8 Development of Mathematical Competencies

Elsbeth Stern

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In structuralistic theories as well as in domain-specific theories of cognitive development, improvement of mathematical competencies is considered to be an important issue. Structuralistic views as developed by Piaget (1950), Case (1985), and recently by Halford (1992) use mathematical problems requiring number conservation or proportional reasoning to demonstrate progress in general information-processing efficiency or cognitive structures. Structuralistic theories of cognitive development have been challenged seriously by nativistic views that see the neonate as preprogrammed to make sense of specific information sources. There is a compelling line of evidence for an innate origin of mathematical knowledge. All aspects guiding the cardinal understanding of numbers, such as counting, seem to be guided more or less by innate principles (Gelman 1990; Wynn 1990). Therefore, the basic principles of mathematics are acquired easily by young preschool children without systematic instruction. In contrast, however, the acquisition of advanced mathematical concepts that are products of cultural evolution takes place during a long and continuous process that requires systematic instruction (Gelman 1991; Resnick 1989). Advanced mathematical understanding necessary for school-based mathematics entails radical restructuring of early cardinal number understanding. Advanced mathematical understanding is based on the mathematical symbol system that is characterized by a dual role: On one hand, symbols are used to describe and model concrete, real-world situations and events, whereas on the other hand they derive their mathematical power from the fact that their intrinsic meaning is divorced from concrete contexts. One can solve problems such as $3 + 5 =$ without referring to real situations in which elements of sets are combined. To have advanced understanding of mathematics means knowing that numbers are understood not only as counting instruments but are also used to describe the relation between sets and symbolic systems.

The importance of advanced mathematical understanding becomes clear when considering mathematical problems that go beyond the counting function of numbers. Three types of mathematical competencies fulfilling this criterion are considered in this chapter. (a) Strategy use in arithmetic: Solving complex arithmetic problems such as $16 + 8 - 8 =$ can be considerably facilitated by using principles such as commutativity and the neutral element. The use of a shortcut strategy instead of a computing strategy requires flexibility in dealing with quantitative symbols. (b) Solving arithmetic word problems: Understanding and solving word problems requires modeling real-life situations with the help of mathematical symbols. In particular, understanding statements that describe the relations between sets such as "Peter has five marbles less than John" requires understanding that the function of numbers goes beyond counting (Stern 1993a). (c) Proportional reasoning: When faced with situations that require considering at least two different units (e.g., situations dealing with speed), principles guiding the counting function of numbers must be ignored; although it is true that larger numbers refer to larger quantities for counting numbers, only the relation between the involved numbers allows inferences about the size of quantities for proportional units.

Mathematics provides an example of a domain in which huge performance differences are obtained at all age levels. Similar to the question of whether between-age differences are due to domain-specific or domain-general differences, within-age differences are considered to be explained by differences in general or in specific abilities. G-factor theories explain individual differences by referring to differences in general processing efficiency. Support for these theories comes from data showing correlations across performances in different domains (e.g., Spearman 1927). The correlation between measures of general intelligence and achievement in mathematics is higher than with most other school subjects (Gustaffson and Balke 1993). There are two main approaches toward explaining what Factor G might be. Low-level models focus on differences in basic information-processing aspects, such as speed and capacity (Jensen 1982). High-level approaches, on the other hand, focus on metacognitive competencies (Sternberg 1985). Both high-level and low-level theories can explain individual differences in mathematics because in this domain very complex facts are described with the help of a very sparse system of symbols. Understanding a sequence of mathematical symbols entails activating complex knowledge structures, which presupposes a high processing efficiency. At the same time, the complexity of most mathematical problems requires metacognitive abilities for planning the problem-solving process.

Domain-specific approaches such as those developed by Thurstone (1938) or recently by Gardner (1983) are based on empirical findings that indicate that individuals may show high competencies in some domains, although they are only average or even below average in other domains. There are good reasons for considering specific factors to explain variance in mathematical achievement: Intelligent persons with poor performance in mathematics exist, as do participants showing the opposite pattern.

Open Research Questions

Theories of general development and theories of individual differences focus on similar distinctions and processes: In both research approaches, domain-specific

competencies are contrasted with general competencies. Moreover, the competencies considered in Factor G theories of intelligence, such as speed, capacity, or processing efficiency, correspond to competencies focused on some neo-Piagetian theories of development. Combining theories of cognitive development and psychometric theories means developing a theory that is able to explain coherently individual differences within and between age groups. According to the theory developed by Anderson (1992), competencies arise from domain-specific modules that are restructured with development. For example, individual differences in understanding rational numbers are caused by individual differences in the speed of restructuring conceptual primitives into more advanced and abstract knowledge structures. Innate differences in general information-processing speed are responsible for within-age level performance differences. Applied to the acquisition of mathematical competencies, this means that participants differ in the speed of restructuring cardinal numbers into more abstract numerical concepts.

