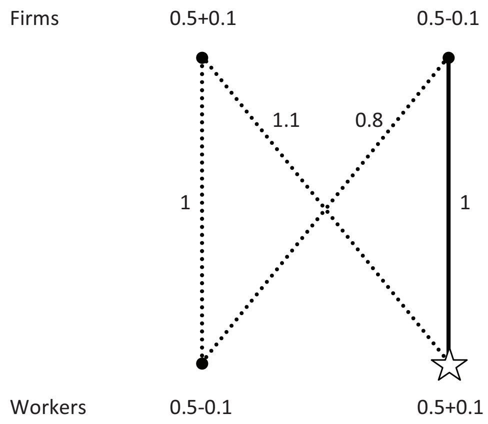


In period $t+3$, suppose that Firm Two is selected who searches but finds no partner, resulting in reduction of demand by 0.1.

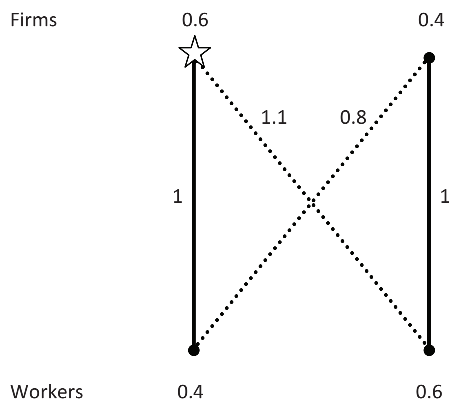


In period $t+4$, suppose that Worker Two is selected who breaks the partnership with Firm One and increases its demand by $\delta = 0.1$, finding a partner in Firm

Two.



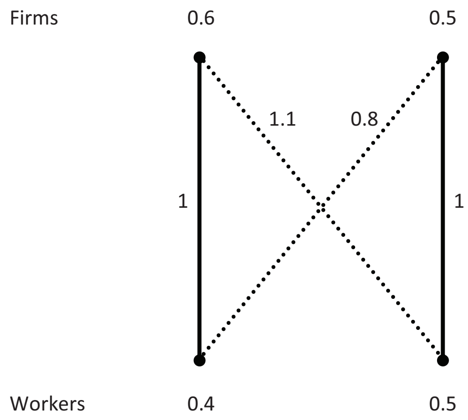
Suppose that Firm One is selected again in *period* $t+5$. This leads to a successful search and match with Worker One without further demand changes.



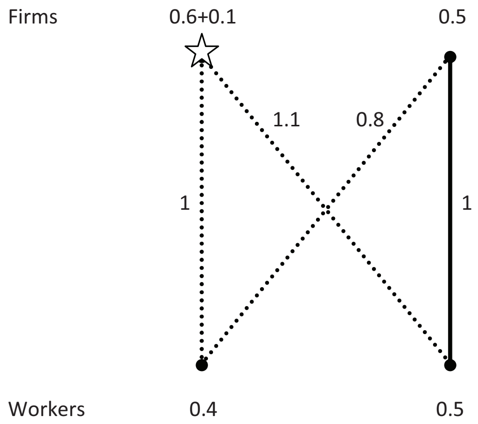
The attained allocation in *period* $t+5$, in contrast to the original allocation from *period* t, is pairwise stable.

Complete search: robust stability

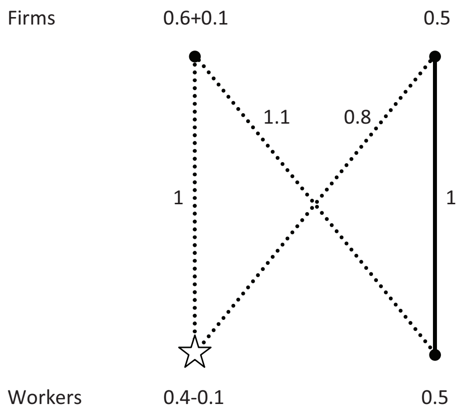
The next situation (illustrated below) is pairwise stable: there exist no alternative matches for any pair such that both players strictly prefer to form that partnership and reallocate the resulting payoffs in any way. Suppose we start at this state in *period* t'. Over the course of a few periods, all payoffs may change when allocations are broken but pairwise stability will not be violated and another stable allocation is eventually reached.



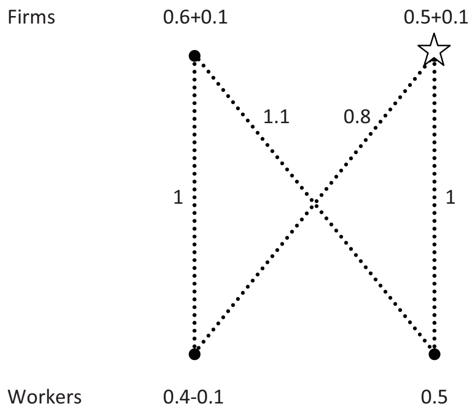
In period $t'+1$, suppose that Firm One is selected who breaks the partnership with Worker One and increases its demand by $\delta = 0.1$, finding no feasible partner to accomodate his demand.



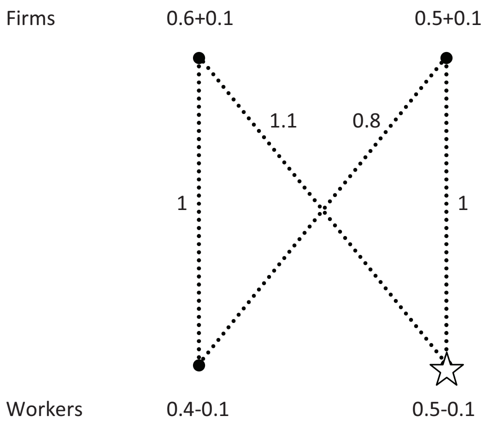
In period $t' + 2$, suppose that Worker One is selected who searches but finds no partner, resulting in reduction of demand by 0.1.



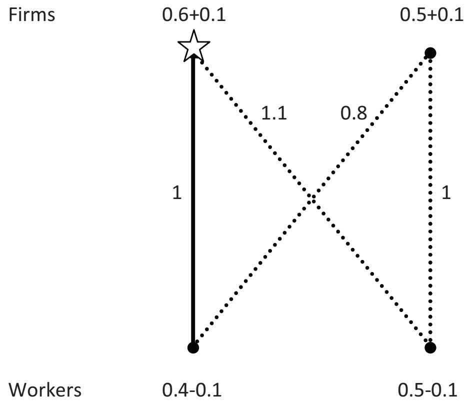
In period $t' + 3$, suppose that Firm Two is selected who breaks the partnership with Firm Two and increases its demand, finding no partner.



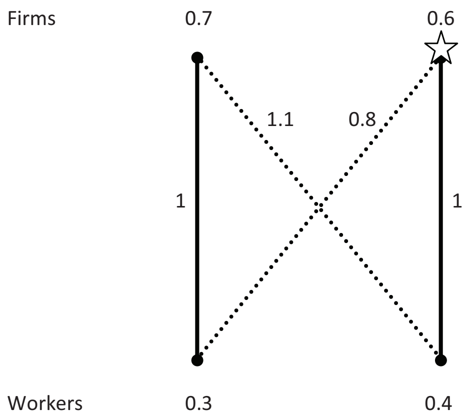
In period $t' + 4$, suppose that Worker Two is selected who searches but finds no partner, resulting in a reduction of demand.



Suppose that Firm One is selected again in *period* $t' + 5$. This leads to a successful search and match with Worker One.



Similarly, suppose Firm Two is selected in *period* $t' + 6$. This leads to a successful search and match with Worker Two and an overall allocation that is pairwise stable again.



3.5 The core

The formulation of the assignment game as well as the following results are due to Shapley and Shubik:

Shapley & Shubik (1972) - Theorem. *The core of any assignment game $G(v, N)$ is nonempty and consists of all optimal assignments with allocations that are pairwise stable.*

Shapley & Shubik (1972) - Corollary 1. *If an assignment is optimal, it is compatible with any pairwise stable allocation, that is, any given pairwise allocation can be supported by any optimal assignment.*

Shapley & Shubik (1972) - Corollary 2. *When an allocation in a given assignment is pairwise stable, then its supporting assignment is optimal.*

Shapley & Shubik's theorem implies that some optimal assignment must form to support a core allocation and that any optimal assignment is compatible with any possible core allocation.[15] The relevant blocking coalitions, therefore, reduce to pairs and optimal assignments become exchangeable to support any core allocation. For the latter reason, most of the literature concerned with stability in assignment games focusses on the allocative aspect.[16] Regarding core assignments, on the other hand, Sotomayor [2003] notes the following:

Sotomayor (2003) - Theorem. *If there is only one optimal assignment, the number of core allocations is infinite.*

Sotomayor (2003) - Corollary. *If the core contains a unique allocation, there is more than one optimal allocation.*

[15] See Roth & Sotomayor [1992] for more discussion.

[16] Sotomayor [2003] p. 261: "every optimal matching is compatible with any core payoff. Because of this, one has been led to think of this game as having only one optimal matching, which explains why the existent theoretic results for the assignment game only concentrate on the payoffs of the players."

With respect to any given partnership (i, j), we can distinguish between two types: those that do not form in any optimal assignment and those that form in some optimal assignment. We will refer to pairs (i, j) that form in some optimal assignment as "optimal pairs."

Optimal pair.

> *The pair (i, j) is optimal if there exists an optimal assignment A such that $a_{ij} = 1$.*

In any core allocation ϕ, $\phi_i + \phi_j = \alpha_{ij}$ for all optimal pairs and $\phi_k + \phi_l \geq \alpha_{kl}$ for all other pairs. When the optimal assignment is unique, there exists only one optimal partner for each player and the number of optimal pairs is exactly n.

When search is complete

With complete search, we will prove that pairwise stable demands are implemented almost surely for any assignment game and that demands are rarely infeasible by very much. (Theorem 6 and Lemma 7) This is because instabilities are transient, even though states with given levels of instability may not be transient.

Theorem 6. *Given any $\gamma > 0$, for any assignment game played with our procedure and complete search, there exists a finite time T_γ such that, for every $t > T_\gamma$,*

$$\mathbb{P}[d_i^t + d_j^t < \alpha_{ij}] < \gamma$$

for all $(i, j) \in F \times W$.

Proof. **Step 1.** To prove the result, we first show that, in any state Z^t, the "instability," as captured by

$$IN(Z^t) = \sum_{(i,j)\in F\times W} \{[\alpha_{ij} - (d_i^t + d_j^t)]\mathbf{1}_{d_i^t+d_j^t<\alpha_{ij}}\},$$

where $\mathbf{1}_{d_i^t+d_j^t<\alpha_{ij}}$ is an indicator equal to one whenever $d_i^t + d_j^t < \alpha_{ij}$, never increases.

In other words, given any state Z^t, the sum of demands $d_i^t + d_j^t$ when $d_i^t + d_j^t < \alpha_{ij}$ of any firm-worker pair (i,j), whether the pair is currently assigned or not, never decreases: $\Delta(d_i^t + d_j^t)\mathbf{1}_{d_i^t+d_j^t<\alpha_{ij}} = [(d_i^{t+1} + d_j^{t+1}) - (d_i^t + d_j^t)]\mathbf{1}_{d_i^t+d_j^t<\alpha_{ij}} \geq 0$ for all $(i,j) \in F \times W$ and for all $Z^t \in \Omega$.

Let us consider the various cases.

Case 1.1. When neither i nor j are selected, no increase or decrease occurs and $\Delta(d_i^t + d_j^t) = 0$.

Case 1.2. When $d_i^t + d_j^t \leq \alpha_{ij}$ and i or j are selected, $\Delta(d_i^t + d_j^t) \geq 0$ because a previously assigned player may increase his demand but not decrease it, and an unassigned player will match someone in a partnership without having to reduce.

Case 1.3. When $d_i^t + d_j^t > \alpha_{ij}$ (that is $d_i^t + d_j^t \geq \alpha_{ij} + \delta$) and i or j are selected, $\Delta(d_i^t + d_j^t) \geq -\delta$ because a previously assigned player may increase his demand but not decrease it, and an unassigned player may reduce by at most δ.

Jointly, the three cases show that $\Delta(d_i^t + d_j^t)\mathbf{1}_{d_i^t+d_j^t<\alpha_{ij}} \geq 0$, which implies that $IN(Z^t)$ never increases. Furthermore, no (d_i^t, d_j^t) with $\Delta(d_i^t + d_j^t) > \alpha_{ij}$ will result in $d_i^{t+1} + d_j^{t+1} < \alpha_{ij}$.

Step 2. We now show that, starting in any Z^t for which some demand pair (d_i^t, d_j^t) is unstable ($d_i^t + d_j^t < \alpha_{ij}$), a path into the core exists that has positive probability bounded away from zero. Together with the previous argument that instability never increases, this ensures that, if we wait long enough, the probability of positive instability goes to zero. In other words, instability is transient.

We claim that, given any state Z^t, the sum of demands $d_i^t + d_j^t$ when $d_i^t + d_j^t < \alpha_{ij}$ for any firm-worker pair (i, j), whether the pair is currently assigned or not, goes up by at least δ with a positive probability in finite time, that is, there exists a finite $t' < \infty$ such that $\mathbb{P}[(d_i^{t+t'} + d_j^{t+t'}) - (d_i^t + d_j^t) \geq \delta | Z^t] > 0$ for all $(i, j) \in F \times W$ and for all Z^t with $d_i^t + d_j^t < \alpha_{ij}$.

Case 2.1. If i and j either form a pair with each other or are both assigned to someone else, that is Z^t is such that $a_{kj}^t = 1$ for some $k \in F$ and $a_{ik}^t = 1$ for some $k \in W$, one of the two is chosen with positive probability $\frac{2}{2n}$ and increases with probability r: $\mathbb{P}[\Delta(d_i^t + d_j^t) = \delta | Z^t] = \frac{2}{2n} r = \frac{r}{n}$.

Case 2.2. If Z^t is such that one is unassigned and the other assigned, the assigned one is chosen with probability $\frac{1}{2n}$ and increases with positive probability r: $\mathbb{P}[\Delta(d_i^t + d_j^t) = \delta | Z^t] = \frac{r}{2n}$.

Case 2.3. If Z^t is such that both are unassigned, the probability that one of the two is selected who will find a partnership in the next period is $\frac{2}{2n}$. That same player is then chosen again in the next period and chooses to increase his demand with probability $\frac{1}{2n} r$. Hence, $\mathbb{P}[(d_i^{t+2} + d_j^{t+2}) - (d_i^t + d_j^t) = \delta | Z^t] = \frac{2}{2n} \frac{r}{2n} = \frac{1}{2} \frac{r}{n^2}$.

For any Z^t with demands $d_i^t + d_j^t < \alpha_{ij}$, the probability that stability increases in two periods is, therefore,

$$\mathbb{P}[IN(Z^{t+2}) - IN(Z^t) \leq -\delta | Z^t] \geq \frac{1}{2} \frac{r}{n^2}.$$

The probability that instability stays constant is

$$\mathbb{P}[IN(Z^{t+2}) - IN(Z^t) = 0|Z^t] \leq 1 - \frac{1}{2}\frac{r}{n^2}.$$

It takes at most $\frac{1}{\delta}$ steps to reach the stability frontier of any possible partnerships because all $\alpha_{ij} \leq 1$. Furthermore, we know that taking such a step every two periods has probability of at least $\frac{1}{2}\frac{r}{n^2}$. Hence, the sequence of reaching the stability frontier of all n optimal partnerships in $2 \times \frac{n}{\delta}$-many periods has positive probability of at least $[\frac{1}{2}\frac{r}{n^2}]^{\frac{n}{\delta}}$ so that

$$\mathbb{P}[IN(Z^t) > 0] \leq [1 - (\frac{1}{2}\frac{r}{n^2})^{\frac{n}{\delta}}]^{\frac{t}{2}},$$

which is smaller than any arbitrarily small γ after $T_\gamma > t = 2\log\gamma \ / \ \log[1 - (\frac{1}{2}\frac{r}{n^2})^{\frac{n}{\delta}}]$. □

Lemma 7. *Given any $\beta, \gamma > 0$, for any assignment game played with our procedure and complete search with step size $\delta \leq \beta\gamma$, there exists a finite time $T_{\beta,\gamma}$ such that, for every period $t > T_{\beta,\gamma}$,*

$$\mathbb{P}[d_i^t + d_j^t > \alpha_{ij} + \beta] < \gamma$$

for all optimal pairs $(i, j) \in F \times W$.

