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Interaction between Knowledge and Contexts on Understanding Abstract Mathematical Concepts

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The purpose of the present study was to investigate the effects of sparse vs real contexts on the understanding of abstract mathematical concepts regarding the interpretation of linear graphs. Four experiments addressed this issue: two focused on children's performance (N = 124 and 35, respectively; mean age about 12 years old) and two focused on adults (N = 165 and 169, respectively). In all four experiments subjects were asked to interpret isomorphic linear graphs. The tasks were embedded in sparse vs real contexts. Taken together, results of all four experiments showed that a sparse context facilitated understanding of abstract mathematical concepts more than real contexts. In addition, content analysis of students' responses showed that students activated different knowledge structures in solving the isomorphic tasks embedded in the various contexts. The theoretical and practical implications of the findings are discussed.

How do children approach isomorphic mathematical problems? Are mathematical problems embedded in real contexts easier to solve than the same problems embedded in sparse contexts? Does the solution of problems embedded in one kind of context facilitate the solution of problems embedded in a different kind of context? Finally, to what extent do contexts exert different effects on children's understanding of abstract mathematical concepts than on adults? The present research addresses these issues by a series of four studies each focusing on isomorphic problems involving linear graphs.

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Current research in cognitive psychology has recognized the important role of the nonmathematical context in facilitating mathematical understanding (e.g., Hembree, 1992). Theoretically, there are three contextual conditions to consider. First, the context is relatively neutral presenting no supportive information nor any confusion, but the subject must disentangle the mathematical situation from it and then proceed to act—this is partially what is being described by Kintsch (1988) and Kintsch and Greeno (1985). Second, the context is both familiar and in some way supportive of the mathematical situation. Hudson's classical ''how many birds would not get worms'' problem (Hudson, 1983) and the research of DeCorte, Verschaffel, and DeWin (1985) both support the result that in some cases words act as supports for the mathematical situation and enhance performance. Finally, the contextual information exerts debilitating effects on performance—this has been consistently reported in studies showing that mathematics word problems are more difficult to solve than their corresponding equations.

Until recently, cognitive-developmental psychologists underscored the relative importance given to ''process'' over ''context.'' In mathematics, for example, it has largely been believed that ''performing operations is void to contextual details. . . . Once a problem is mathematically formulated, the answer can be obtained through automatic processing that is devoid of context'' (Janvier, 1990, p. 188). (See Nunes, 1993, for an interesting discussion of this issue.) Yet, intensive research has shown that often this is not the case. Carraher, Carraher, & Schliemann (1985), for example, showed that children who were street venders in Brazil were able to solve rather difficult mathematical problems relating to their selling activities, but they had considerable difficulties in solving the same problems in the sparse context of the paper-and-pencil world of school. In a later study that also focused on Brazilian children, Saxe (1991) compared mathematics performance of schooled nonsellers and unschooled sellers. He found that ''the children construct different kinds of mathematics knowledge. The schooled child makes use of a specialized knowledge of the number orthography as a central feature of mathematical problem solving, whereas the unschooled child does not, relying instead on specialized knowledge of numerical representations linked to the currency system. Furthermore, in structuring solutions to each of the problem solving tasks—arithmetic, ratio, comparison, and mark-up— . . . children across groups differed in the way they accomplished these problems. These differences were linked to the specialization of practice-linked conventions or school-linked algorithms cognitive forms which . . . were interwoven with the way in which the mathematical problems were presented to the two groups'' (p. 151). In a similar vein, Lave (1988) and Greeno (1989) demonstrated that people were much more successful in solving quantitative problems corresponding to everyday activities than problems embedded in sparse contexts. These studies show, therefore, that knowledge and cognition are not independent of the contextual situation in which they are activated. In real, meaningful, and goal-related situations, rather than in sparse contexts, the solvers were more often able to solve mathematical problems involving quantities.

The cognitive interaction between context and knowledge on mathematical problem solving has been found not only in studies focusing on different physical contexts (e.g., Carraher et al., 1985; Greeno, 1989; Lave, 1988; Nunes, 1993; Saxe, 1991), but also in studies investigating various kinds of story contexts. Hembree (1992) distinguished between different kinds of story styles categorized by different criteria: concrete vs abstract; factual vs hypothetical; familiar vs unfamiliar; imaginative vs ordinary; and personalized vs impersonal. Using a meta-analysis technique, Hembree (1992) calculated the Effect-Size of story contexts by computing the difference between the mean performance on problems embedded within familiar and unfamiliar contexts and dividing the difference by the standard deviation of the population. He showed that familiar contexts were linked with better performance more often than unfamiliar contexts ($Effect - Size = .40$) and that problems embedded within concrete settings, or imaginative stories were associated with Effect-Sizes that were marginally significant ($Effect - Size = .14$ and $.17$, respectively). All other contexts did not affect mathematical performance.