It is the goal of this chapter to shed light on the impact of general and domain-specific abilities on longitudinal development of individual differences in advanced mathematical competencies between and within age levels by addressing the following issues:

1. *The impact of age-related developmental levels on mathematical competencies.* If the general cognitive competencies obtained at a particular age level have a strong impact on performance across many specific competencies as predicted by structuralistic theories, high correlations between general and specific measures within the same measurement point will be obtained. If, however, domain-specific performance is determined by successive acquisition and restructuring of specific knowledge, high correlations between similar measures presented at different measurement points are expected.
2. *The impact of preschool performance on later mathematical competencies.* If interindividual performance differences in mathematics are caused by the speed with which basic concepts are restructured into more abstract and advanced concepts, performance differences measured in preschool are expected to predict later mathematical performance because participants who began restructuring basic concepts earlier are expected to be able to develop more advanced concepts. A question of interest in this context is whether general or specific competencies have a stronger impact on later mathematical achievement.
3. *The impact of competencies in elementary school on proportional reasoning.* As mentioned before, the acquisition of proportional reasoning is considered to be an important transition in all theories of cognitive development, and therefore this issue merits further research. Individual differences in proportional reasoning in middle grades might be determined by performance differences in domain-specific knowledge in elementary school. Because advanced mathematical knowledge is developed from restructured basic competencies, the earlier such competencies are acquired, the earlier they might be restructured. According to structuralistic theories of development, however, advanced general cognitive abilities are considered to be the necessary precondition for proportional reasoning. Therefore, a substantial within-measurement point correlation between general abilities and proportional reasoning is expected.
4. *The impact of broader abilities on mathematical performance.* Differences in domain-specific knowledge cannot be expected to be the only source of within- and between-

age level differences in mathematical competencies. Low-level abilities such as speed of information processing as well as high-level abilities such as metacognitive skills and crystallized intelligence might have an impact on mathematical competencies. Moreover, besides such general competencies, former studies on mathematical literature abilities have shown an impact of spatial abilities on mathematical performance. The age level at which different kinds of abilities will influence mathematical performance is analyzed.

Method

Participants

In every wave of the LOGIC study, measures of mathematical competencies were presented. However, for the following reasons, the full sample of children could not be considered in the analyses. As mentioned in the introduction of this volume, not all children entered school in the same year. As the time spent on attending school has clear effects on mathematical competencies, only the children who had entered school by 1987 were considered in the following analyses. As measures of word-problem solving were presented only in the SCHOLASTIC sample, analyses considering these measures were based only on 110–120 participants.

Measures of Mathematical Competencies

In the LOGIC study, we focused on the acquisition of mathematic competencies that demand numerical understanding beyond a cardinal understanding of numbers. Three problem types were considered: (a) word problems dealing with the comparison of sets, (b) numerical problems requiring the use of elaborated strategies, and (c) problems dealing with proportions.

Word Problems Dealing with the Comparison of Sets. The first time that children's numerical understanding goes beyond the cardinal understanding of numbers might be when they understand the quantitative comparison. Understanding sentences such as "John has three marbles more than Peter" requires the understanding that the difference between the sets is not a concrete, existing set of elements, but it rather describes the relation between the two sets. Quantitative comparison is generally not understood before entering school (Stern 1993b). Arithmetic word problems dealing with the comparison of sets are more difficult than problems dealing with the exchange, combination, and equalization of sets (Riley, Greeno, and Heller 1983; Stern 1993b). A well-known result first published by Hudson (1983) and replicated several times (Davis-Dorsey, Ross, and Morrison 1991; Stern 1993b) illustrates young children's difficulties: Although nearly all children solve problems such as "There are five birds and three worms. How many birds won't get a worm," fewer than 20% of the participants solve the problem if it ends with the question "How many more birds than worms are there?" An explanation for these differences might be

that children have difficulties accessing the appropriate mathematical model (Stern 1993a, 1993b; Stern and Lehrndorfer 1992). Although the question "How many birds won't get a worm?" asks for a concrete, countable set, the question "How many more birds than worms are there?" asks for the relation between two sets. Understanding that numbers can be used not just for counting but also for describing the relation between sets might be the first step to an extended mathematical understanding. Stern (1993b) has found several lines of evidence indicating that understanding and solving word problems dealing with the comparison of sets is the precondition for developing advanced mathematical competencies such as understanding rational numbers.