Proof. From Theorem 6, we know that, for every period $t > T_\gamma$, $\mathbb{P}[d_k^t + d_l^t < \alpha_{kl}] < \gamma$ for all $(k, l) \in F \times W$.

Take a state Z^t with $d_k + d_l \geq \alpha_{kl}$ for all (k, l). Suppose there exists some optimal firm-worker pair (i, j) with $d_i + d_j > \alpha_{ij}$. This implies that at least one of the two partners must be unassigned and dissatisfied. Otherwise, Z^t could not be pairwise stable or (i, j) could not be an optimal pair.

The probability, $\mathbb{P}[\Delta(d_i^t + d_j^t) = \delta | Z^t] \leq \frac{1}{2n} r$, is, therefore, an upper bound on the probability of increase. At most one of two partners of an optimal pair may increase.

The probability, $\mathbb{P}[\Delta(d_i^t + d_j^t) = -\delta | Z^t] \geq \frac{1}{2n} a\delta$, is a lower bound on the probability of decrease. At least one of two partners of an optimal pair, who currently must demand at least δ, may decrease.

Hence, in expectation,

$$\mathbb{E}[\Delta(d_i^t + d_j^t) | Z^t] \leq -\frac{\delta}{2n}(a\delta - r).$$

From state Z^t,

$$\mathbb{E}[d_i^{t+1} + d_j^{t+1} - \alpha_{ij} | Z^t] \leq \mathbb{E}[d_i^t + d_j^t - \alpha_{ij} | Z^0] - \frac{\delta}{2n}(a\delta - r).$$

Starting at any initial state Z^0,

$$\mathbb{E}[d_i^t + d_j^t - \alpha_{ij}] \leq \max\{\delta, \mathbb{E}[d_i^{t-1} + d_j^{t-1} - \alpha_{ij} | Z^0] - \frac{\delta}{2n}(a\delta - r)\},$$

which by iteration and substituting $c \equiv \frac{\delta}{2n}(a\delta - r)$ becomes

$$\mathbb{E}[d_i^t + d_j^t - \alpha_{ij} | Z^0] \leq \max\{\delta, [(d_i^0 + d_j^0 - \alpha_{ij}) - ct]\},$$

which is larger than δ until after time $T_{\beta,\gamma}$ when $(d_i^0 + d_j^0 - \alpha_{ij}) - ct \leq \delta$. This implies $T_{\beta,\gamma} \geq \frac{1}{c}(d_i^0 + d_j^0 - \alpha_{ij} - \delta)$.

Rearranging for the Markov inequality, it follows that, for any Z^0 such that $\delta \leq \beta\gamma$ and $t > T_{\beta,\gamma}$,

$$\mathbb{P}[d_i^t + d_j^t > \alpha_{ij} + \beta] < \frac{\delta}{\beta} = \gamma, \tag{3.1}$$

which holds for all optimal pairs. $\square$

Note that completeness of search ensured that search always lead to matching if there were feasible partners. Specifying the exact selection mechanism, e.g. equiprobable selection, is not necessary for the results to hold as long as search is complete and the searcher succeeds in finding a partner. Note further that the above arguments hold true for constant degrees of stickiness, s, if $1 - s > a\delta$.

3.6 Equity

So far, no long-run observations have been made beyond convergence into the core. Fifty-fifty for any pair that forms in an optimal assignment may seem a split like many other splits, either pairwise stable or not. Players' loss aversion, however, creates tendencies that favour equity. Indeed, when "fifty-fifty is unconstrained for a pair," we find that our procedures with complete search implement splits close to fifty-fifty most of the long-run time.

Fifty-fifty is unconstrained for a pair (i, j)**.**

> *Fifty-fifty is unconstrained for a pair* (i, j) *if, for any optimal assignment* A *and core allocation* ϕ*, the alternative allocation* ϕ' *with* $\phi'_{ij} = (\frac{\alpha_{ij}}{2}, \frac{\alpha_{ij}}{2})$ *for* (i, j)*, holding fixed the assignment,* A*, and allocations of the other pairs, is also a core allocation.*

The above condition implies two important facts:

(1) In any optimal assignment A, $a_{ij} = 1$: the optimal assignments may vary with respect to the partnerships amongst the other players but not with respect to (i, j). Otherwise, we could not reallocate within (i, j) while holding fixed

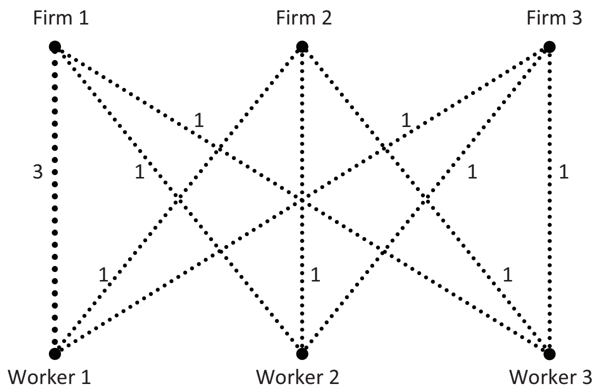
the assignment and all allocations of the other pairs without violating pairwise stability.

(2) In any allocation ϕ such that $\phi_k + \phi_l \geq \alpha_{kl}$ for all (k,l), $\phi_i < \phi_j \Rightarrow \phi_{i'} + \phi_j > \alpha_{i'j}$ for all $(i', j) \neq (i, j)$: whoever holds the higher demand for a split of (i, j) has no outside alternatives. This holds when $\phi_k + \phi_l \geq \alpha_{kl}$ for all (k,l), whether ϕ is a core allocation or not.[17]

Consider the following example where fifty-fifty is unconstrained for the first firm-worker pair.

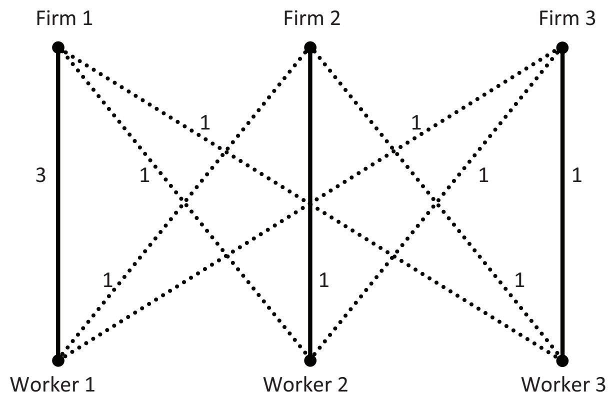
Example 3.2. *n firms and n workers play the assignment game $G(v, \{F \cup W\})$ with $\alpha_{ij} = 1$ for all $(i,j) \in F \times W$ except for $\alpha_{11} = 3$.*

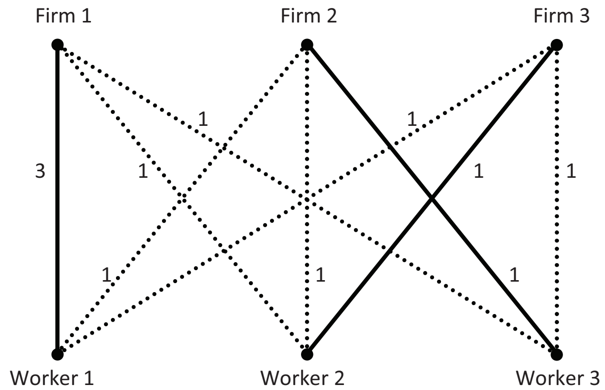
The following graph illustrates for $n = 3$.



[17]We will prove this fact in Lemma 8.

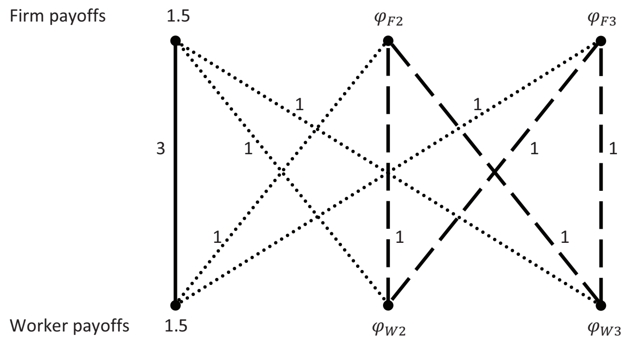
In any optimal assignment, Firm One and Worker One must be matched. Both below assignments are optimal.





In the core, firms and workers share in a way that is pairwise stable. Suppose the partial arrangement $\{\phi_{F2}, \phi_{W2}, \phi_{F3}, \phi_{W3}\}$ is pairwise stable.[18] When Firm One and Worker One share fifty-fifty, neither Firm One nor Worker One can improve upon 1.5 by partnering someone else because any alternative partnership pays at most one (see below).

[18] $\phi_k + \phi_l = 1$ for all $(k, l) \in (F2, F3) \times (W2, W3)$.



Given any core arrangement, we could reallocate the shares of Firm One and Worker One to fifty-fifty, leaving the assignment and allocations in the rest of the market unchanged, to obtain another core outcome.

Lemma 8. *Given any $\beta, \gamma > 0$, for any assignment game played with our procedure with complete search and step size $\delta \leq \frac{\beta\gamma}{2}$, there exists a time $T_{\beta,\gamma}$ such that, for every $t > T_{\beta,\gamma}$,*

$$\mathbb{P}[|d_i^t - d_j^t| > \beta] < \gamma$$

for any pair (i, j) for which fifty-fifty is unconstrained.

Proof. From Theorem 6 and Lemma 7, we know that, after finite time, $\mathbb{P}[d_k^t + d_l^t < \alpha_{kl}] < \gamma$ for all pairs (k, l) and $\mathbb{P}[d_i^t + d_j^t > \alpha_{ij} + \beta] < \gamma$ for all optimal pairs (i,j). Furthermore, from Theorem 6, we know that once $d_k^t + d_l^t \geq \alpha_{kl}$ for some (k,l) in period t, $\mathbb{P}[d_k^{t+m} + d_l^{t+m} < \alpha_{kl}] = 0$ for all future $m = \{1, 2, ...\}$.

Consider the set of pairwise stable states $Z^{\hat{t}}$ when $d_k^{\hat{t}} + d_l^{\hat{t}} \geq \alpha_{kl}$ for all pairs (k,l) and pair (i,j) forms with $d_i^{\hat{t}} + d_j^{\hat{t}} = \alpha_{ij}$ for which fifty-fifty is unconstrained. Note that this set contains the core states because optimal pairs in the rest of society may or may not form and that single-period sequences in that chain may take longer in the original chain (we indicate time in the subsequence by $\widehat{\ }$). The following four facts are true for the subsequence:

First, if $d_i^{\hat{t}} < d_j^{\hat{t}}$, $\quad d_{i'}^{\hat{t}} + d_j^{\hat{t}} > \alpha_{i'j}$ for all $(i',j) \neq (i,j)$.[19] Second, when $d_i^{\hat{t}} \leq d_j^{\hat{t}}$, no transfer larger than δ from i to j may take place before (i,j) forms again. Third, when $d_i^{\hat{t}} < d_j^{\hat{t}}$, transfers from j to i may be larger than δ but cannot be as large as to reverse the relative positions of the two. Fourth, when $d_i^{\hat{t}} < d_j^{\hat{t}}$, the transfer scenario from j to i is more likely than the transfer scenario from i to j.

Jointly, the four facts reveal an inherent equity-bias, irrespective of whether pairs in the rest of the market form optimally or not. As a result, players i and j will share close to fifty-fifty with high probability in the long run.

Claim 1. *For any $Z^{\hat{t}}$ with $d_i^{\hat{t}} < d_j^{\hat{t}}$,*

$$d_{i'}^{\hat{t}} + d_j^{\hat{t}} > \alpha_{i'j}$$

for all $(i'j) \neq (i,j)$.

Suppose there exists a state $Z^{\hat{t}}$ such that $d_i^{\hat{t}} < d_j^{\hat{t}}$ and $d_{i'}^{\hat{t}} + d_j^{\hat{t}} = \alpha_{i'j}$ for some $(i',j) \neq (i,j)$.

Without violating $d_k^{\hat{t}} + d_l^{\hat{t}} > \alpha_{kl}$ for all (k,l), we can reduce, one by one, any $d_k^{\hat{t}}$ for which $d_k^{\hat{t}} + d_{l'}^{\hat{t}} > \alpha_{kl'}$ for all (k,l') and any $d_l^{\hat{t}}$ for which $d_{k'}^{\hat{t}} + d_l^{\hat{t}} > \alpha_{k'l}$ for all

[19] Note this comes straight from the definition of fifty-fifty being unconstrained for (i,j).

(k', l), until no such $\widehat{d_k^t}$ or $\widehat{d_l^t}$ exists. This resulting allocation is a core allocation because it is efficient and pairwise stable. Since $\widehat{d_{i'}^t} + \widehat{d_j^t} = \alpha_{i'j}$ for some $(i', j) \neq (i, j)$ in that allocation, we cannot reallocate to fifty-fifty amongst (i, j) without violating pairwise stability. This contradicts our supposition. Hence, $\widehat{d_{i'}^t} + \widehat{d_j^t} > \alpha_{i'j}$ for all $(i'j) \neq (i, j)$ in any $\widehat{Z^t}$ with $\widehat{d_i^t} < \widehat{d_j^t}$.

Claim 2. *For any $\widehat{Z^t}$ with $\widehat{d_i^t} \leq \widehat{d_j^t}$,*

$$\mathbb{P}[\widehat{d_j^{t+1}} > \widehat{d_j^t} + \delta | \widehat{Z^t}] = \mathbb{P}[\widehat{d_i^{t+1}} < \widehat{d_i^t} - \delta | \widehat{Z^t}] = 0.$$

We start with $\widehat{d_i^t} \leq \widehat{d_j^t}$ for (i, j) and $\widehat{d_k^t} + \widehat{d_l^t} \geq \alpha_{kl}$ for all (k, l).

Suppose j, the player who demands more, increases to $\widehat{d_j^t} + \delta > \frac{\alpha_{ij}}{2}$. Claim One implies that only (i, j) can possibly satisfy j and needs to form in order for j to be able to increase further. For that, i has to reduce. This establishes $\mathbb{P}[\widehat{d_j^{t+1}} > \widehat{d_j^t} + \delta | \widehat{Z^t}] = 0$ and, conversely, $\mathbb{P}[\widehat{d_i^{t+1}} < \widehat{d_i^t} - \delta | \widehat{Z^t}] = 0$.

Claim 3. *For any $\widehat{Z^t}$ with $\widehat{d_i^t} < \widehat{d_j^t}$,*

$$\mathbb{P}[\widehat{d_j^{t+1}} < \frac{\alpha_{ij}}{2} - \delta | \widehat{Z^t}] = \mathbb{P}[\widehat{d_i^{t+1}} > \frac{\alpha_{ij}}{2} + \delta | \widehat{Z^t}] = 0.$$

Claim One implies that any demand for more than $\frac{\alpha_{ij}}{2}$ when $\widehat{d_k^t} + \widehat{d_l^t} \geq \alpha_{kl}$ for all (k, l), by either i or j, can only be satisfied when (i, j) forms. Starting in $\widehat{Z^t}$ with $\widehat{d_i^t} < \widehat{d_j^t}$ ($\Rightarrow \widehat{d_j^t} - \widehat{d_i^t} \geq 2\delta$), transfers from j to i, therefore, cannot reverse the positions of the players (cannot exceed $\frac{\widehat{d_j^t} - \widehat{d_i^t}}{2} + \delta$) before i and j form again. This establishes $\mathbb{P}[\widehat{d_j^{t+1}} < \frac{\alpha_{ij}}{2} - \delta | \widehat{Z^t}] = 0$ and, conversely, $\mathbb{P}[\widehat{d_i^{t+1}} > \frac{\alpha_{ij}}{2} + \delta | \widehat{Z^t}] = 0$.