The powerful effects of contextual stories have not been limited to a certain age group. Stern and Lenderhofer (1992) and Stern (1993), for example, showed the facilitative effects of rich, meaningful, and goal-related contexts on kindergartners' and first-graders' solution of word problems involving addition and subtraction. Davis-Dorsey, Ross, and Morrison (1991) demonstrated the effects of personalizing mathematical word problems and rewording them for explicitness on second- and fifth-graders' solutions. Hembree (1992) reported similar findings for students in fourth to twelfth grades. It seems that real, meaningful, and goal-related contexts substantially facilitate solvers' ability to construct a mental representation of the mathematical model of a given problem, which in turn influences mathematical performance (Anand & Ross, 1987; Greeno, 1989; Mayer, 1983; Resnick, 1987, 1989; and Schoenfeld, 1985).

Three types of mechanisms may explain the effects of the context on cognition: a real context may facilitate semantic comprehension, enhance the use of particular strategies, or activate knowledge structures that allow for more/less efficient processing (Ceci & Roazzi, 1993). With regard to the last cited mechanism, the assumption is that in the domain of mathematics there are two kinds of knowledge structures: quantitative related to the mathematical entities, and situational related to the physical entities within a problem (Hall, Kibler, Wenger, & Truxaw, 1989). When the two kinds of knowledge structures are in accord, the real, meaningful, and goal-related context helps the construction of a mental representation that supports the mathematical model and intimately links to better mathematical performance than the sparse context. When, however, the two kinds of knowledge structures are not in accord or when the solver does not know how to bridge them, he or she may construct a mathematical model that is simplistic, constructed of unconnected elements.

Three examples may illustrate this point: one relates to subtraction, the second to multiplication, and the third to linear graphs.

Fischbein (1992) explains that in practical terms, to subtract means to compare two sets and determine the noncommensurate portion or to remove from a set of elements a subset and to determine what remains (the container model). Although the container model fits many situations involving subtraction, there are situations that contradict it, such as: subtracting from zero or subtracting with regrouping when one works in the written numerical representation with base ten. When children are asked to subtract from zero, they often answer that $0 - a = 0$ because nothing can be drawn from an empty container. When children have to subtract a ''bigger'' digit from a ''smaller'' one (e.g., in the problem $52 - 37 = ?$), they often reverse the order of the digits (i.e., when $a < b$, $a - b$ becomes $b - a$) because in practical terms the subset has to be smaller than the whole set. Similarly, children who do not possess the part-whole schema, cannot solve some problems involving quantitative comparisons.

The second example refers to multiplication. It is well known that people who conceive multiplication as repeated addition cannot understand why the product of regular fractions is smaller than each factor. Although the repeated addition model fits common situations involving natural numbers, it cannot be applied to situations involving fractions. In particular the repeated addition model does not fit problems that require to calculate a portion of a number (e.g., how much is half of eight?) or the area of a rectangle with sides smaller than one.

Finally, the third example relates to the conceptual domain of functions and graphs (Leinhardt, Zaslavsky, & Stein, 1990; Mevarech & Kramarski, in press; Schoenfeld, Smith, & Arcavi, 1993). In mathematical terms, a linear graph ($y = mx + b$) is determined by its slope (m) and y intercept (b). Yet, many students tend to comprehend graphs pointwise (Leinhardt et al., 1990). Although this way fits many situations, it is unsuitable in situations of continuous functions. When children were asked, for instance, to construct a graph representing the situation: ''the more time Sarah studies for tests, the better her grades are,'' many marked only one, single point in the Cartesian system that corresponds to the highest possible grade (A or 100) and large amount of time (e.g., six hours). In explaining their construction, these children argue that they use only one point because ''practically, in a series of events the last event is the most important one'' (Mevarech & Kramarski, in press).