To find out which children understand the quantitative comparison at a very early age and what consequences early understanding of the quantitative comparison might have on later mathematical understanding, we presented arithmetic word problems dealing with the comparison of sets in Waves 5, 6, and 7 (second, third, and fourth grade, respectively)[1] in a written test. In each test, there were 10 comparison problems: 2 were one-step problems, such as "John has five marbles. He has two marbles less than Peter. How many marbles does Peter have?" and 8 were multiple-step problems, such as "John has five marbles. He has two marbles less than Peter. How many marbles do John and Peter have together?" or "Susan and Mary have 14 dolls altogether. Susan has 2 dolls less than Mary. How many dolls does Mary have?"

For each wave, the number of correct problems, defined as the correct answer and the correct equation, was scored.

Strategy Use in Arithmetic. The use of simplifier strategies to solve arithmetic problems also requires an advanced understanding of mathematics. The use of simplifier strategies, such as $8 + 5 = 8 + 2 + 3$, is based purely on symbol manipulation. Detached from any concrete context, symbols can be manipulated in different ways with the only constraint being that the sum total remains constant. In all waves, two kinds of problems requiring the use of simplifier strategies were presented: inversion problems and estimation problems.

1. *Inversion problems.* Only a few children younger than 10 use shortcut strategies to solve inversion problems such as $a + b - b =$ (Bisanz and LeFevre 1990; Stern 1992), although children at this age level can be expected to have mastered the principles underlying the shortcut strategy: commutativity and the neutral element. In a computer-based procedure presented in Wave 5 (second grade), Wave 6 (third grade), and Wave 7 (fourth grade), problems requiring facilitating strategies were presented. A problem (e.g., $35 + 8 - 8$) was presented in the middle of a computer screen, and the correct answer was presented in one corner of the screen while an incorrect answer was presented in another corner. Children had to press a button corresponding to the correct answer, and if no button was pressed within 9 s, the next problem was presented. The time required to solve the problem was measured. A maximum of 9 s was given.
2. *Estimation problems.* With the same procedure, the use of the estimation strategy was investigated. Estimation problems were one-step subtraction and addition problems

[1] Only the LOGIC children who were also in the SCHOLASTIC study were given these tasks.

with one number being larger than 20. The wrong answer was always a number that contradicted basic mathematical principles. For example, in a subtraction problem, the incorrect answer presented might be larger than the numbers used in the equation. Participants who first checked whether one of the two possible answers contradicted basic principles of mathematics did not have to compute an answer.

The number of correctly solved problems and the mean solution times for correct answers were scored. Alpha analyses indicated that the score "number of correctly solved problems" was more reliable at all measurement points than scores based on solution times. Participants who did not use strategies failed in most cases to solve the problem within the required time and, therefore, were not given credit for a correct answer.

Proportional Reasoning. Dealing with proportions requires giving up principles that guide the understanding of counting: Larger numbers do not always refer to larger quantities than smaller numbers. Being able to reason proportionally means knowing that one has to consider at least two sources of information before drawing a conclusion. Adolescents' difficulties with understanding proportions, decimal numbers, and fractions are well documented. The most frequent mistake made is that larger numbers are considered to refer to larger values than smaller numbers; for instance, that $6/8 > 6/7$ (Hiebert 1986).

Proportional reasoning in elementary- and middle-grade children is a well-researched field: Several standardized tasks have been developed, and several studies describe five steps leading to correct proportional reasoning (Case 1978; Karplus, Pulos, and Stage 1983; Noelting 1980). Using as an example a task in which one has to determine which of two beverages made from glasses of raspberry juice and glasses of water will taste more intensive, we define the stages as follows. (a) In the stage of *isolated centration*, participants only consider whether raspberry juice was added or not. (b) In the stage of *unidimensional comparison*, participants ignore the amount of water and only consider which beverage contains an absolute greater amount of raspberry juice. (c) In the stage of *bidimensional comparison*, both dimensions are considered, but without quantification. The children only compare the quantity of water and juice in each beverage, and they pick the beverage having an excess of juice over water. When this is the case for both beverages, participants guess. (d) In the stage of *bidimensional comparison with quantification*, children compute the difference between water and juice for each beverage and choose the beverage with the larger difference. (e) Only in the stage of *ratio comparison* do participants use division strategies to find out in which beverage the proportion of juice is larger.