Claim 4. *For any $\widehat{Z^t}$ with $\widehat{d_i^t} < \widehat{d_j^t}$,*

$$\mathbb{P}[\widehat{d_j^{t+1}} \geq \widehat{d_j^t} + \delta | \widehat{Z^t}] \quad \text{and} \quad \mathbb{P}[\widehat{d_i^{t+1}} \leq \widehat{d_i^t} - \delta | \widehat{Z^t}] \quad \leq \frac{r}{2n} \frac{a\widehat{d_i^t}}{2n},$$

$$\mathbb{P}[\widehat{d_j^{t+1}} \leq \widehat{d_j^t} - \delta | \widehat{Z^t}] \text{ and } \mathbb{P}[\widehat{d_j^{t+1}} \leq \widehat{d_j^t} - \delta | \widehat{Z^t}] \quad \geq \frac{r}{2n}\frac{a\widehat{d_j^t}}{2n}.$$

j increases with probability $\frac{r}{2n}$. Subsequently, if no $j' \neq j$ exists for whom $\widehat{d_i^t} + \widehat{d_{j'}^t} = \alpha_{ij'}$, i decreases with probability $\frac{a\widehat{d_i^t}}{2n}$. If some $j' \neq j$ exists for whom $\widehat{d_i^t} + \widehat{d_{j'}^t} = \alpha_{ij'}$, i decreases with probability less than $\frac{a\widehat{d_i^t}}{2n}$. This establishes $\mathbb{P}[\widehat{d_j^{t+1}} = \widehat{d_j^t} + \delta | \widehat{Z^t}] \leq \frac{r}{2n}\frac{a\widehat{d_i^t}}{2n}$ and $\mathbb{P}[\widehat{d_i^{t+1}} = \widehat{d_i^t} - \delta | \widehat{Z^t}] \leq \frac{r}{2n}\frac{a\widehat{d_i^t}}{2n}$.

Note that both, $\mathbb{P}[\widehat{d_j^{t+1}} = \widehat{d_j^t} + \delta | \widehat{Z^t}]$ and $\mathbb{P}[\widehat{d_i^{t+1}} = \widehat{d_i^t} - \delta | \widehat{Z^t}]$, are less than $\frac{r}{2n}\frac{a\widehat{d_i^t}}{2n}$ when there exists some $j' \neq j$ for which $\widehat{d_i^t} + \widehat{d_{j'}^t} = \alpha_{ij'}$. In this case, i could possibly increase and j could decrease (several times) before (i, j) forms again.

i increases with probability $\frac{r}{2n}$. It follows from Claim One that no $(i', j) \neq (i, j)$ exists such that $\widehat{d_{i'}^t} + \widehat{d_j^t} \leq \alpha_{i'j}$. Hence, j reduces with probability $\frac{a\widehat{d_j^t}}{2n}$. This establishes $\mathbb{P}[\widehat{d_j^{t+1}} \leq \widehat{d_j^t} - \delta | \widehat{Z^t}] \geq \frac{r}{2n}\frac{a\widehat{d_j^t}}{2n}$ and $\mathbb{P}[\widehat{d_j^{t+1}} \leq \widehat{d_j^t} - \delta | \widehat{Z^t}] \geq \frac{r}{2n}\frac{a\widehat{d_j^t}}{2n}$.

Jointly, the four claims imply that, given any $\widehat{Z^t}$ with $\widehat{d_i^t} < \widehat{d_j^t}$,

$$\mathbb{E}[\widehat{d_j^{t+1}} - \widehat{d_i^{t+1}} | \widehat{Z^t}] \leq (\widehat{d_j^t} - \widehat{d_i^t}) - \frac{\delta r a}{2n^2}(\widehat{d_j^t} - \widehat{d_i^t}).$$

Since $\widehat{d_i^t} < \widehat{d_j^t} \Rightarrow (\widehat{d_j^t} - \widehat{d_j^t}) \geq 2\delta$,

$$\mathbb{E}[\Delta(\widehat{d_j^t} - \widehat{d_i^t}) \ | \ \widehat{Z^t}] \leq -\frac{\delta r a}{2n^2} 2\delta.$$

Starting in any $\widehat{Z^0}$, iteration and substituting $c = \frac{\delta^2 r a}{n^2}$ yield

$$\mathbb{E}[|\widehat{d_i^t} - \widehat{d_j^t}|] \leq \max\{2\delta, |\widehat{d_i^0} - \widehat{d_j^0}| - c\widehat{t}\},$$

which is smaller than 2δ for any $\widehat{t} > \widehat{T_{\beta,\gamma}}$ when $|\widehat{d_i^0} - \widehat{d_j^0}| - c\widehat{t} \leq 2\delta$ with $\widehat{T_{\beta,\gamma}} \geq \frac{1}{c}(|\widehat{d_i^0} - \widehat{d_j^0}| - 2\delta)$.

Rearranging for the Markov inequality, it holds that, for any $\widehat{Z^0}$ with $\delta \leq \frac{\beta\gamma}{2}$, after $\widehat{t} > \widehat{T_{\beta,\gamma}}$,

$$\mathbb{P}[|d_i^{\widehat{t}} - d_j^{\widehat{t}}| \geq \beta] \leq \gamma. \tag{3.2}$$

□

When fifty-fifty is not in the core for an optimal pair, these players won't split equally in the long run. They may, however, still end up splitting quite equitably, pairwise stable outside alternatives permitting. The positive efficiency, stability and equity results, however, may break down when search is incomplete.

When search is incomplete

The completeness of search, p, determines the long-run probability that players are misassigned or not assigned at all.

Proposition 9. *Given any assignment game played with our procedure and search completeness, p, the proportion of time that any pair (i,j) forms is at most $\frac{p}{(1-p)r+p}$ in the long run.*

Proof. Decompose the set of states, Ω, into $\Omega_{a_{ij}=1}$ and $\Omega_{a_{ij}=0}$, denoting the states in which a given optimal pair (i,j) either forms or not. Let $\mu_{a_{ij}=1}$ be the probability of being in $\Omega_{a_{ij}=1}$ and $1 - \mu_{a_{ij}=1} = \mu_{a_{ij}=0}$ the probability of being in $\Omega_{a_{ij}=0}$.

If $\frac{r(1-p)}{n}$ is a lower bound on leaving $\Omega_{a_{ij}=1}$ and $\frac{p}{n}$ an upper bound on entering $\Omega_{a_{ij}=1}$, then

$$\mu_{a_{ij}=1} \times (1 - \frac{r(1-p)}{n}) + (1 - \mu_{a_{ij}=1}) \times \frac{p}{n} = \frac{p}{(1-p)r+p} \geq \mu_{a_{ij}=1},$$

i.e. $\frac{p}{(1-p)r+p}$ is indeed an upper bound on $\mu_{a_{ij}=1}$.

Now, we prove the bounds.

Lower bound on leaving $\Omega_{a_{ij}=1}$: We leave $\Omega_{a_{ij}=1}$ if one of the two players is selected (with probability $2\frac{1}{2n}$) who breaks the partnership (with probability r) and then fails to find the same partner again, even if still feasible, (with probability $1-p$). $\frac{r(1-p)}{n}$ is, therefore, a lower bound on leaving $\Omega_{a_{ij}=1}$. This is true for all $Z \in \Omega_{a_{ij}=1}$ irrelevant of how many feasible alternatives exist.

Upper bound on entering $\Omega_{a_{ij}=1}$: We enter $\Omega_{a_{ij}=1}$ if one of the players is selected (with probability $2\frac{1}{2n}$) who may be unassigned and find the other (with probability p). If either player is assigned somewhere else, entering $\Omega_{a_{ij}=1}$ will be less likely. $\frac{p}{n}$ is, therefore, an upper bound on entering $\Omega_{a_{ij}=1}$. $\square$

When search is too incomplete, any one pair in particular (even if optimal) forms with small probability $\frac{p}{(1-p)r+p}$. If few optimal assignments exist, this implies that optimal pairs rarely form and that the pairwise stable set will not be implemented in the long-run.

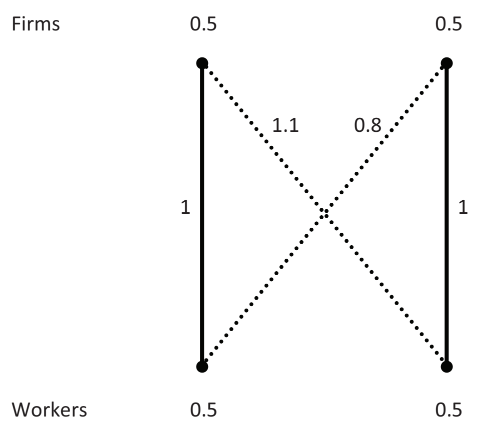
Appendix

A comparison with Jackson & Wolinsky [1996]

Jackson & Wolinsky [1996] consider a network pairwise stable if it is robust to one-link deviations. As one-link deviations, they permit unilateral link-cutting by an assigned player and bilateral link-creation by two free agents. Both actions, however, are not permitted at the same time: two agents that are assigned elsewhere cannot both break their links and form a new link together.

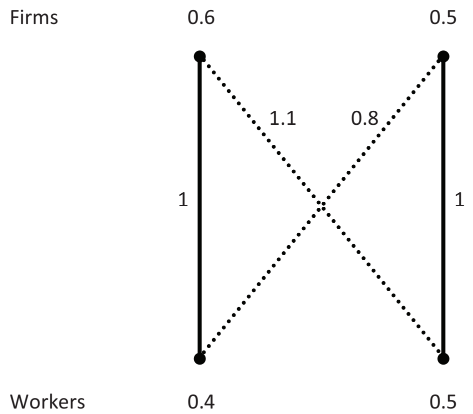
Based on our procedure, the probabilistic stability of partnerships can be assessed in terms of one-link deviations.

Below, we consider a state from Example 3.1 that is, in terms of one-link deviations, pairwise stable. No agent has an interest to cut his link unilaterally and no two free agents exist. That state, however, is not pairwise stable in a matching sense because Firm One and Worker Two prefer to break both of their links at the same time and form a pair. We will show that, in our procedure, such a state is also probabilistically relatively unstable. In the calculations, we consider the equiprobable pair selection case. Given a searcher's reached set of possible partners, $\mathcal{S}$, $\mathcal{F} \subseteq \mathcal{S}$ of which feasible, each feasible pair forms with probability $\frac{1}{|\mathcal{F}|}$.



Actively, anybody's increase breaks his own existing pair because it can not accommodate any higher demands. Furthermore, the existing pairs are also cut if Firm One and Worker Two match. Pair-separation, therefore, has an active and passive component: $\mathbb{P}[a_{11}^{t+1} = 0|Z^t] = \mathbb{P}[a_{22}^{t+1} = 0|Z^t] = \frac{r}{2} + \frac{r}{4}p$. The partnership of Firm Two and Worker One will not form because current demands are too high: $\mathbb{P}[a_{21}^{t+1} = 1|Z^t] = 0$. Firm One and Worker Two match if either one breaks his current pair and finds the other: $\mathbb{P}[a_{12}^{t+1} = 1|Z^t] = \frac{r}{2}p$.

Below, we consider a pairwise stable state, both in terms of one-link deviations and in a matching sense. We will show that, compared to the above state which was pairwise instable in a matching sense, this state is also probabilistically more stable. This extra stability is due to pair-separation being less likely and new pair-formation having probability zero.



Any further increase breaks all feasibilities and no new pair can form: $\mathbb{P}[a_{12}^{t+1} = 1|Z^t] = \mathbb{P}[a_{21}^{t+1} = 1|Z^t] = 0$. Anybody's active increase breaks his partnership but no pair is passively rematched: $\mathbb{P}[a_{11}^{t+1} = 0|Z^t] = \mathbb{P}[a_{22}^{t+1} = 0|Z^t] = \frac{r}{2}$.

Comparing the probabilistic stability of stable (core) and instable states, we note that the probability of active pair-separation is always positive but higher when unstable in a matching sense. Furthermore, core-stable states are robust against formation of suboptimal pairs because search will occur at demand levels that preclude suboptimal pairs from forming. In unstable states, formation of new pairs may have positive probability but it may cause existing, optimal partnerships to separate.

Chapter 4

Evolutionary coalitional games

Abstract

We study dynamic coalitional games in characteristic function form played by agents who feel their way to the gains from cooperation by forming different coalitions. Based solely on each individual's own assessment of how well he did in the past, the players vary their coalitional commitments and adjust their demands. For balanced games, the procedures lead to the implementation of the core. For games with an empty core, the grand coalition breaks down and inefficient subcoalitions continue to form in the long run. Inherent tendencies towards equity are revealed but inequities persist where higher coalitional alternatives exist.

JEL classifications: C71, C73, D83

Keywords: cooperative game theory, core, evolutionary game theory, (completely uncoupled) learning, stability

4.1 Introduction

In cooperative game theory, a rich literature exists that supports stable and equitable outcomes, both normatively and in terms of implementations via bargaining mechanisms. Of particular interest has been the study of persistent inequalities due to coalitional participation constraints. Ideally, outcomes are robust against coalitional deviations and equitable at the same time. Aumann & Maschler [1964] describe the basic admixture of normative (equity) and positive (stability) considerations of these outcomes:

> "*if all things are equal*, it is *fair* to divide the profits equally. [...] If *all things* are <u>not</u> equal, people will still be happy with their coalition if they agree that the *stronger* partners will get more."[1]

In other words, it is fair to split equally amongst all unless some have a good reason (in terms of their position in the game) to get more. A number of solutions with this flavour have been proposed. Aumann & Maschler [1964] propose the bargaining set of allocations against which no justified objection by any subgroup of the population exists. Selten [1972] proposes the equal division core of allocations that are robust against deviation by subgroups who split equally amongst themselves. Under standard assumptions, both Aumann & Maschler's [1964] bargaining set and Selten's [1972] equal division core contain the core because the relevant deviation constraints are more lax in these solutions and, therefore, permit more equitable allocations.[2] Inequitable core allocations, on the other hand, are still contained in the bargaining set and in the equal division core. Additional arguments are needed to identify outcomes in the core.

[1] Aumann & Maschler [1964], p. 446.

[2] The core and bargaining set of convex games coincide (Maschler, Peleg & Shapley [1972]).

The algorithm by Dutta & Ray [1989] selects a particular equitable allocation. For the class of convex games, the algorithm implements a constrained Lorenz-maximal allocation that is unique and generally lies on the frontiers of the core, of the bargaining set and of the equal division core. From that allocation, no Pigou-Dalton transfer reduces inequity further without violating participation or incentive compatibility constraints. Generalisations and characterisations of these "constrained egalitarian" outcomes have been proposed.[3]

Constrained egalitarian solutions are not only appealing for the admixture of stability and equity but also because, compared to other (more inequitable) solutions, they fare well in light of experimental evidence (e.g. Miller [1980], Selten [1987]). We contribute to the literature by providing a fully dynamic bargaining model based on trial and error; learning in a basic form. Only recently have evolutionary bargaining models been extended to coalitional games. The existing models extend the strategy space of evolutionary bargaining models without coalitions to allow for the formation of subcoalitions. This generalisation poses substantial problems because the underlying bargaining game is essentially nonstrategic and is not based on the unilateral deviation principle of noncooperative game theory. In our models, players do not have a strategic model at all. Instead, agents adjust their demands incrementally and change their coalitions locally, based solely on their own assessments of how well they did in the past.

The long-run dynamics of the procedures generally support outcomes that are stable and equitable. The process is inherently geared towards stable outcomes, and once stability is achieved, towards equity within the constraints posed by coalitional incentives. Instabilities posed by groups with superior coalitional alternatives tend to vanish in the long run. In balanced games, the core is implemented most of the time. In games with an empty core, long-run ineffi-

[3] A more detailed discussion will follow in this chapter (pp. 104-105).

ciencies persist. Once the procedure has reached the core, tendencies towards equity are revealed but halted if superior coalitional alternatives exist.

Structure

The remainder of this introduction discusses related literature, provides a synopsis of the model and sketches the results. In the following two sections, we set up the model and go through the dynamics of an example. Section Four contains formal results concerning stability and efficiency. Section Five analyses the dynamics leading to equity and illustrates the findings at hand of an example.

Coalitions and constrained social optima

Ray's [2007] book provides an excellent review of the game-theoretic literature on coalition formation. The book discusses cooperative solution concepts, foundations of coalitional games and noncooperative implementations of coalition formation. The present work complements the existing literature with an evolutionary cooperative model.

Chapters Two and Three of this thesis develop evolutionary procedures for cooperative bargaining and assignment games, that is, for cooperative games with limited coalition structures. In cooperative bargaining, the whole society has to reach a cooperative outcome jointly, coalitions cannot form locally. Loss-averse behaviour and proportional demand-revisions induce internal bargaining dynamics that favour equity. Over time, this equity bias leads to implementation of equitable outcomes most of the time.