Summarizing from the results reported by Noelting (1980) and Karplus et al. (1983), participants begin to make bidimensional comparisons at about 7 years of age, bidimensional comparisons with quantification at about 10 years of age, and ratio comparison only after 15 years of age. In the longitudinal study, we focused on the transition from the bidimensional comparison (Stage 3) to the bidimensional comparison with quantification (Wave 5). In Waves 7 (Grade 4), 8 (Grade 5), and

9 (Grade 6), problems were presented that could be solved correctly by using the strategies typical for Stages 2–4. The problems were embedded in different context stories, such as testing the taste of mixed raspberry, estimating the weight of pieces of cheese differing in size, or collecting money for a good purpose.

Although they differ in their superficial structure, problems measuring arithmetic strategy use, understanding the quantitative comparison, and proportional reasoning are all similar in that they are based on elaborated mathematical understanding. Mathematical comprehension beyond counting is required to solve each type of problem. Therefore, close connections between these kinds of problems are expected. There are good reasons for assuming that an early use of elaborated strategies and early competencies in solving quantitative comparison problems might be good predictors of early competencies in proportional reasoning. Moreover, it may even be that mathematical performance in preschool is a good predictor of later performance in mathematics: Children who acquire basic mathematical competencies earlier than others may use this knowledge for developing more elaborated knowledge structures at an earlier time than other children.

Measures of Mathematical Competencies in Preschool

Counting Abilities. Long before entering formal school, children spontaneously acquire counting abilities at about age 4 (Wynn 1990). Differences in counting abilities in early childhood might predict differences in mathematical performance in elementary school. The earlier children learn to count, the earlier they might understand that counting is not the only function of numbers, and therefore they extend their mathematical understanding. In Wave 1 of the longitudinal study, counting abilities were observed: The children were asked to count sets of sizes varying between one and five.

Number Conservation. In Piaget's (1950) theory, the number-conservation test was considered to measure the transition from the preoperational stage to the concrete operational stage. Longitudinal studies by Stevenson and Newman (1986), however, indicated performance in number conservation to be a good predictor of mathematical performance in elementary school. Mastering the number-conservation task means understanding that verbal expressions such as "more than" and "less than" refer to the given number of elements in a set rather than to the spatial expansion of the elements. Thus, the number-conservation task might be an indicator of early quantitative reasoning ability rather than of the general cognitive level. In Wave 1 and Wave 3, children were presented with number-conservation problems.

Estimation of Quantities. Another measure of early quantitative abilities was quantity estimating, which is part of a German school-readiness test developed by Kern (1971). Children were presented with a set of three to nine small cubes and had to determine the number without counting them. Although this test was developed long before sophisticated theories of knowledge representation existed, a post hoc theoretical basis of this test is that it measures the efficiency with which visual information is transformed into mathematical symbols.

Numerical Abilities Measured with the Hannover–Wechsler–Intelligence Scale for Preschool Children (HAWIVA). In Wave 2 and in Wave 4, the Numerical Abilities subtest in the HAWIVA (Eggert 1978) was regarded as a measure of numerical abilities. In this subtest, participants had to solve addition and subtraction word problems.

Measures of General Cognitive Abilities

In each wave, measures of general cognitive abilities were presented. Three types of general measures were considered.

Nonverbal Intelligence. The Columbia Mental Maturity Scale (CMMS, Burgemeister, Blum, and Lorge 1972) was presented in Waves 1, 3, and 5. In Waves 7 and 9, a German version of the Raven test (Weiss and Osterland 1979) was presented.

Measures of Basic Information Processing. Measures of capacity and speed were presented as follows. (a) Capacity, the number span was measured in Waves 4 and 6 with a subtest of the Hamburg–Wechsler–Intelligence Scale for School Children (HAWIK, Tewes 1983). Word span was measured in Wave 3. (b) Speed, paper-and-pencil measures of speed were presented in Waves 6 and 8. Under time pressure, participants either had to relate numbers to digits or had to mark signs with particular features.

Verbal Intelligence. The General Knowledge, Analogies, and General Understanding subtests of the HAWIK and HAWIVA were combined and considered to be a measure of verbal intelligence.

Spatial Abilities. Spatial abilities are considered to have an influence on mathematical competencies. In Wave 8, Subtests 7 (folding) and 8 (field dependance) of the German Intelligence Test PSB (Prüfsystem für Schul- und Bildungsberatung [Testing System for educational counseling]), developed by Horn and Cattell (1966), were used. The PSB test is based on Thurstone's (1938) multiple-intelligence theory.

Results

Growth of Competencies

For all mathematical tests presented at different age levels, an increase in performance level was obtained, as expected.

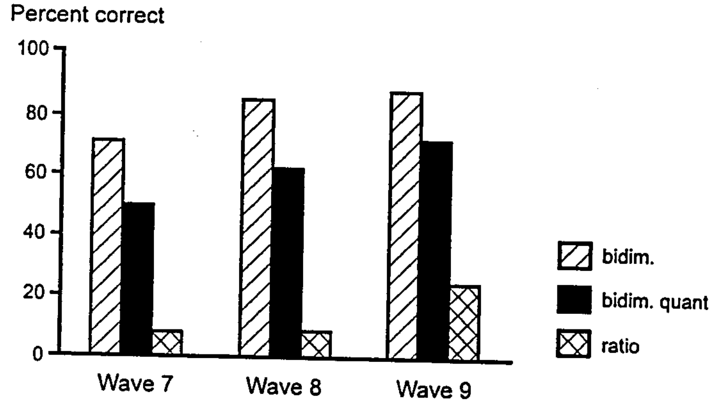

Figure 1. Percentage of correctly solved proportional reasoning problems separated by problem types: (a) Problems that could be solved correctly by qualitative bidimensional comparison, (b) problems that could be solved correctly by quantitative bidimensional comparison (difference strategy), and (c) problems that could only be solved correctly by the use of the ratio strategy.

Number Conservation. In Wave 1, only 10% of the participants passed the number-conservation task, whereas in Wave 3, 56% of the participants mastered it.

Word-Problem Solving. The mean solution rate for compare problems was .36 (SD = .24) in Wave 5, .47 (SD = .32) in Wave 6, and .56 (SD = .28) in Wave 7. The correlations between the three measurement points were substantial (Waves 5–6 = .61; Waves 6–7 = .73; Waves 5–7 = .54, $p < .001$).

Strategy Use. The mean solution rate for problems requiring the use of facilitating strategies was .32 (SD = .27) in Wave 5, .45 (SD = .28) in Wave 6, and .76 (SD = .18) in Wave 7. The correlations between the three waves, although significant, were lower than for word-problem solving (Waves 5–6 = .43; Waves 6–7 = .50; and Waves 5–7 = .32, $p < .01$).

Proportional Reasoning. In accordance with other reported results (e.g., Noelting 1980), a transition from unidimensional comparison to bidimensional comparison took place. Figure 1 depicts the mean solution rates for problems requiring the use of different strategies. The results indicate that the bidimensional comparison strategy was used frequently in Wave 7, but the bidimensional strategy with quantification was used less frequently. In Wave 8, in contrast, the bidimensional strategy with quantification was quite common, but the ratio strategy was very uncommon. Even in Wave 9, only a few of the problems requiring the ratio strategy were solved correctly. The results depicted in Figure 1 indicate reduced variance in Waves 8 and 9. The correlations between the three waves were significant, but rather low between Waves 8 and 9, indicating that many participants were in a transition phase (Waves 7–8 = .39; Waves 8–9 = .28; and Waves 7–9 = .38, $p < .01$).

The Development of Individual Differences

The Impact of Age on Mathematical Competencies. A principal-components factor analysis with the 31 measures was conducted to find out whether different measures presented at the same age level show higher correlations than similar measures presented at different age levels. If this is the case, the analysis will produce age-level factors; that is, different measures presented at the same measurement time will load on the same factor. The analysis explained 70% of the variance and revealed the nine factors depicted in Table 1. There were no age-level factors at all, and measures presented at different age levels showed high loadings on the same factor. This result does not support models of development nor of individual differences that emphasize the impact of general abilities on mathematical competencies.

The Impact of Preschool Performance on Later Mathematical Competencies. Separate path analyses for strategy use, word-problem solving, and proportional reasoning were conducted to obtain information about the impact of general abilities and numerical competencies obtained in preschool on mathematical performance later during school. The results depicted in Figure 2 indicate that the general and specific preschool measures had a considerable impact. The results show that performance on the number-conservation task and the estimation-of-quantities task obtained in Wave 3 has a strong impact on later mathematical performance. Given that both tasks contained only a few items and were therefore less reliable than intelligence tests, their impact is especially remarkable. Knowing at an early age that verbal expressions used in number-conservation tasks such as "more than" and "less than" refer to the number of elements of sets rather than to their spatial extension seems to be helpful in understanding situations involving quantitative comparison and proportions. Participants who were good estimators of set sizes before they entered school were inclined to develop conceptually based computing strategies in the middle of elementary school.