In assignment games, many disjoint partnerships form and cooperation may continue amongst some pairs when others separate. Players continue to experiment with different arrangements and partners. In the long-run, our procedures implement optimal assignments and pairwise stable allocations when search is complete. Within partnerships, players will share close to fifty-fifty most of the long-run time unless one or both of the partners have more profitable outside alternatives.

In both previous chapters, the tendencies towards equity stem from the same internal bias: as partnerships reach long-run stable possibility frontiers, that is, as they reach stable and efficient outcomes, transfers from rich to poor are more likely than the other way around. Over time, this favours stable and efficient outcomes with higher levels of equity. Equity, however, is constrained by outside alternatives and ensuing pairwise instabilities.

In this chapter, we develop learning procedures for general coalitional games and extend our dynamic apparatus to environments where coalitions of any size may form. The results from the previous chapters generalise. For balanced games, unblockable allocations are implemented (the core). In games with empty cores, long-run inefficiencies persist. Inherent equity-tendencies are halted by coalitional participation incentives.

Our model complements several strands of literature concerned with implementation of core-stable outcomes.

Noncooperative core implementation. Lagunoff [1994] proposes a mechanism in which players sequentially announce coalition plans that have to be ratified by the other players. Proposals that are ratified by everybody, in equilibrium, must satisfy all coalitions, or else some coalition could unsettle the current agreement. Hence, only core allocations are supported as Nash equilibria. Lagunoff's [1994] model is based on earlier work by Kalai, Postlewaite & Roberts [1979]. Alternative mechanisms are proposed by Chatterjee, Dutta, Ray & Sengupta [1993], Perry & Reny [1994], Serrano [1995] and Serrano &

Vohra [1997]. In all models, the set of core allocations is implemented via the set of (subgame-perfect) Nash equilibria of (multistage) bargaining games played by perfectly rational and completely informed players.

Dynamic coalition formation. Konishi & Ray [2003] propose a model in which a fixed population of perfectly rational agents with common knowledge and beliefs plays the same cooperative game repeatedly. Groups of players continue to be selected and are given the opportunity to reconfigure their coalitional agreements. The selected groups decide what to do based on their current inner allocation and on the present value of alternative configurations. Groups reconfigure to alternative structures if they promise higher present values. No reconfigurations will occur if no deviation promises a higher present value. For balanced games, the core is implemented when agents are sufficiently farsighted.

Evolutionary noncooperative implementations. Evolutionary models relax the rationality and information assumptions and allow for bounded rationality. Extending the evolutionary bargaining model without coalitions by Young [1993], Agastya [1997], [1999], Arnold & Schwalbe [2002], Rozen [2010], [2010a] and Newton [2010] allow for coalitional response dynamics. The dynamics reveal that groups of players who currently receive less than they are able to achieve as separate subgroups will eventually coordinate away from the others and jointly improve their positions. For balanced games, this leads to core implementation after all coalitional possibilities are realised. Within the core, drifts towards equity are revealed (Agastya [1999], Rozen [2010a], Newton [2010]). In contrast to Young's [1993] original bargaining model without coalitions, however, equity amongst individuals is not necessarily favoured in the long run because individuals in different positions of the game may have different coalitional alternatives. The exact nature of the long-run stable allocations depends crucially on how strategies are specified when coalitions form.[4]

The basic difficulty of this strand of models is to formulate, from a cooperative game which is essentially nonstrategic, the relevant players in terms of individuals and coalitions that consist of individuals. With both types of players (individuals and coalitions), strategies need to be associated even though one is a member of the other. Any such cooperative-strategic association "is only one non-cooperative representation of the cooperative game" and the "lack of 1-1 correspondence between cooperative and non-cooperative games is likely to prohibit any generalized application of stochastic stability arguments to cooperative game theory."[5]

Constrained egalitarianism. Constrained egalitarian outcomes of convex games are implemented by the Dutta-Ray algorithm (Dutta & Ray [1989]) and by variants and generalisations thereof (Dutta [1990] and Dutta & Ray [1991]) for more general classes of games. Mechanisms algorithmically select outcomes with both appealing stability and equity qualities. In these algorithms, groups of players with maximal average payoffs, for which no superior coalitional alternatives exist, are recursively selected. Within each such group, payoffs are shared Lorenz-maximally until no coalition exists for which superior alternatives exist and all subgroups share Lorenz-maximally. For the class of convex games, the Dutta-Ray [1989] algorithm selects the unique Lorenz-maximal allocation in the core. That outcome is also selected by application of the Rawlsian criterion on the core and is Rawls-maximal (Arin & Inarra [2001]). Moreover, it coincides with the coalitional Nash bargaining solution (Compte & Jehiel [2010]) and is obtained by core-constrained maximisation of

[4]The implied levels of equity for general classes of games in Newton [2010] differ substantially from those in Agastya [1999] and Rozen [2010a] because different strategies are assigned with coalitions in the respective models.

[5]Newton [2010], p. 2.

any symmetric additive strictly concave social welfare function or minimisation of any symmetric additive strictly convex inequity measure (Hougaard, Peleg & Thorlund-Petersen [2001]).

For the larger class of balanced games, the Dutta & Ray [1991] algorithm may fail to pick an allocation in the core when the game is nonconvex. Constrained egalitarian solutions in the core have been characterised in Klijn, Slikker, Tijs & Zarzuelo [2000], Arin & Inarra [2001], Hougaard, Peleg & Thorlund-Petersen [2001], Koster [2002], Hokari [2002], Jaffray & Mongin [2003], Arin, Kuipers & Vermeulen [2003], Hougaard, Peleg & Osterdal [2005], Hougaard & Osterdal [2010] and Compte & Jehiel [2010].

Model synopsis and sketch of results

Each period, a cooperative game is played by a large population of agents who make demands as conditions for joining coalitions. When demands are globally feasible, coalitions form and players may get their demand. When some demands are infeasibly high, some feasible coalitions may still form and continue cooperation locally but the players with infeasibly high demands will be left out until they reduce sufficiently.

The central assumption concerning behaviour is that it is based solely on each individual's own assessment of how well he did in the past. In the current period, decision-making is completely uncoupled from the others' actions, differing solely according to whether a player receives payoffs that exceed his aspirations (player is satisfied) or not (player is dissatisfied). Satisfied players in successful cooperative partnerships occasionally break their current coalition in order to try and search for a coalition that pays more. This experimentation comes at the risk of causing global or local cooperative failure. Players whose demands are not satisfied become dissatisfied. Dissatisfied players search for coalitions to accommodate their demands and reduce their demands if left out of coalitions for extended periods of time.

As in the previous chapters, this behaviour takes place in a decentralised and anonymous market place where players are poorly informed about others current and past actions and payoffs. Recall the example of electronic market makers (such as *Priceline.com*) in the context of one-to-one and many-to-one matching (see Chapter Three, pp. 59-60). Of course, many environments allow for more complex matchings than pairs in assignments. Hotels using an electronic market maker, for example, are repeatedly matched to many customers but one customer is usually matched to none or only one hotel. The model of this chapters generalises the procedures from the previous cooperative environments to general cooperative games.

The basic ingredients of the behavioural model are incremental adjustments, loss aversion and search. Adjustments are made locally and in small increments. During cooperative failure, players are more likely to make attempts at restoring cooperation the greater their loss. During search, players screen cooperative alternatives. When search is complete, feasible coalitions are identified and coordinated upon.

Over time, two elements shape the outcomes that are long-run stable. On the one hand, coalitional participation incentives gear the process towards stability, that is, towards allocations that are unblockable. This leads to implementation of the core for games with nonempty cores and to persistent inefficiencies for games with an empty core. On the other hand, due to loss aversion, the procedures reveal an inherent drift towards equity. These equity-tendencies, however, are halted if subgroups hold superior coalitional alternatives to others. In other words, stability and equity are both inherently favoured when not in conflict. When conflicting, the stability tendencies trump those favouring equity.

4.2 Evolutionary cooperative games

A fixed population of players, $N = \{1, ..., n\}$, repeatedly plays a cooperative game in characteristic function form. Each period, some coalition structure (a partition of N) forms, players make demands and receive payoffs from joining coalitions that accommodate their demands.

Cooperative game. *$G(v, N)$ is the cooperative game with $v : C \to \mathbb{R}$ for all $C \subseteq N$ and $v(\emptyset) = 0$. v is normalised so that $v(i) = 0$ for all $i \in N$.*

We assume superadditivity which captures the notion that coalition formation is beneficial.

Superadditivity.

> *v is superadditive if $v(C \cup C') \geq v(C) + v(C')$ for all $C, C' \subset N$ such that $C \cap C' = \emptyset$.*

Let $\phi = \{\phi_1, ..., \phi_n\}$ be an allocation of payoffs to all players. If ϕ is such that no blocking coalition exists for which deviation and internal rearrangement may Pareto-improve their position, we say such an allocation is group-stable.

Group stability.

> *ϕ is group-stable if $\sum_{i \in S} \phi_i \geq v(S)$ for all $S \subseteq N$.*

Core.

> *ϕ is in the core if, and only if, $\sum_{i \in N} \phi_i = v(N)$ and ϕ is group stable.*[6]

Due to Bondereva [1963] and Shapley [1967] is the result that the necessary and sufficient condition for nonemptiness of the core is balancedness.

[6] Note that the notion of group-stability in our definition of the core is usually referred to as "unblockability." Since our model is not based on the usual story of objections and deviations, group stability is a more appropriate wording in our context.

Balancedness.

$F = \{S_1, ..., S_m\}$ is a balanced family of coalitions $S \subseteq N$ if there exists a vector of balancing weights $\lambda = (\lambda(S_1), ..., \lambda(S_m))$, such that $\lambda(S) > 0$ for all $S \in F$ and, for any player i, $\sum_{S \in F: i \in S} \lambda(S) = 1$.

v is balanced if, and only if, for all balanced families F with balancing vector λ, it is true that $v(N) \geq \sum_{S \in F} \lambda(S)v(S)$.

Dynamics

States. $Z^t = (\rho^t, \phi^t, d^t)$ is a state of our process in period t, specifying a coalition structure ρ^t (partition of N), a vector of payoffs $\phi^t = (\phi_1^t, ..., \phi_n^t)$ for players 1 to n and a vector of demands $d^t = (d_1^t, ..., d_n^t)$ for players 1 to n. Demands and payoffs are nonnegative real numbers. The set of all states is Ω.

Satisfaction. When a player's payoff matches or exceeds his demand, he becomes satisfied: i is *satisfied* in period t if $\phi_i^{t-1} \geq d_i^{t-1}$. Otherwise, when his share of the surplus falls short of his demand, he becomes *dissatisfied*.

Payoffs. Given any current coalition structure, non-singleton coalitions $S \in \rho^t$ pay their members their demands, all others are singletons and receive zero: $\phi_i^t = d_i^t$ for all $i \in S: \ S \in \rho^t$ and $|S| > 1$, $\phi_j^t = 0$ for all other j.

Our process is driven by two types of actions. On the one hand, players hold demands and occasionally revise them. On the other hand, when temporarily without coalition, players search for coalition partners. Each period, one player is selected (with probability $\frac{1}{n}$) and allowed to revise his demand and search for a match. The other players don't search and their demands remain unchanged.

Search. During search, a player addresses possible coalition partners. Depending on the current demand levels, some coalitions may be found that are feasible. Players may search when satisfied and when dissatisfied. Let t be the current period. A satisfied player ($\phi_i^{t-1} \geq d_i^{t-1}$) searches for a coalition to pay his new, higher demand $d_i^t > d_i^{t-1}$. A dissatisfied player ($\phi_i^{t-1} < d_i^{t-1}$) searches with his previous demand d_i^{t-1}.

We consider the case when search is complete, that is, when all players with whom the searcher can feasibly form a coalition are reached.[7] Given a current demand vector, d^t, including the searcher's own demand, d_i^t, there is a set of feasible coalitions, $\mathcal{F}$, such that, for all $S \in \mathcal{F}$, $i \in S$ and $\sum_{i \in S} d_i^t \leq v(S)$, and $\sum_{i \in S'} d_i^t > v(S')$ for all $S \notin \mathcal{F}$. If $\mathcal{F}$ is nonempty, the search of i will find it and some $S \in \mathcal{F}$ will form.

Demand transitions. The selected player increases or decreases his demand, dependent on whether he is satisfied or not.

When a satisfied player is selected, he breaks his current partnership with some small probability, r, to demand an extra δ. With this new demand, he searches for a new coalition. With probability $1 - r$, he remains inactive, his current partnership and demands unchanged.

When a dissatisfied player is selected, he immediately searches for a coalition to accommodate his demand. If he finds no match at the end of his search, he demands δ less than previously with a high probability $1-s_i^t$. With probability s_i^t, he does not decrease his demand.

We assume that the probability r, with which a satisfied player breaks an agreement in order to demand more is constant and small. The degree of stickiness, s_i^t, features loss aversion, interpreted in the sense that players who suffer greater loss during cooperative failure are more likely to adjust their

[7]See Chapter Three (pp. 67-68) for more detail on complete search and the incomplete search case (pp. 93-94).

demands. We assume loss aversion in constant linear form $1 - s_i^t = ad_i^{t-1}$ for some small a for all players i and for all time periods t.[8]

Coalitional transitions. The coalition structure, ρ^t, remains unchanged if no player searches. ρ^t changes when a player searches and finds a new coalition.

Suppose Z^t is the state in period t with coalition structure ρ^t. Suppose player i is selected in period $t+1$.

If i is dissatisfied in period $t+1$, he searches with demand $d_i^{t+1} = d_i^t$ and the demand vectors of periods t and $t+1$ are identical. Given the demand vector, none to several coalitions involving the searcher may be feasible. If not all are infeasible, some coalition S such that $i \in S$ may form with $\sum_{i \in S} d_i^t \leq v(S)$. When no coalition $S \subseteq N$ exists such that $i \in S$ and $\sum_{i \in S} d_i^t \leq v(S)$, all coalitions remain unchanged and i will remain a singleton coalition. If a nonempty set $\mathcal{F}$ of coalitions exists such that $i \in S$ and $\sum_{i \in S} d_i^t \leq v(S)$ for all $S \in \mathcal{F}$, we assume some coalition in $\mathcal{F}$ forms.[9] All $C \in \rho^t$ with $C \cap S = \emptyset$ remain unchanged. All individuals in $C \setminus S$, for all $C \in \rho^t$ with $C \cap S \neq \emptyset$, break into singletons.

If i is satisfied in period $t+1$, he may increase his demand to $d_i^{t+1} = d_i^t + \delta$ and search with that higher demand. A new coalition structure may result where some new coalition S such that $i \in S$ may form with $\sum_{i \in S} d_i^t \leq v(S)$ and other coalitions break down. If no coalition $S \subseteq N$ exists such that $i \in S$ and $\sum_{i \in S} d_i^{t+1} \leq v(S)$, all members of i's original coalition C end up as singletons, all other coalitions remain unchanged. If $\mathcal{F}$ is nonempty, some $S \in \mathcal{F}$ forms randomly. All coalitions $S' \in \rho^t$ with $S' \cap S = \emptyset$ remain unchanged. All

[8] See Chapter Two (pp. 27-35) for more detail.

[9] Again, we need that the searcher i finds some coalition, no matter which one, and will be satisfied with probability one next period. Many possible selection mechanisms for which coalition forms when many are feasible are possible (e.g. equiprobable, most efficient, largest, etc.). The results concerning core-stability are robust to these alternative coalition selection mechanisms.

members of i's original coalition C who are not contained in the newly formed S will be left as singletons. For all coalitions S' for whom $S' \cap S \neq \emptyset$, where some coalition members have been snatched away by formation of S, the individuals left in $S' \setminus S$ break into singletons.

Illustration. *Consider the cooperative game $G(v, N)$ with $N = \{1, 2, 3, 4\}$, $v(C) = 6$ for $|C| \geq 3$, and $v(C) = 0$ for all other $C \subset N$. Say the current state is Z^t with coalition structure $\rho^t = \{1, (2, 3, 4)\}$, payoffs $\phi^t = \{0, 2, 2, 2\}$ and demands $d^t = \{1, 2, 2, 2\}$.*

The following table illustrates a possible state in *period t*. Player One is a singleton (the bracket indicates the singleton player), demanding one and receiving zero. The other players form a three-player coalition (indicated by no brackets), demanding and receiving two.

Partition	[Player 1]	Player 2	Player 3	Player 4
Demands	1	2	2	2
Payoffs	0	2	2	2

The following tables illustrate the three states that may follow when Player One is selected in *period* $t+1$, depending on which of the feasible coalitions forms. Player One is dissatisfied and searches with demand $d_i^t = 1$ (the searcher is underlined in the tables below). He will find feasible coalitions in $(1, 2, 3)$, $(1, 2, 4)$ and $(1, 3, 4)$ and one of the following three states will be played (again, the bracket indicates the singleton player, the other players form a three-player coalition).

Partition	**Player 1**	Player 2	Player 3	Player 4
Demands	1	2	2	2
Payoffs	1	2	2	0

Partition	**Player 1**	Player 2	Player 3	Player 4
Demands	1	2	2	2
Payoffs	1	2	0	2

Partition	**Player 1**	Player 2	Player 3	Player 4
Demands	1	2	2	2
Payoffs	1	0	2	2

Suppose instead that, *in period* $t+1$, Player Two who is currently satisfied is selected and increases his demand to three. Now, only coalitions $(1,2,3)$ and $(1,2,4)$ are jointly feasible with Player Two's new demand. Hence, one of two states as illustrated in the tables below will be played.

Partition	Player 1	**Player 2**	Player 3	Player 4
Demands	1	3	2	2
Payoffs	1	3	2	0

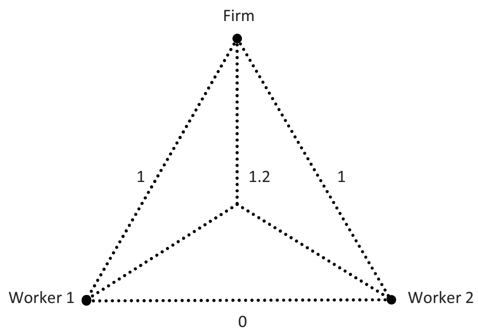
Partition	Player 1	**Player 2**	Player 3	Player 4
Demands	1	3	2	2
Payoffs	1	3	0	2

Markov properties. Since, the transition probability between any two states $\mathbb{P}(Z^{t+1}|Z^t, Z^{t-1}, ..., Z^0)$ is $\mathbb{P}(Z^{t+1}|Z^t)$, the process is Markov. For simplicity, we shall assume that all coalitional worths lie on a δ-grid including the initial states so that the chain moves on that grid only.

The set of all states Ω is finite. Because the chain satisfies standard properties, a unique stationary equilibrium distribution exists.

4.3 Evolving play

Employer-employee game. *$G(v, N)$ is such that $N = \{F, W_1, W_2\}$, $v(N) = 1.2$, $v(F, W_1) = v(F, W_2) = 1$, $v(W_1, W_2) = 0$ and $v(i) = 0$ for all $i \in N$.*[10]



A Firm may either produce with one of two workers or with both.[11] With both workers, the Firm generates a profit of 1.2, which is 0.2 higher than the unit-profit he makes partnering only one of the workers. Firm without workers and workers without Firm are worthless.

The core of this game is nonempty and consists of all allocations ϕ such that the grand coalition forms with $\sum_{i \in N} \phi_i = 1.2$ and $\phi_F + \phi_{W_i} \geq 1$ for both firm-worker pairs. The maximum allocation to the Firm is everything, the minimum 0.8. The maximum to a worker is 0.2, the minimum 0. Clearly, the

[10] Example 7.6. from Ray [2007], pp. 124-125. Note this game is not convex: $v(N) - v(F, W_1) = 0.2 < v(F, W_2) - v(F) = 1$. For nonconvex games, the Dutta-Ray algorithms generally fail to select an allocation in the core. In this example, an inefficient outcome is selected, one of the two firm-worker pairs forms, leaving the other worker out.

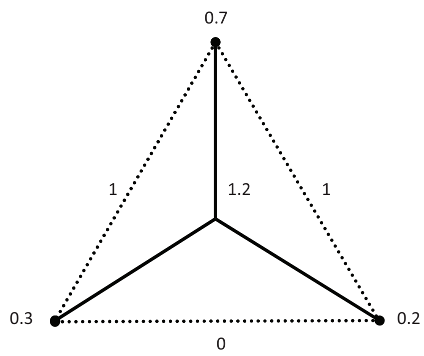
[11] We can also think of them as representing many players in terms of management and unions.

equal split is not in the core.

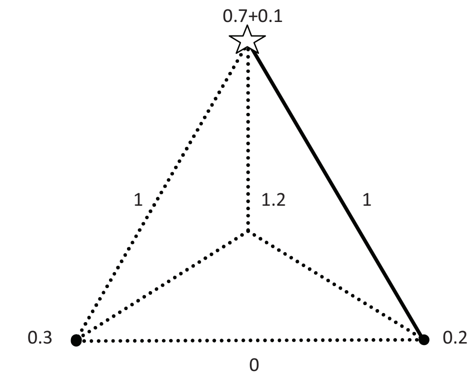
What outcomes do our procedures implement?

A transition into the core

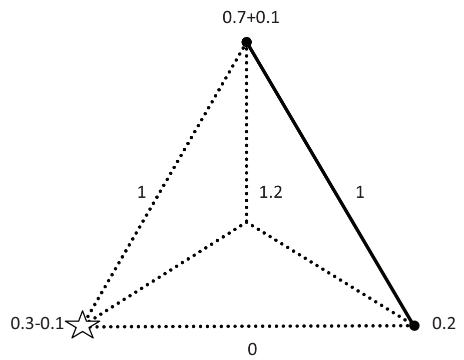
At period t, the following graph describes a possible state of the game, dashed lines indicating coalitions that could form and full lines indicating coalitions that form. The numbers next to the lines indicate that coalition's worth. The numbers on the edges indicate what players currently demand and, in the case that they are in coalitions, also get. Here, N has formed and all players get their respective demands. However, the current allocation is blockable because Firm and Worker Two (currently receiving 0.7 and 0.2 respectively) strictly prefer to form a partnership (worth 1) and distribute the extra 0.1 amongst themselves.



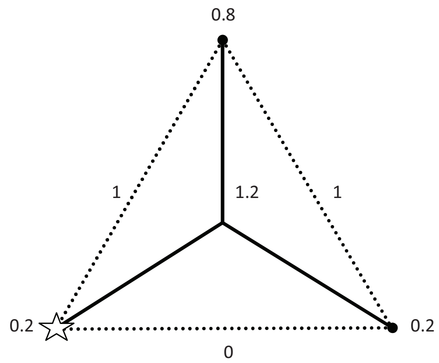
In period $t + 1$, some player is selected at random. Let us suppose that the Firm is selected (the selected player is indicated by the star) who breaks the partnership with both workers and increases its demand by $\delta = 0.1$, finding a partner in Worker Two. That leaves Worker One without coalition and dissatisfied.



In period $t + 2$, suppose that Worker One is selected, who searches but finds no partner at his current demand, and reduces his demand by 0.1 at the end of the period.

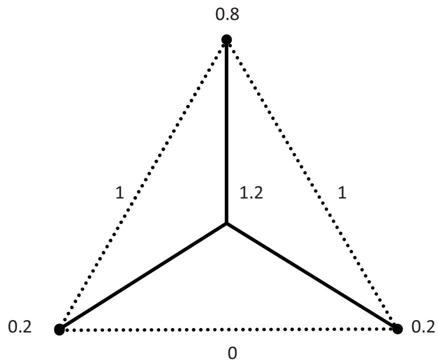


In period $t+3$, a second selection of Worker One already leads to successful search and re-formation of the grand coalition. In contrast to the initial period-t allocation, the resulting allocation is in the core: there exists no firm-worker pair that can do better.

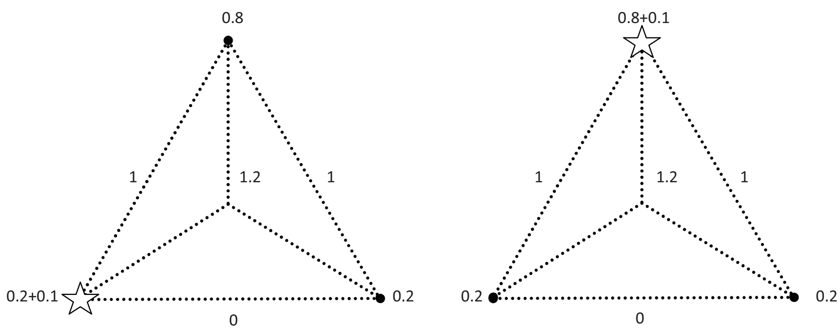


Stability in the core

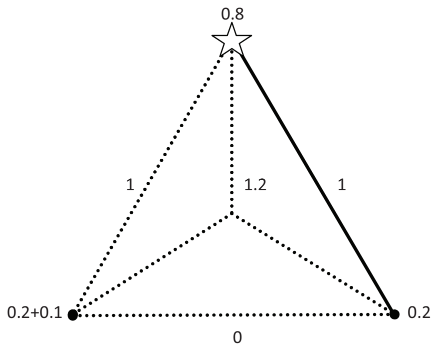
Now suppose we start *in period* t' at a core state as in period $t+3$ above.



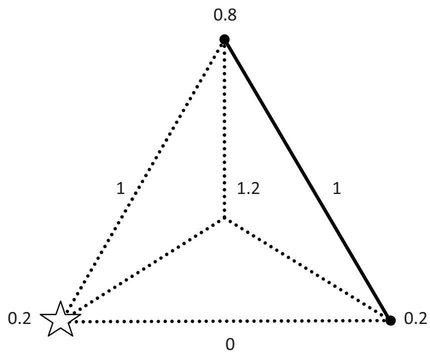
In period $t' + 1$, an increase by any of the three players is equally likely and will break the current agreements, whether made by a worker or by the Firm. Suppose either Firm or Worker One break the agreement. In either case, the searcher can find no partners at the current demand levels to continue cooperation. No coalitions will form as a result.



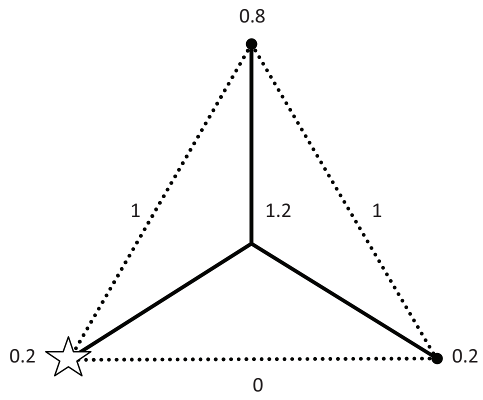
Which coalitions are feasible *in period* $t' + 2$ depends on who increased in the previous period. Suppose that Worker One increased in the previous period, in which case the pair consisting of the Firm and Worker Two can form now if, suppose, the Firm is selected.



In period $t' + 3$, suppose that Worker One is selected and, because his demand is still not feasible in any coalition, decreases back to 0.2.

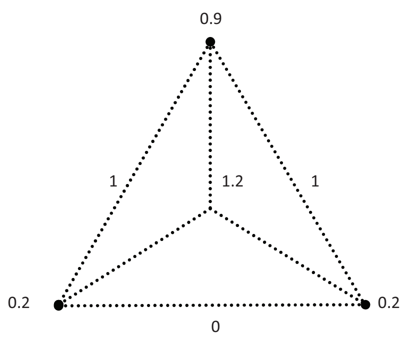


In period $t' + 4$, suppose that Worker One is selected again. Now, the grand coalition is feasible again and we can move back to the original allocation.



Equity in the core

Consider what happens in the following situation when the grand coalition and both firm-worker pairs are infeasible.



With loss aversion, the player with the highest demand is most likely to reduce. In this game, due to asymmetry in the core, this will always be the Firm when all coalitions are infeasible and all allocations are group-stable. However, due to asymmetry in the core, the process will not support the equal split.

4.4 Long-run behaviour

In finite time, our procedures converge almost surely into a set in and close to the core when it is nonempty and, thereafter, continue to move therein. When the core is empty, the process will move through inefficient blocking coalitions and never satisfy all players in the long run. Existing equity biases are halted once coalitional participation constraints are reached.

Theorem 10. *Given any $\gamma > 0$, for any cooperative game played with our procedure, there exists a finite time T_γ such that, for every $t > T_\gamma$,*

$$\mathbb{P}[\sum_{i \in S} d_i^t < v(S)] < \gamma$$

for all $S \subseteq N$.

Proof. **Step 1.** To prove the result, we first show that, for any state Z^t at any time t, "group instability," as captured by

$$IN(Z^t) = \sum_{S \subseteq N} \{[v(S) - \sum_{i \in S} d_i^t] \mathbf{1}_{\sum_{i \in S} d_i^t < v(S)}\},$$

where $\mathbf{1}_{\sum_{i \in S} d_i^t < v(S)}$ is an indicator equal to one whenever $\sum_{i \in S} d_i^t < v(S)$, never increases.

For any state Z^t, define $\Delta(\sum_{i \in S} d_i^t) = \sum_{i \in S} d_i^{t+1} - \sum_{i \in S} d_i^t$ for all $S \subseteq N$. Note that coalition S need not have formed in Z^t to calculate this. For any $S \subseteq N$ in any state Z^t, the following bounds on $\Delta(\sum_{i\in S} d_i^t)$ can be established.

Case 1.1. When no $i \in S$ is selected, $\Delta(\sum_{i\in S} d_i^t) = 0$.

Case 1.2. When $\sum_{i\in S} d_i^t \leq v(S)$ and some $i \in S$ is selected, $\Delta(\sum_{i\in S} d_i^t) \geq 0$ because a satisfied player may increase his demand but not decrease and a dissatisfied player is guaranteed to find a coalition that can satisfy him without having to reduce.

Case 1.3. When $\sum_{i\in S} d_i^t > v(S)$ (i.e. $\sum_{i\in S} d_i^t \geq v(S) + \delta$) and some $i \in S$ is selected, $\Delta(\sum_{i\in S} d_i^t) \geq -\delta$ because any reduction is at most δ.

As $\Delta(\sum_{i\in S} d_i^t) \geq 0$ is true whenever $\sum_{i\in S} d_i^t \leq v(S)$ for any $S \subseteq N$ in any state Z^t, group instability indeed never increases: $\Delta(IN(Z^t)) \leq 0$. Furthermore, no transition from $\sum_{i\in S} d_i^t > v(S)$ to $\sum_{i\in S} d_i^t < v(S)$ is possible.

Step 2. We now show that, from any Z^t with demands d^t such that $\sum_{i\in S} d_i^t < v(S)$ for some $S \subseteq N$, a finite path to group stability exists that has positive probability bounded away from zero. Jointly with the previous argument that group instability never increases, this ensures that the probability of group instability goes to zero over time. In other words, instability is transient.[12]

For any state Z^t with $\sum_{i\in S} d_i^t < v(S)$ for some $S \subseteq N$, the following bounds on $\mathbb{P}[\sum_{i\in S} d_i^{t+t'} - \sum_{i\in S} d_i^t \geq \delta]$ for some $t' < \infty$ can be established.

Case 2.1. If at least one $i \in S$ is currently satisfied, $\mathbb{P}[\Delta(\sum_{i\in S} d_i^t) = \delta] \geq \frac{r}{n}$.

Case 2.2. If all $i \in S$ are dissatisfied, the probability that at least one is satisfied next period, holding all demands fixed, is at least $\frac{1}{n}$ because S is feasible and may form when one of them is selected. Then, in period $t+1$, the probability that he increases is $\frac{r}{n}$.

[12]Note that higher levels of instability are transient but not states with a given level of instability, that is, we may move back and forth in between states with the same level of instability.