However, although the number-conservation task and the estimation-of-quantities task both contributed to later mathematical performance, they were only moderately related to each other ($r = .32$, $p < .01$), and therefore it was not possible to combine them into a latent variable. Mathematical competencies measured in preschool seem to be quite task specific and may be restructured into broader abilities only later in development. The earlier participants acquire mathematical competencies such as number conservation and estimation of quantities, the greater the chance seems to be for developing knowledge structures that allow them to cope with more sophisticated mathematical problems.

Number estimation may measure efficiency in combining information and mathematical symbols. Participants showing good performance in this task may be able to transform visual information into symbolic information, and vice versa. This ability may help in switching between symbolic and visual representations, and representing a problem visually may help with shortcut strategies. In addition, solving problems

Table 1. *Factor Loadings of the Principal-Component Factor Analysis*

Variable	Factor 1: Verbal intelligence	Factor 2: Mathematical abilities	Factor 3: Number sense	Factor 4: Nonverbal intelligence	Factor 5: Number sense	Factor 6: Capacity	Factor 7: Nonverbal intelligence	Factor 8: Proportional reasoning	Factor 9: Number conservation
NC1	.09	.06	.13	.07	.06	.11	.03	.03	**.82**
NC3	.26	.18	.07	.09	**.47**	.08	.08	.34	−.22
EQ3	.19	−.01	**.42**	.10	**.40**	.19	.32	.26	−.15
MA2	.32	.26	.15	**.53**	.15	.19	.02	.15	−.11
MA4	.22	−.02	**.50**	.08	**.52**	.24	−.14	.07	−.00
MA6	.10	.30	.23	.10	**.63**	.10	−.08	−.01	.21
MA9	.19	**.48**	**.47**	.15	.22	.05	.01	.24	.13
WP5	.30	**.63**	**.39**	.18	.24	−.00	.02	.16	−.00
WP6	.24	**.77**	.23	.21	.10	.11	.05	.09	.02
WP7	.15	**.45**	.24	.34	**.59**	.01	.14	−.12	.07
PR7	.28	.28	.27	.20	.17	.13	−.06	**.58**	.08
PR8	.08	.18	.09	.05	−.01	.05	−.07	**.77**	.03
PR9	.20	**.60**	.05	−.21	.30	.05	.10	.22	−.14
AA5	.04	.24	**.68**	.09	.29	.20	.12	−.00	.06
AA6	.13	.12	**.77**	.02	.13	.06	.21	.05	.08
AA7	.13	.32	**.66**	.11	−.09	−.05	.03	.15	.08
CAN4	.10	.17	.08	.11	.04	**.84**	.03	.07	.11
CAN6	.06	.00	.10	.15	.15	**.84**	.09	.16	.03
CAS3	.33	**.38**	.10	.19	.02	**.55**	−.15	−.21	.01
DS6	−.00	.00	.16	.16	−.14	−.03	**.81**	−.19	−.09

Table 1. *(Cont.)*

Variable	Factor 1: Verbal intelligence	Factor 2: Mathematical abilities	Factor 3: Number sense	Factor 4: Nonverbal intelligence	Factor 5: Number sense	Factor 6: Capacity	Factor 7: Nonverbal intelligence	Factor 8: Proportional reasoning	Factor 9: Number conservation
NVI1	.17	.00	.16	**.75**	.01	.18	.12	−.00	.10
NVI2	.02	.31	.05	**.44**	.14	.13	**.35**	.10	−.02
NVI3	.07	.14	−.10	**.61**	**.43**	.06	.14	.18	.20
NVI5	.11	**.35**	.05	.34	.13	.17	.31	.29	.05
NVI6	.18	.33	.09	.10	.14	.03	**.74**	.04	.18
NVI7	.17	**.68**	.17	.12	.08	.20	.34	.16	.08
NVI9	.04	**.54**	.25	.27	.02	.20	.23	.23	.31
VI1	**.67**	.12	.21	**.47**	−.05	.04	.05	−.11	−.23
VI2	**.69**	.17	.12	.33	−.08	.01	.01	.12	−.16
VI4	**.84**	.08	.07	.11	.24	.16	.01	.04	.12
VI6	**.84**	.16	.12	.01	.17	.10	.16	.16	.21
VI9	**.76**	.31	.13	−.04	.19	.12	.08	.20	.14

Note: The numbers at the end of the variable names refer to the wave. Factor loadings > .35 are bold. NC = Number conservation; EQ = Estimation of quantities; MA = Mathematical abilities; WP = Word problems; PR = Proportional reasoning; AA = Arithmetic abilities; CAN = Capacity numbers; CAS = Capacity sentences; DS = Digit–Symbol Test; NVI = Nonverbal intelligence; VI = Verbal intelligence.