The probability of being at least δ closer to group stability after two periods is, therefore,

$$\mathbb{P}[IN(Z^{t+2}) - IN(Z^t) \leq -\delta] \geq \frac{r}{n^2}$$

and the probability of being equally far away from group stability is

$$\mathbb{P}[IN(Z^{t+2}) - IN(Z^t) = 0] \leq 1 - \frac{r}{n^2}.$$

Since $v(N) \geq v(S)$ for all $S \subset N$, $\frac{v(N)}{\delta}$ is the maximum number of steps necessary to reach the frontier of any of the 2^n coalitions. Hence, there is a path with positive probability of $[\frac{r}{n^2}]^{2^n \frac{v(N)}{\delta}}$ that group stability is reached from any Z^t after $2^{n+1}\frac{v(N)}{\delta}$ periods. The probability of group-instability after t periods is, therefore,

$$\mathbb{P}[IN(Z^t) > 0] \leq [1 - [\frac{r}{n^2}]^{2^n \frac{v(N)}{\delta}}]^{\frac{t}{2}},$$

which is smaller than any arbitrarily small γ after $T_\gamma > t = 2\log\gamma \ / \ \log[1 - (\frac{r}{n^2})^{2^n \frac{v(N)}{\delta}}]$. □

Corollary 11. *Given any $\gamma > 0$, for any unbalanced game played with our procedure, there exists a finite time T_γ such that, for every $t > T_\gamma$,*

$$\mathbb{P}[\sum_{i \in N} d_i^t \leq v(N)] < \gamma.$$

Proof. Theorem 10 implies that, for every $t > T_\gamma$, $\mathbb{P}[\sum_{i \in S} d_i^t < v(S)] < \gamma$ for all $S \subseteq N$. This holds whether the game is balanced or unbalanced (whether the core is nonempty or empty). When the core is empty, there exists no allocation or demand vector, d, such that $\sum_{i \in S} d_i \geq v(S)$ for all $S \subseteq N$ and $\sum_{i \in N} d_i = v(N)$. Hence, $\mathbb{P}[\sum_{i \in N} d_i \leq v(N)] < \gamma$. Any S, for which

$\mathbb{P}[\sum_{i\in S} d_i^t \leq v(S)] \geq \gamma$ after time $t > T_\gamma$, must be a strict subset of N. $\square$

Lemma 12. *Given any $\beta, \gamma > 0$, for any balanced cooperative game played with our procedures with step size $\delta \leq \beta\gamma$, there exists a finite time $T_{\beta,\gamma}$ such that, for every $t > T_{\beta,\gamma}$,*

$$\mathbb{P}[\sum_{i\in N} d_i^t > v(N) + \beta] < \gamma.$$

Proof. For balanced games, $\sum_{i\in N} d_i^t > v(N)$ implies that the grand coalition cannot form and that there must exist some nonempty set of dissatisfied players $S \subseteq N$ such that, for all $i \in S$, $\sum_{j\in C} d_j^t > v(C)$ for all $C \subseteq N$ with $i \in C$: nobody in S has any coalitional opportunities at the current demand level. For any $i \in S$, $d_i^t \geq \delta$ because any demand of zero can be satisfied in any coalition including his own singleton.

The probability of reduction for all players $i \in S$ is uniformly bound by

$$\mathbb{P}[\Delta(d_i^t) = -\delta | Z^t] \geq \frac{1}{n} a\delta.$$

The probability of increase for all players $j \notin S$ is uniformly bound by

$$\mathbb{P}[\Delta(d_i^t) = \delta | Z^t] \leq \frac{1}{n} r.$$

In expectation, the expected change in total demands is, therefore,

$$\mathbb{E}[\Delta(\sum_{i\in N} d_i^t) | Z^t] \leq -\frac{\delta}{n}(|S|a\delta - (n - |S|)r),$$

which, for our choices of small experimentation compared to stickiness, is negative. In the worst case, when all but one player who demands just δ are satisfied, we still have a negative drift of size c where $c = \frac{\delta}{n}(a\delta - (n-1)r)$ when r and $a\delta$ are such that $\frac{a\delta}{r} > n - 1$. Starting in state Z^t with $\sum_{i \in N} d_i^t > v(N)$,

$$\mathbb{E}[\sum_{i \in N} d_i^{t+1} | Z^t] \leq \sum_{i \in N} d_i^t - c.$$

Starting in an initial state Z^0 with $\sum_{i \in N} d_i^t > v(N)$, this yields

$$\mathbb{E}[\sum_{i \in N} d_i^t - v(N)] \leq \max\{\delta; \sum_{i \in N} d_i^0 - v(N) - tc\},$$

which, after iterated substitution, is larger than δ until time $t > T_{\beta,\gamma}$ when $\sum_{i \in N} d_i^0 - v(N) - tc \leq \delta$. This implies $T_{\beta,\gamma} \geq \frac{1}{c}(\sum_{i \in N} d_i^0 - v(N) - \delta)$.

Rearranged for the Markov inequality, when $\delta \leq \beta\gamma$ and for any Z^0, it is true that, for every $t > T_{\beta,\gamma}$,

$$\mathbb{P}[\sum_{i \in N} d_i^t - v(N) > \beta] \leq \frac{\delta}{\beta} \leq \gamma.$$

$\square$

We have now established how outcomes close to the core of balanced games are implemented after finite time; how group-stable arrangements form and how payoffs settle close to their core constraints most of the time. When the game is unbalanced, the coalitional opportunities jointly preclude formation of the grand coalition which means inefficient subcoalitions continue to form in the long run.

4.5 Dynamics within the core

Once stability within the core is reached, the procedures move within the set of core-stable outcomes. Any two allocations in the core are reached from one another with positive probability via a series of transfers. Note that implementation of group stability is independent of the exact behavioural specifications during dissatisfaction and of the exact coalition selection mechanism during search. What matters is that search is complete, that is, that satisfied searchers with feasible coalition alternatives will find a coalition that satisfies their demands and that dissatisfied searchers without feasible coalition alternatives reduce with positive probability. With a constant degree of stickiness of $(1-s) \geq a\delta$ too, for example, Theorem 10 and Lemma 12 hold. What is determined by loss aversion are tendencies governing the inner distribution of payoffs within the coalitions: loss aversion creates biases towards equity.

Recall the results for cooperative bargaining games without subcoalitions from Chapter Two as a benchmark: Theorem 2 and Proposition 4 have shown that, for cooperative bargaining, outcomes close to equal splits of the surplus are played most of the long-run time. For a general cooperative game, $G(v, N)$, however, it follows directly from Theorem 10 that the equal split is not implemented when there exists some $S \subset N$ with $\frac{v(S)}{|S|} > \frac{v(N)}{n}$, that is, when some subcoalition has superior average worth. Equal splits are stable in the long run when no subcoalitions have average worths above the population. Further analysis is required for general games where equity is constrained by group instabilities.

Constrained equity

Consider the following example of a convex game.

Four-class society. *$G(v,N)$ is such that $N = \{U, M_U, M_L, W\}$ with ten identical members in U, forty identical members in M_U, fifty identical members in M_L and one hundred identical members in W. Say all coalitions are inessential except for $v(N) = 700$, $v(W) = v(M_L) = v(M_U) = v(U) = 100$.*

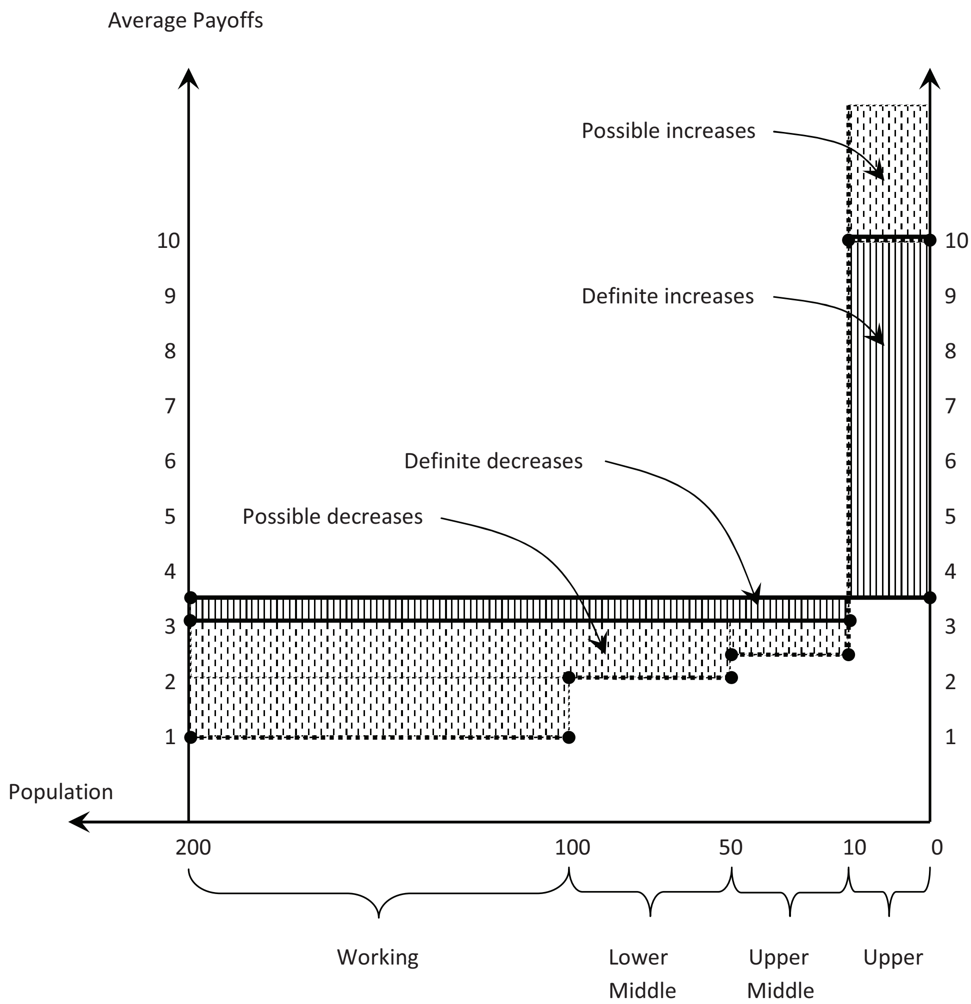
A society of two hundred members with ten members in the Upper Class, forty in the Upper Middle Class, fifty in the Lower Middle Class and one hundred in the Working Class can arrange social contracts involving some or all of the classes. Subsets of each class cannot form essential coalitions. The core of this game contains all efficient arrangements of the grand coalition paying each player a nonnegative share so that each class receives no less than its worth collectively.

Any class on its own gets hundred, which translates into an average of ten in the Upper Class, an average of 2.5 in the Upper Middle Class, an average of two in the Lower Middle Class and an average of one in the Working Class. All classes together attain a total of seven hundred, which is more than the classes are worth separately. Seven hundred for all two hundred players gives an average of 3.5. The equal split is not in the core because the average worth of the Upper Class ($\frac{v(U)}{10} = 10$) lies above the population average ($\frac{v(N)}{200} = 3.5$).

Theorem 10 and Lemma 12 imply that, almost surely after finite time, each class participation constraint is satisfied and that demands are close to globally feasible most of the time.

Let us consider the scenario when we start with everybody getting an equal share. The below graphic illustrates the areas in which increases and decreases will definitely and may possibly follow subsequently.

Class averages: The graph illustrates instabilities from the equal split.



The state where the total surplus is split equally (everybody gets 3.5) is not stable. The Upper Class will quickly break off because its members can guarantee themselves higher payoffs by cooperation amongst themselves. Total Upper Class payoffs will then increase beyond one hundred before cooperating with other classes at all.

As a result, the workers and middle classes get dissatisfied often and reduce towards their own classes' possibility frontiers. When their joint demands fall below six hundred (the residual of the worth of the grand coalition to what the Upper Class is worth by itself), cooperation amongst all groups may become sustainable again. Once all class participation constraints are satisfied and global feasibility is restored, we are in the core. Transfers in between classes and within classes continue. High levels of equity may be expected in the long run because equity biases are revealed where all constraints are slack (as in the drifts of Chapter Two). Equity beyond core constraints, however, is not supported (Theorem 10).

For two known reasons, it is reasonable to expect that outcomes close to the Dutta-Ray solution will be favoured in the long run. First, we know that equity biases globally exist if all class participation constraints are slack (as in the drifts of Chapter Two) but not if constraints are reached (as with pairwise instabilities in Chapter Three). Second, for fixed shares of the other classes, the same drifts (as Chapter Two) favour equity within each class. Jointly, these two factors suggest that our results from previous chapters could generalise. For convex games, we may conjecture that the procedure is such that inequity tends to decrease over time until it settles at the unique core-constrained inequity minimum which is the Dutta-Ray solution. In order to prove this with the methodology from previous chapters, we need to establish a global drift towards equity away from states with substantial levels of inequity above the core-constrained minimum. Moreover, only small adverse tendencies away from equity may be observed at the constraints.

Large adverse tendencies

For general games, adverse tendencies at or close to the constraints, unfortunately, need not be small. In contrast to the models of previous chapters, it is no longer warranted that, from any state, global drifts towards equity exist or that, when the process is at or close to constraints, only very small increases in the VMR need to be considered before we are ensured to re-enter regions with equity bias. Indeed, the adverse tendencies along the stability frontiers of general coalitional games may be large.

We shall illustrate these large adverse tendencies at hand of the above Four-Class Society example.

Suppose we start at the Dutta-Ray solution where each member of the Upper Class gets exactly ten and all others an equal share of the residual of $\frac{600}{190} \approx 3.16$. Note that, because the upper call is at its constraint and because all others already share the surplus equally, any permitted transfer will result in a higher VMR. The Upper Class blocks all transfers from the Upper Class to players in other classes, while transfers within each class and from lower classes to the Upper Class are possible. This implies that, at the Dutta-Ray solution, a tendency away from equity exists. In contrast to previous applications in earlier chapters, the resulting adverse tendencies need not be small. If, for example, starting at the Dutta-Ray solution, a member of the Upper Class increases (say Player One), demands become globally infeasible and someone needs to decrease to restore feasibility. If some other member of the Upper Class decreases (say Player Two), global feasibility is restored and the Upper Class is back to its constraint. However, most permitted transfers from that state will increase inequity again. In fact, only one possible transfer, the one back from Player One to Player Two, decreases inequity. Overall, an adverse tendency away from equity prevails. Suppose Player One increases and Player

Two decreases again so that the resulting state has no equity bias again because most possible transfers increase inequity. Indeed, a series of transfers may take place where players in the Upper Class continue to get more before a state with equity drift is again reached. Series of such transfers increase inequity substantially.

Because the adverse tendencies away from equity at the constraints need not be small, the drift methodology of the previous chapters needs to be extended.

Constrained equity in two-to-one matching

To illustrate why and what kind of additional arguments are needed, recall the Employer-Employee game of two-to-one matching:

Employer-employee game. *$G(v, N)$ is such that $N = \{F, W_1, W_2\}$, $v(N) = 1.2$, $v(F, W_1) = v(F, W_2) = 1$, $v(W_1, W_2) = 0$ and $v(i) = 0$ for all $i \in N$.*

Note the game is balanced but nonconvex.[13] In the core of this game, all payoffs are nonnegative and sum up to 1.2 so that each firm-worker pair gets at least one. In any core allocation, the Firm gets at least 0.8 and that each worker gets at most 0.2. The equal split is not in the core. The most equitable allocation in the core (identifiable, for example, by minimisation of the VMR subject to core-constraints) pays $(0.8, 0.2, 0.2)$ to (F, W_1, W_2).

Theorem 10 and Lemma 12 imply that, almost surely after finite time, each firm-worker pair gets at least one and that total demands are equal to or not much higher than 1.2 most of the time. Based on drift analyses of the VMR alone, however, we cannot predict probable levels of (in)equity in the long run even though an equity bias is inherent to the procedure. Further analysis of

[13]The marginal contribution of each worker is higher in his firm-worker pair than in the grand coalition.

equity amongst the two workers in isolation, however, confirms the conjecture that outcomes close to $(0.8, 0.2, 0.2)$ are played most of the long-run time.

Suppose both workers demand almost the same. When workers' demands lie close and each one below 0.2, the Firm demands more than 0.8 so that neither firm-worker pair is feasible. When total demands are also infeasible, the Firm (holding the highest demand) is most likely to reduce which drives the Firm's demand towards 0.8.

Consider the set of all states Z with $\sum_{i\in S} d_i \geq v(S)$ for all $S \subseteq N$ that will be played almost surely after finite time (Theorem 10). The following two facts are true for any such Z:

(1) When the two workers hold substantially different demands, the inequity in the workers' demands tends to decrease.

$$\frac{(d_{W_1} - d_{W_2})^2}{d_{W_1} + d_{W_2}} > \delta \Rightarrow \mathbb{E}[\Delta(\frac{(d_{W_1} - d_{W_2})^2}{d_{W_1} + d_{W_2}})|Z^t = Z] < 0 \tag{4.1}$$

(2) When both workers hold similar demands, adverse tendencies away from the demand equity are small.

$$\frac{(d_{W_1} - d_{W_2})^2}{d_{W_1} + d_{W_2}} \leq \delta \Rightarrow \Delta(\frac{(d_{W_1} - d_{W_2})^2}{d_{W_1} + d_{W_2}}) < 2\delta \tag{4.2}$$

With (1) negative drifts and (2) small adverse tendencies, Equations 4.1 and 4.2 jointly imply (see Theorem 2 and Corollary 3) that the workers receive almost the same payoff most of the long-run time.[14]

It is one of our main research avenues to extend the drift methodology that we have developed in this thesis to analyse general games. Our aim is to identify regular classes of games for which outcomes close to constrained egalitarian

[14]See the Appendix for the drift calculations.

solutions are implemented most of the time.[15] Extending the methodology may involve analysis of waiting and return times.

Appendix

Drift calculations

Equation 4.1.

In any state Z with $\sum_{i\in S} d_i \geq v(S)$ for all $S \subseteq N$, each worker can either be satisfied or dissatisfied. W.l.o.g., say $d_{W_1} > d_{W_2}$.[16]

If both are satisfied,

$$\mathbb{E}[\Delta(\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}})|Z^t = Z] = \tfrac{r}{2}\tfrac{(d_{W_1}+\delta-d_{W_2})^2}{d_{W_1}+\delta+d_{W_2}} + \tfrac{r}{2}\tfrac{(d_{W_1}-d_{W_2}-\delta)^2}{d_{W_1}+d_{W_2}+\delta} - r\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} = -\tfrac{r\delta}{d_{W_1}+d_{W_2}+\delta}(\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} - \delta),$$

which is negative when

$$\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} > \delta.$$

If both are dissatisfied,

$$\mathbb{E}[\Delta(\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}})|Z^t = Z] = \tfrac{ad_{W_1}}{2}\tfrac{(d_{W_1}-\delta-d_{W_2})^2}{d_{W_1}+\delta-d_{W_2}} + \tfrac{ad_{W_2}}{2}\tfrac{(d_{W_1}-d_{W_2}+\delta)^2}{d_{W_1}+d_{W_2}-\delta} - \tfrac{a(d_{W_1}+d_{W_2})}{2}\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} = -\tfrac{a\delta}{2}\tfrac{d_{W_1}+d_{W_2}}{d_{W_1}+d_{W_2}-\delta}(\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} - \delta),$$

which is negative when

$$\tfrac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} > \delta.$$

If Worker One is dissatisfied and Worker Two is satisfied,

[15] As the two examples in this section have shown, convexity alone need not be necessary or sufficient.

[16] Note this implies Worker Two cannot be satisfied when Worker One is dissatisfied.

$$\mathbb{E}[\Delta(\frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}})|Z^t = Z] =$$

$$\frac{ad_{W_1}}{2}(\frac{(d_{W_1}-\delta-d_{W_2})^2}{d_{W_1}+\delta-d_{W_2}} - \frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}}) + \frac{r}{2}(\frac{(d_{W_1}-d_{W_2}-\delta)^2}{d_{W_1}+d_{W_2}+\delta} - \frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}}) =$$

$$-\frac{a\delta}{2}\frac{d_{W_1}+d_{W_2}}{d_{W_1}+d_{W_2}-\delta}(\frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} - \delta(d_{W_1} + d_{W_2})),$$

which is negative when

$$\frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} > \delta$$

because $d_{W_1} + d_{W_2} < 1$.

Equation 4.2**.**

When both workers get the same $d_{W_1} = d_{W_2}$, $\frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} = 0$.

Define $\frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}} = 0$ when $d_{W_1} = d_{W_2} = 0$.

Demands change by δ. For any given change in one of the demands, $\Delta(\frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}})$ is large when $d_{W_1} + d_{W_2}$ small.

Starting with $d_{W_1} = d_{W_2} = 0$, one increase yields $\Delta(\frac{(d_{W_1}-d_{W_2})^2}{d_{W_1}+d_{W_2}}) = \frac{\delta^2}{\delta} = \delta$.

Chapter 5

Summary

An ambitious strand of game-theoretic endeavour is concerned with the general aim of "naturalising the social contract" (as pursued in Skyrms [1996], Young [1998], Binmore [2005]). This strand of literature provides game-theoretic foundations and evolutionary explanations of the social contract and its norms: "All social contracts that exist, or that could come to exist, must arise by some kind of natural process."[1]

Stable social arrangements may feature significant levels of efficiency and equity and evidence suggests that equity-considerations are important. Existing explanations of equity rely on inherent social preferences or social norms. In this thesis, we propose evolutionary procedures for cooperative games that suggest that more primitive learning than previously considered may also play a role in arriving at such arrangements. In our procedures, tendencies towards equity begin once social arrangements reach long-run stable possibility frontiers, not because social preferences or norms are in place but because because loss-averse dynamic bargaining favours equity naturally. When growth comes to a standstill, the allocative issue of securing shares of a growing pie is replaced by a purely rival bargaining over shares of a fixed total. It turns out

[1]Skyrms in his abstract of talk "Naturalising the Social Contract" given on 15th November 2010 at *LSE Choice Group*.

that, simply put, transfers from rich to poor are eventually more likely than transfers from poor to rich. Over time, this leads to high levels of equity as constrained by coalitional participation incentives.

We consider behaviour that is based on trial and error; learning in a basic form. We apply these learning procedures to cooperative games where coalitions form amongst agents whose demands are compatible.

The results provide dynamic support for equitable and stable solutions in different cooperative environments. The models are driven by individual explorations based on introspective adaptations of aspirations. This type of learning makes sense in environments that are so complex that individuals do not take strategic issues into account. Instead, all actions are made as incremental corrections of behaviour that depend on each individual's own assessment of how well he did in the past. Agents who demand too much of a cooperative agreement receive nothing and eventually respond by reducing their demands. Agents who currently receive what they demand occasionally try to get more.

In cooperative bargaining, high levels of long-run equity arise naturally. Outcomes close to equal splits of the surplus are played most of the time as the cooperative possibility frontier reaches its long-run position. In the short and intermediate run, shifts in the possibility frontier may cause significantly more complex adjustment dynamics. In particular, inequity may temporarily rise as a consequence of growth.

Assignment games in large markets allow the formation of many partnerships. An agent searches for partners and adjusts his demands based solely on his own experience. Individuals on both sides of the market behave in the same way and neither market has an intrinsic advantage. Assignments vary through continued matching, separating and rematching. Players shop around for partners and are randomly matched. Over time, assignments become more efficient and partners agree on allocations that are more stable more often. Once optimal assignments and pairwise stable allocations have been reached, a tendency towards equity is revealed. In the long run, optimal partners share close to fifty-fifty most of the time unless one or both partners have better outside alternatives.

In general coalitional games where arbitrary coalitions may form, individuals explore different coalitional alternatives, continuing to adjust their demands and varying their coalitional allegiances. For balanced games, core outcomes are implemented once groups of players have realised their coalitional worths. When the core is empty, inefficiencies persist. The procedures reveal that equity within and in between coalitions is favoured but constrained by coalitional participation incentives.

In all applications, we find that very simple learning procedures result in complex cooperative agreements with an appealing admixture of positive (stability) and normative (equity) criteria. This complements the theoretical findings concerning social norms when players learn in environments where more information is available.

Our technical apparatus uses auxiliary inequity measures such as variance-to-mean ratio and coefficient of variation. These measures are not only convenient summaries of a state's equity but also permit the study of convergence in a simple way. In the long run, our procedures get close to cooperative solutions most of the time but do not converge in a standard sense.

As regards recommendations for mechanism design, a few points are noteworthy. First, the implementation of core-stable outcomes depends on complete search. Switching from existing partners to equally valuable short-run alternatives must be easy, that is, lock-in and contractual costs must be negligible and should be encouraged. Even if switching to new coalition structures is largely random and does not lead to efficient temporary outcomes, mechanisms with easy switching ensure efficiency and stability most of the long-run time. If switching to partners is too costly, the necessary experimentation to find the right partner eventually may be precluded and inefficient assignments may persist. Alternative mechanisms that may be more efficient in the short-run but prevent easy switching, on the other hand, may display persistent long-run inefficiencies. Second, recommendations turn the other way when the core is empty. Complete search and easy switching will lead to persistent long-run inefficiencies and it may instead be socially preferable to limit search and communication. Efficient but blockable outcomes may persist when coordinated deviations of subcoalitions are difficult. Third, equity in our models stems from the behavioural components of the interaction itself. Instead of more complex social norms or social preferences, this suggests that mechanisms designed to reinforce loss aversion or costs levied proportionally should lead to very equitable allocations.

Several natural extensions suggest themselves. We would like to explore the interplay and tradeoff between rate of experimentation and stickiness in more detail. This would provide more information about the length of search, which would be crucial to understand the inefficiencies and dynamics of the process in the intermediate run. For the same reason, it is also interesting to get a more precise measure of the effectiveness and speed of search with different behavioural specifications regarding adjustment and search. Naturally, we would also like to pursue the analysis of asymmetries amongst players and allow behaviour to vary over time. Work on these issues would begin with simulations.

In extending the models, we want to allow for a wider array of experimentation, concession and search-match mechanisms. It is also natural to apply the learning procedures to games with externalities and multiple membership. As sketched in Chapter Three, exploring our learning in networks is a natural candidate.

Two major avenues are empirical and experimental work. We should begin with evaluations of empirical trends in light of the models' implications and estimate waiting times implied by the model for parameter values consistent with data. We should also find applications where our behavioural rules make most sense and data is already available. Promising applications are common-pool resources, and in large and disconnected labour markets. We should then do experiments involving many players and test our theory in the laboratory. Extending the existing experimental work, we would like to collect evidence for behavioural patterns in repeated cooperative situations amongst more than just a few players.

We should probably start experimenting with assignment games and then extend to other matching markets. The experiments by Charness, Corominas-Bosch & Fréchette [2007] suggest that the structure of assignment games, regarding stability and coalition structure, allows to investigate several interesting questions in detail. The structure of the core of assignment games is convenient and outcomes are easy to interpret. We are particularly interested in three main aspects. First, we want to find evidence for probing upward and downward stickiness. We want to test if patterns revealed in bilateral bargaining also hold in assignment games and would like to understand what kind of experimentation and stickiness we find. Second, we are interested to see if and how optimal assignments and stable allocations are reached. Finally, we want to know what kind of equity or inequity biases are revealed and how cooperative agreements vary over time.

Bibliography

M. Agastya, "Adaptive Play in Multiplayer Bargaining Situations," *Review of Economic Studies 64*, 411-26, 1997.

M. Agastya, "Perturbed Adaptive Dynamics in Coalition Form Games," *Journal of Economic Theory 89*, 207-233, 1999.

J. Alexander & B. Skyrms, "Bargaining with Neighbors: Is Justice Contagious?" *The Journal of Philosophy 96*, 588-598, 1999.

J. Alexander, "Evolutionary Explanations of Distributive Justice," *Philosophy of Science 67,* 490-516, 2000.

J. Alexander, "Evolutionary Game Theory," *Stanford Encyclopedia of Philosophy,* 2009.

A. Alkan, "Nonexistence of stable threesome matchings," *Mathematical Social Science 19*, 207-209, 1988.

J. Arin & E. Inarra, "Egalitarian solutions in the core," *International Journal of Game Theory 30*, 187-193, 2001.

J. Arin, J. Kuipers & D. Vermeulen, "Some characterizations of egalitarian solutions on classes of TU-games," *Mathematical Social Sciences 46*, 327-345, 2003.

T. Arnold & U. Schwalbe, "Dynamic coalition formation and the core," *Journal of Economic Behavior and Organization 49*, 363-380, 2002.

R. Aumann & M. Maschler, "The bargaining set for cooperative games," in *Advances in Game Theory, Annals of Mathematics Studies 52,* M. Dresher, L. S. Shapley & A. Tucker (eds.), Princeton University Press, 443-476, 1964.

R. Axelrod & W. Hamilton, "The Evolution of Cooperation," *Science 211*, 1390-1396, 1981.

R. Axelrod, *The Evolution of Cooperation,* Basic Books, 1984.

R. Axtell, J. Epstein & H. P. Young, "Emergence of Class Norms in a Multi-Agent Model of Bargaining," in *Social Dynamics,* S. Durlauf & H. P. Young (eds.), MIT Press, 2001.

M. Baïou & M. Balinski, "Many-to-many matching: stable polyandrous polygamy (or polygamous polyandry)," *Discrete Applied Mathematics 101*, 1-12, 2000.

M. Baïou & M. Balinski, "Admissions and recruitment," *American Mathematical Monthly 110,* 386-399, 2003.

M. Balinski, "Les enjeux du recrutement," *Pour la Science 288,* 64-70, 2001.

M. Balinski & D. Gale, "On the Core of the Assignment game," in *Functional Analysis, Optimization and Mathematical Economics*, L. J. Leifman (ed.), Oxford University Press, 274-289, 1987.

T. F. Bewley, *Why Wages Don't Fall During a Recession,* Harvard University Press, 1999.

K. G. Binmore, *Game Theory and the Social Contract, Volume 1: Playing Fair,* MIT Press, 1994.

K. G. Binmore, *Game Theory and the Social Contract, Volume 2: Just Playing,* MIT Press, 1998.

K. G. Binmore, *Natural Justice,* Oxford University Press, 2005.

K. G. Binmore, A. Rubinstein & A. Wolinsky, "The Nash Bargaining Solution," *RAND Journal of Economics 17,* 176-188, 1986.

K. G. Binmore, L. Samuelson & H. P. Young, "Equilibrium selection in bargaining models," *Games and Economic Behavior 45,* 296-328, 2003.

K. G. Binmore, A. Shaked & J. Sutton, "An outside option experiment," *Quarterly Journal of Economics 104,* 753-770, 1989.

W. Blomquist, *Dividing the waters,* ICS Press, 1992.

W. Blomquist, E. Schlager & T. Heikkia, *Common waters, diverging streams,* Resources for the Future Press (DC), 2004.

O. N. Bondareva, "Some Applications of Linear Programming Methods to the Theory of Cooperative Games" (In Russian), *Problemy Kybernetiki 10*, 119-139, 1963.

H. Brems, *Quantitative Economic Theory; a synthetic approach,* Wiley, 1968.

H. Brems, "From the Years of High Theory: Frederik Zeuthen (1888-1959)," *History of Political Economy 8*, 400-411, 1976.

G. W. Brown, "Iterative Solutions of Games by Fictitious Play," in *Activity Analysis of Production and Allocation,* T. C. Koopmans (ed.), Wiley, 1951.

R. Bush & F. Mosteller, *Stochastic Models of Learning,* Wiley, 1955.

G. Charness, M. Corominas-Bosch & G. Fréchette, "Bargaining and network structure: An experiment," *Journal of Economic Theory 136*, 28-65, 2007.

K. Chatterjee, B. Dutta, D. Ray & K. Sengupta, "A non-cooperative theory of coalitional bargaining," *Review of Economic Studies 60,* 463-477, 1993.

K.-S. Chung, “On the Existence of Stable Roommate Matchings,” *Games and Economic Behavior 33*, 206-230, 2000.

O. Compte & P. Jehiel, “The Coalitional Nash Bargaining Solution,” *Econometrica,* forthcoming, 2010.

M. Corominas-Bosch, “Bargaining in a network of buyers and sellers,” *Journal of Economic Theory 115,* 35-77, 2004.

H. Crössmann & R. Tietz, “Market behavior based on aspiration levels,” in *Lecture Notes in Economics and Mathematical Systems 213*, R. Tietz (ed.), 170-185, 1983.

H. Dalton, “The measurement of the inequality of incomes,” *Economic Journal 30*, 348-461, 1920.

S. DellaVigna, ”Psychology and Economics: Evidence from the Field,” *Journal of Economic Literature 47*, 315-372, 2009.

G. Demange & D. Gale, “The strategy of two-sided matching markets,” *Econometrica 53,* 873-988, 1985.

G. Demange, D. Gale & M. Sotomayor, “Multi-item auctions,” *Journal of Political Economics 94,* 863-872, 1986.

B. Dutta, “The egalitarian solution and reduced game properties in convex games,” *International Journal of Game Theory 19*, 153-169, 1990.

B. Dutta & D. Ray, “The concept of egalitarianism under participation constraints,” *Econometrica 57*, 615-635, 1989.

B. Dutta & D. Ray, “Constrained egalitarian allocations,” *Games and Economic Behaviour 3*, 403-422, 1991.

F. Echenique & J. Oviedo, “Core many-to-one matchings by fixed point methods,” *Journal Economic Theory 115*, 358-376, 2004.