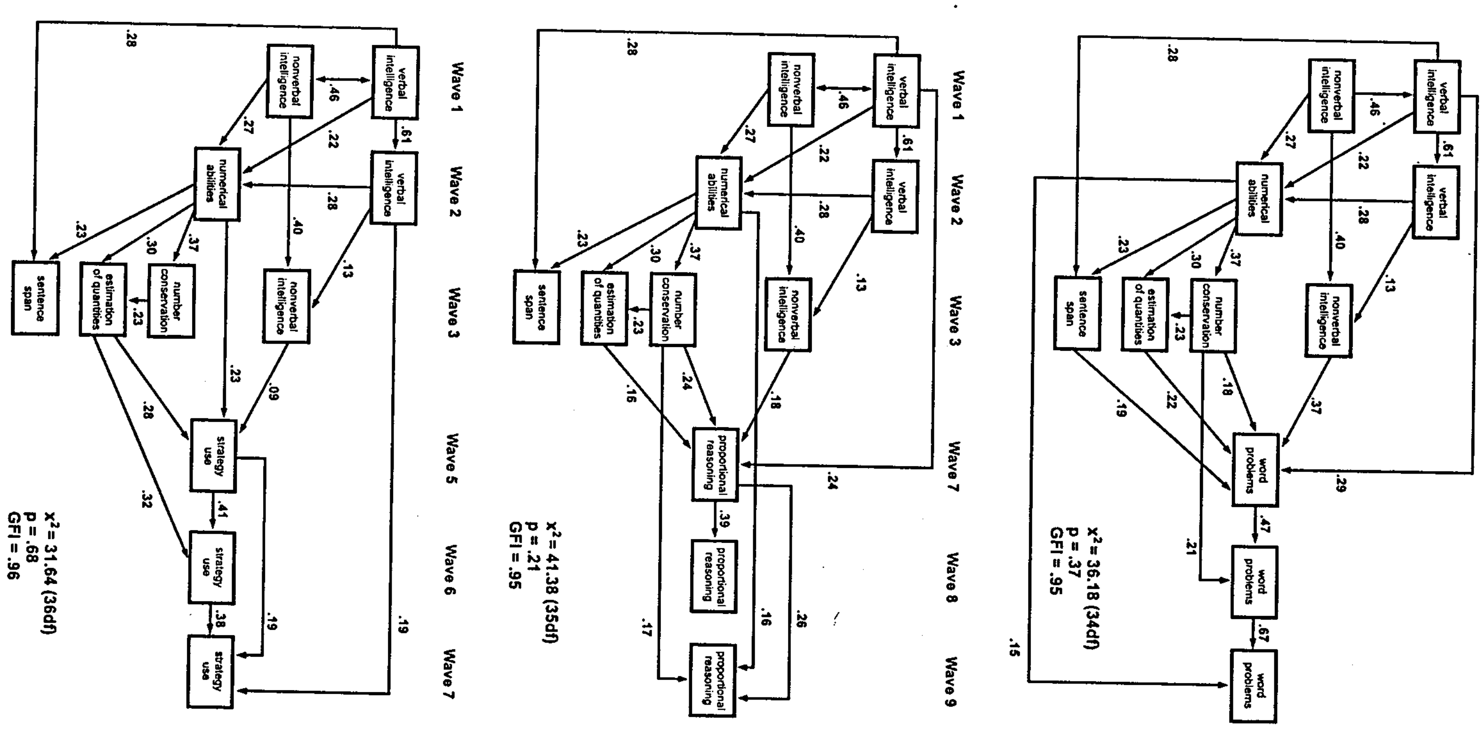

Figure 2. Results of the path analyses.

presented verbally, such as arithmetic word problems, may be facilitated if one is flexible in switching between visual and symbolic representation.

The number-conservation task may measure an ability to consider quantitative features rather than spatial extension features when verbal information about quantities is given. Participants who solved the conservation task at an early age were rather good at solving mathematical problems dealing with quantitative comparison (i.e., dealing with quantitative information).