F. Echenique & J. Oviedo, “A theory of stability in many-to-many matching markets,” *Theoretical Economics 1*, 233-273, 2006.

T. Ellingsen, “The Evolution of Bargaining Behavior,” *Quarterly Journal of Economics 112*, 581-602, 1997.

S. Fatima, M. Wooldridge & N. Jennings, “A Comparative Study of Game Theoretic and Evolutionary Models of Bargaining for Software Agents,” *Artificial Intelligence Review 23,* 185-203, 2005.

E. Fehr & K. Schmidt, “A theory of fairness, competition, and cooperation,” *The Quarterly Journal of Economics 114*, 817-868, 1999.

F. Forges, “The ex ante incentive compatible core of the assignment game,” *Mathematical Social Sciences 47,* 135-151, 2004.

D. Foster & H. P. Young, "Regret testing: Learning to play Nash equilibrium without knowing you have an opponent," *Theoretical Economics 1*, 341-367, 2006.

J. Gale, K. Binmore & L. Samuelson, "Learning to be Imperfect: The Ultimatum Game," *Games and Economic Behavior 8,* 56-90, 1995.

D. Gale & L. S. Shapley, "College admissions and the stability of marriage," *American Mathematical Monthly 69*, 9-15, 1962.

D. B. Gillies, "Solutions to General Non-Zero-Sum Games," in *Contributions to the Theory of Games*, Vol. 4, H. Kuhn & A. Tucker (eds.), Princeton University Press, 47-85, 1959.

C. Gini, *Variabilità e mutabilità,* Tipografia di Paolo Cuppini, Bologna, 1912.

S. Hart & A. Mas-Colell, "Uncoupled Dynamics Do Not Lead to Nash Equilibrium," *American Economic Review 93*, 1830-1836, 2003.

S. Hart & A. Mas-Colell, "Stochastic Uncoupled Dynamics and Nash Equilibrium," *Games and Economic Behavior 57,* 286-303, 2006.

J. Hatfield & P. Milgrom, "Auctions, matching and the law of auctions matching and the law of aggregate demand," *American Economic Review 95*, 913-935, 2005.

H. Heckhausen, "Motivationsanalyse der Anspruchsniveau-Setzung," *Psychologische Forschung 25,* 118-154, 1955.

Z. Hellman, "Bargaining Set Solution Concepts in Repeated Cooperative Games," *Discussion Paper Series 523*, Center for Rationality and Interactive Decision Theory, Hebrew University, Jerusalem, 2009.

T. Hokari, "Monotone-path Dutta-Ray solutions on convex games," *Social Choice and Welfare 19*, 825-844, 2002.

F. Hoppe, "Erfolg und Mißerfolg," *Psychologische Forschung 14,* 1-62, 1931.

J.L. Hougaard & L.P. Osterdal, "Monotonicity and social welfare optima," *Games and Economic Behaviour*, forthcoming, 2010.

J.L. Hougaard, B. Peleg & L.P. Osterdal, "The Dutta-Ray solution on the class of convex games: A generalization and monotonicity properties," *International Game Theory Review 7*, 431-442, 2005.

J.L. Hougaard, B. Peleg & L. Thorlund-Petersen, "On the set of Lorenz maximal imputations in the core of a balanced game," *International Journal of Game Theory 30*, 147-165, 2001.

M. O. Jackson & A. Wolinsky, "A strategic model of social and economic networks," *Journal of Economic Theory 71,* 44-74, 1996.

J. Jaffray & P. Mongin, "Constrained egalitarianism in a simple redistributive model," *Theory and Decision 54*, 33-56, 2003.

D. Kahneman & A. Tversky, “Prospect Theory: An Analysis of Decision Under Risk,” *Econometrica 47*, 263-91, 1979.

E. Kalai, A. Postlewaite & J. Roberts, “A group incentive compatible mechanism yielding core allocations,” *Journal of Economic Theory 20*, 13-22, 1979.

U. Kamecke, “Non-cooperative matching games,” *International Journal of Game Theory 18,* 423-431, 1989.

R. Karandikar, D. Mookherjee, D. Ray & F. Vega-Redondo, “Evolving Aspirations and Cooperation,” *Journal of Economic Theory 80*, 292-331, 1998.

R. A. Kerr, “Geothermal Tragedy of the Commons,” *Science* (12 July), 134-135, 1991.

B. Klaus & F. Klijn, “Paths to Stability for Matching Markets with Couples,” *Games and Economic Behavior 85*, 154-171, 2007.

F. Klijn, M. Slikker, S. Tijs & J. Zarzuelo, “The egalitarian solution for convex games: some characterizations,” *Mathematical Social Sciences 40*, 111-121, 2000.

F. Kojima & M. Ünver, “Random Paths to Pairwise Stability in Many-to-Many Matching Problems: A Study on Market Equilibration,” *International Journal of Game Theory 36*, 473-488, 2006.

H. Konishi & D. Ray, “Coalition formation as a dynamic process,” *Journal of Economic Theory 110*, 1-41, 2003.

H. Konishi & M. Ünver, “Credible group stability in many-to-many matching problems,” *Journal of Economic Theory 129*, 57-80, 2006.

M. Koster, “Hierarchical constrained egalitarianism in TU-games,” *Mathematical Social Sciences 43*, 251-265, 2002.

L. Kranich, A. Perea & H. Peters, “Core Concepts for Dynamic TU Games,” *International Game Theory Review 7*, 43-61, 2005.

S. Kuznets, “Economic Growth and Income Inequality,” *American Economic Review 45*, 1-28, 1955.

R. Lagunoff, “A simple non-cooperative core story,” *Games and Economic Behavior 7*, 54-61, 1994.

R. D. Luce & H. Raiffa, *Games and decisions: introduction and critical survey,* Wiley, 1957.

E. Marchi & J. Oviedo, “The core of a further m-sided assignment game,” *European Journal of Operational Research 98*, 617-625, 1997.

J. Marden, H. Young, G. Arslan & J. Shamma, “Payoff-based dynamics for multi-player weakly acyclic games,” *Department of Mechanical and Aerospace Engineering Working Paper,* UCLA, http://www.its.caltech.edu/ ~marden/ Payoff_Based_Learning_Algorithms.pdf, 2007.

M. Maschler, B. Peleg & L. Shapley, "The kernel and bargaining set of convex games," *International Journal of Game Theory 2*, 73-93, 1972.

C. E. Miller, "Coalition formation in characteristic function games: Competitive tests of three theories." *Journal of Experimental Social Psychology 16,* 61-76, 1980.

H. Moulin, *Axioms of Cooperative Decision Making*, Monograph of the Econometric Society, Cambridge University Press, 1988.

H. Moulin, *Cooperative Micro-economics: A Game Theoretic Introduction,* Princeton University Press, 1995.

H. Moulin, *Fair Division and Collective Welfare,* MIT Press, 2003.

R. B. Myerson, *Game Theory: Analysis of Conflict,* Harvard University Press, 1991.

J. Nash, *Non-Cooperative Games,* Ph.D. thesis, Princeton University, 1950a.

J. Nash, "The Bargaining Problem," *Econometrica 18*, 155-162, 1950.

J. von Neumann & O. Morgenstern, *Theory of Games and Economic Behavior,* Wiley, 1947.

J. Newton, "Non-cooperative convergence to the core in Nash demand games without random errors or convexity assumptions," Ph.D. thesis, University of Cambridge, 2010.

J. O'Looney, *Economic Development and Environmental Control,* Quorum Press, 1995.

E. Ostrom, *Governing the Commons: The Evolution of Institutions for Collective Action,* Cambridge University Press, 1990.

E. Ostrom, R. Gardner & J. Walker, *Rules, Games, and Common-Pool Resources,* University of Michigan Press, 1994.

J. Oviedo, "The Core of a Repeated n-Person Cooperative Game," *European Journal of Operational Research 127*, 512-524, 2000.

D. Perez-Castrillo & M. Sotomayor, "A Simple Selling and Buying Procedure," *Journal of Economic Theory 103*, 461-474, 2002.

M. Perry & P. Reny, "A non-cooperative view of coalition formation and the core," *Econometrica 62*, 795-817, 1994.

A. C. Pigou, *Wealth and Welfare*, Macmillan, 1912.

J. Potters & H. Reijnierse, "Gamma-component additive games," *International Journal of Game Theory 24*, 49-56, 1995.

B. Pradelski & H. P. Young, "Learning Efficient Nash Equilibria in Distributed Systems," *Department of Economics Working Paper 480,* University of Oxford, 2010.

M. Pycia, "Many-to-One Matching with Complementarities and Peer Effects," *Penn State Working Paper*, 2007.

T. Quint, "The core of an m-sided assignment game," *Games and Economic Behavior 3*, 487-503, 1991.

T. Quint, "Characterization of cores of assignment games," *International Journal of Game Theory 19,* 413-420, 1991.

H. Raiffa, "Arbitration schemes for generalized two-person games," in *Contributions to the Theory of Games,* Vol. 2, H. Kuhn, A. Tucker & M. Dresher (eds.), Princeton University Press, 361-387, 1953.

D. Ray, *A game-theoretic perspective on coalition formation,* Oxford University Press, 2007.

D. Ray & R. Vohra, "A Theory of Endogenous Coalition Structures," *Games and Economic Behavior 26*, 286-336, 1999.

S. Rochford, "Symmetrically pairwise-bargained allocations in an assignment market," *Journal of Economic Theory 34*, 262-281, 1984.

A. E. Roth, "Bargaining experiments," in *Handbook of Experimental Economics,* J. Kagel & A. E. Roth (eds.), Princeton University Press, 1995.

A. E. Roth, U. Rothblum & J. Vande Vate, "Stable Matchings, Optimal Assignments, and Linear Programming," *Mathematics of Operations Research 18*, 803-828, 1993.

A. E. Roth, T. Sönmez & U. Ünver, "Pairwise kidney exchange," *Journal of Economic Theory 125*, 151-188, 2005.

A. E. Roth & M. Sotomayor, *Two-Sided Matching: A Study in Game Theoretic Modeling and Analysis*, Cambridge University Press, 1990.

A. E. Roth & M. Sotomayor, "Two-sided matching," in *Handbook of Game Theory with Economic Applications*, Volume 1, R. Aumann & S. Hart (eds.), 485-541, 1992.

A. E. Roth & J. Vande Vate, "Random Paths to Stability in Two-Sided Matching," *Econometrica 58*, 1475-1480, 1990.

K. Rozen, "Conflict Leads to Cooperation in Nash Bargaining," *mimeo,* Yale University, 2010.

K. Rozen, "Conflict Leads to Cooperation in Nash Bargaining: Supplemental Result on Evolutionary Dynamics," *web appendix,* Yale University, 2010a.

A. Rubinstein, "Perfect Equilibrium in a Bargaining Model," *Econometrica 50*, 97-109, 1982.

A. Rubinstein, "A Bargaining Model with Incomplete Information about Time Preferences," *Econometrica 53*, 1151-1172, 1985.

M. Saez-Marti & J. W. Weibull, "Clever Agents in Young's Evolutionary Bargaining Model," *Journal of Economic Theory, Elsevier 86*, 268-279, 1999.

L. Samuelson, *Evolutionary Games and Equilibrium Selection,* MIT Press, 1997.

H. Sauermann & R. Selten, "Anspruchsanpassungstheorie der Unternehmung," *Zeitschrift für die Gesamte Staatswissenschaft 118*, 577-597, 1962.

R. Selten, "Equal Share Analysis," *Contributions to Experimental Economics,* Vol. 3, H. Sauermann (ed.), 130-165, 1972.

R. Selten, "Equity and coalition bargaining in experimental three-person games," in *Laboratory experimentation in economics,* A. E. Roth (ed.), Cambridge University Press, 42-98, 1987.

R. Serrano, "A market to implement the core," *Journal of Economic Theory 67*, 285-294, 1995.

R. Serrano & R. Vohra, "Non-cooperative implementation of the core," *Social Choice and Welfare 14* 513-525, 1997.

L. S. Shapley, "Notes on the n-Person Game III: Some Variants of the von Neumann-Morgenstern Definition of Solution," *RM-670*, The Rand Corporation, Santa Monica (CA), 1951.

L. S. Shapley, "A Value for n-Person Games," in *Contributions to the Theory of Games*, Vol. 2, H. Kuhn & A. Tucker (eds.), Princeton University Press, 307-317, 1953.

L. S. Shapley, "On Balanced Sets and Cores," *Naval Research Logistics Quarterly 14*, 453-460, 1967.

L. S. Shapley, "Cores of convex games," *International Journal of Game Theory 1,* 11-26, 1971.

L. S. Shapley & M. Shubik, "The Assignment Game I: The Core," *International Journal of Game Theory 1*, 111-130, 1972.

J. Shogren, "Fairness in bargaining requires a context: An experimental examination of loyalty," *Economics Letters 31*, 319-323, 1989.

T. Sjöström, "Ståhl's bargaining model," *Economics Letters Volume 36*, 153-157, 1991.

B. Skyrms, *Evolution of the Social Contract,* Cambridge University Press, 1996.

M. Sotomayor, "The Lattice Structure of the Set of Stable Outcomes of the Multiple Partners Assignment Game," *International Journal of Game Theory 28*, 567-583, 1999.

M. Sotomayor, "Some further remark on the core structure of the assignment game," *Mathematical Social Sciences 46,* 261-265, 2003.

M. Sotomayor, “Implementation in the many-to-many matching market,” *Games and Economic Behavior 46*, 199-212, 2004.

I. Ståhl, *Bargaining theory,* Stockholm School of Economics, 1972.

H. von Stackelberg, *Marktform und Gleichgewicht,* Springer, 1934.

G. L. Thompson, “Auctions and market games,” in *Essays in Game Theory and Mathematical Economics in honor of Oskar Morgenstern,* R. Aumann ed., BIW Mannheim, 181-196, 1981.

R. Tietz, “An Experimental Analysis of Wage Bargaining Behavior,” *Zeitschrift für die gesamte Staatswissenschaft 131,* 44-91, 1975.

R. Tietz & O. Bartos, “Balancing of aspiration levels as fairness principle in negotiations,” *Lecture Notes in Economics and Mathematical Systems 213,* R. Tietz (ed.), 52-66, 1983.

R. Tietz & H. Weber, “On the nature of the bargaining process in the Kresko-game,” in *Contributions to experimental economics,* Vol. 3, H. Sauermann (ed.), 305-334, 1972.

R. Tietz & H. Weber, “Decision behavior in multi-variable negotiations,” in *Contributions to experimental economics,* Vol. 7, H. Sauermann (ed.), 60-87, 1978.

R. Tietz, H. Weber, U. Vidmajer & C. Wentzel, “On Aspiration Forming Behavior in Repetitive Negotiations,” in *Contributions to experimental economics,* Vol. 7, H. Sauermann (ed.), 88-102, 1978.

S. Tijs & R. Branzei, “Additive Stable Solutions on Perfect Cones of Cooperative Games,” *International Journal of Game Theory 31*, 469-474, 2002.

E. Thorndike, “Animal Intelligence: An Experimental Study of the Associative Processes in Animals,” *Psychological Review 8*, 1898.

A. Tversky & D. Kahneman, “Loss Aversion in Riskless Choice: A Reference-Dependent Model,” *The Quarterly Journal of Economics 106,* 1039-1061, 1991.

F. Vega-Redondo, *Evolution, Games, and Economic Behaviour,* Oxford University Press, 1996.

H. Weber, “On the theory of Adaptation of Aspiration Levels in a Bilateral Decision Setting,” *Zeitschrift für die gesamte Staatswissenschaft 132*, 582-591, 1976.

J. W. Weibull, *Evolutionary Game Theory*, MIT Press, 1995.

S. Wright, *Evolution: Selected papers,* University of Chicago Press, 1986.

H. P. Young, “An evolutionary model of bargaining,” *Journal of Economic Theory 59*, 145-168, 1993.

H. P. Young, *Equity in theory and practice*, Princeton University Press, 1994.

H. P. Young, *Individual Strategy and Social Structure: An Evolutionary Theory of Institutions*, Princeton University Press, 1998.

H. P. Young, "Conventional Contracts," *Review of Economic Studies 65*, 773-792, 1998a.

H. P. Young, "Learning by trial and error," *Games and Economic Behavior 65*, 626-643, 2009.

F. Zeuthen, *Problems of Monopoly and Economic Warfare,* Routledge & Kegan Paul, 1930.