The results indicate that the general abilities measured by verbal and nonverbal intelligence also contribute to mathematical performance. Taken together, our results show that different types of mathematical competencies correlate within and between different age levels. Early competencies acquired without systematic schooling have an effect on the acquisition of mathematical competencies in elementary school.

The Impact of Competencies in Elementary School on Proportional Reasoning. A regression analysis was conducted to find out what variables predict proportional reasoning in fifth and sixth grade because considerable individual performance differences are obtained at this age level. Although some participants already used ratio strategies, others still relied on bidimensional comparison. The results, depicted in Table 2, show that word-problem solving in Wave 7 is the best predictor of proportional reasoning in Wave 9 and that performance on intelligence tests and basic information-processing tasks assessed in Waves 8 and 9 play only a minor role. This result again shows the impact of domain-specific competencies. Good performance on word-problem solving 2 years earlier was more helpful for proportional reasoning than high general abilities at the same measurement point. The results emphasize the importance of long-term knowledge acquisition.

The Impact of Broader Abilities on Mathematical Performance. The former analyses suggested a strong impact of domain-specific competencies on individual differences in mathematical competencies within and between different age levels. However, general competencies might additionally contribute to the explanation of individual differences in mathematics. In a stepwise regression analysis, performance at the end of elementary school (Wave 7) in the three types of mathematical competencies considered in this chapter was predicted by verbal and nonverbal intelligence measures in Waves 1–7, as well as by measures of basic information processing (word capacity, Wave 3; number capacity, Waves 4 and 6; and speed, Wave 6). In addition, the measures of spatial abilities were included.

The results, depicted in Table 3, indicate that performance in verbal and nonverbal intelligence tests at all age levels had a clear impact on mathematical competencies. However, our results do not support any impact of basic information-processing efficiency, such as processing speed and capacity. Thus, the influence of general factors may be due to high-level abilities such as metacognitive strategies rather than to basic processing efficiency.

Table 2. *Results of the Stepwise Regression Analyses*

Predictor	Proportional reasoning
Verbal intelligence	
Wave 4	—
Wave 6	—
Nonverbal intelligence	
Wave 5	1
Wave 7	1
Stragtegy use	
Wave 5	1
Wave 6	—
Wave 7	—
Proportional reasoning	
Wave 7	3
Word-problem solving	
Wave 5	2
Wave 6	—
Wave 7	25
R^2	33

Notes: Proportional reasoning in Grade 6 (Wave 9) was predicted by general and specific competencies measured during elementary school time. The percentage of explained incremental variance is depicted.

Final Conclusions

What do our longitudinal data tell us about sources of development in mathematical competencies and sources of individual differences in mathematics? Our results clearly suggest a considerable domain-specific impact on differences within and between age levels. However, there is an additional impact of high-level general abilities at all age levels. The results indicate that with growing age, the relation between mathematical performance and general abilities increases, especially for nonverbal intelligence. However, our results clearly indicate that high-level rather than low-level abilities influence mathematical performance. Reductionistic approaches to intelligence such as those suggested by Jensen (1982) or Anderson (1992) are not supported at all by our data.

Emphasizing the domain-specific impact on differences between and within different age levels does not at all mean that mathematical competencies are isolated knowledge structures that develop independently from other competencies. To the contrary, mathematical competencies are developed by using mathematical language to describe situations and events in quite different domains. Rich knowledge about the world provides opportunities to use mathematical language for modeling and

Table 3. *Results of Stepwise Regression Analyses*

Predictor	Criteria: Strategy use	Word-problem solving	Proportional reasoning
Verbal intelligence			
Wave 1	—	—	—
Wave 2	—	2	1
Wave 4	—	1	3
Wave 6	—	1	1
Nonverbal intelligence			
Wave1	1	—	2
Wave 3	1	1	1
Wave 5	—	—	1
Capacity sentences			
Wave 3	—	1	7
Number conservation			
Wave 3	—	1	2
Number estimation			
Wave 3	—	1	1
Strategy use			
Wave 5	2	—	1
Wave 6	26	—	27
Word-problem solving			
Wave	—	1	—
Wave 6	7	54	—
R^2	38	63	47

Note: Strategy use, word-problem solving, and proportional reasoning at the end of elementary school in Grade 4 (Wave 7) was predicted by general competencies. The percentage of explained incremental variance is depicted.

simulating real-life situations. By doing so, one might not only extend subject-matter knowledge but also acquire metastrategies that generally help one deal with complex and abstract problems.

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