Given these studies, it is questionable whether real contexts can prove important when the task is rich with abstract mathematical concepts that are apart from concrete realities or specific objects. Smith, Langston, and Nisbett (1992) identified four interrelated factors that make a code or representation abstract. ''It can: (a) contain relatively few meaning components (this is the sense that color is more abstract than red); (b) contain variables (such as p and q in modus ponens); (c) have a high degree of generality; and (d) be relatively nonperceptual.'' Since many people reason using abstract knowledge structures inferred from ordinary life experience (Cheng & Holyoak, 1985), there is reason to suppose that in the lack of fine-grained structures that determine the properties of the mathematical objects and that contradict the misconceptions, people would use pragmatic reasoning schemes that are highly context bound. In this case, the real context may even exert debilitating effects on the solution because it may divert the subjects' attention from the crucial factors of the mathematical structure to the nonmathematical information presented by the context. At present, however, most of the studies that investigated cognitive interaction between knowledge and context focused on problems that were not particularly rich in abstract mathematical concepts. These studies typically examined the basic mathematical operations: addition, subtraction, multiplication, and division, for which the very nature of everyday contexts may indeed facilitate the activation of knowledge structures appropriate for solving such problems. Little is known, however, on the effects of real vs sparse contexts on the understanding of abstract mathematical concepts. The first suggestive evidence of the facilitative effects of a sparse context on understanding of abstract mathematical concepts comes from a study limited to the concepts of successive divisions and limit (Stern & Mevarech, 1996). This study showed that children intuitively solved problems better under a sparse context than under real contexts. Stern and Mevarech explained that the incongruity between the situational contexts—everyday situations, and the mathematical concepts—successive divisions and limit, led children to argue in a pragmatic way under the real context and in a theoretical way only under the sparse context.

The fact that the very nature of successive division and limit contradicts everyday experience raises the question of the extent to which sparse vs real contexts would exert different effects on students' understanding of other abstract mathematical concepts. The present study was designed, therefore, to investigate the cognitive interaction between knowledge and context on students' interpretation of linear graphs. Graphs were selected because: (a) although they are abstract, they are nonetheless relevant to everyday experience—people read graphs in newspapers, advertisements, TV, etc.; (b) people often encounter graphs both in and out of school situations; and (c) the findings may have both theoretical and practical implications. From a theoretical perspective, investigating the role of problem contexts on understanding linear graphs is likely to allow firm inference about the nature of situational knowledge. In addition, from a practical perspective, understanding how children function in different contexts may contribute to the design and manipulation of instructional conditions.

EXPERIMENT 1

The purpose of Experiment 1 was to examine the cognitive interaction between knowledge and contexts on students' ability to interpret linear graphs. Research on graph interpretation has shown that the task requires considerable expertise (Bell & Janvier, 1981; Clement, 1989; Leinhardt, et al., 1990; Vinner, 1983; Wainer, 1992) or training (McDermott, Rosenquist, & van Zee, 1987). Difficulties with graphing have been identified at all levels of education, from elementary school to an honor section of a calculus-based university physics course (McDermott et al., 1987). The most prevalent difficulties are: discriminating between the slope and the height of a graph, matching narrative information with relevant features of the graph, relating one type of graph to another, separating the shape of a graph from the path of the motion, considering graphs as pictures, and connecting graphs to the real world (Leinhardt et al., 1990; McDermott et al., 1987).

Difficulties in interpreting graphs have been observed at three levels of information processing: (a) an elementary level involving data extraction (e.g., What was the company income in 1985?); (b) an intermediate level involving trends seen in parts of the data (e.g., Between 1980 and 1985, what was the rate of change in the company income?); and (c) a comprehensive level involving an understanding of the deep structure of the data, usually comparing trends while seeing groupings (e.g., ‘‘Which companies show the same pattern of growth?’’) (Wainer, 1992, p. 16). It is possible that people who are at the elementary level of graph processing do not understand the deep structure of the ‘‘Cartesian connection’’ (Schoenfeld et al., 1993, p. 87). Their graph interpretation is characterized by reference to unconnected elements, a lack of understanding the meaning of a slope as representing the rate of change, and a relatively simple focus on specific points such as the *y* intercepts (Schoenfeld et al., 1993).

Although previous research has shown student difficulty in interpreting linear graphs embedded in real contexts (e.g., Leinhardt et al., 1990), no comparisons were made between students’ performance on problems representing the same graphs but embedded in different contexts. We termed such problems ‘‘isomorphic problems’’ because they ‘‘have similar solution procedures but different story contexts’’ (Weaver III & Kintsch, 1992, p. 419). Furthermore, relative to real contexts, graphs embedded in sparse contexts were administered to more expert students as part of the assessment of knowledge regarding linear functions (e.g., Vinner, 1983). Such students are usually older, more trained, more experienced, and have more competence in solving abstract mathematical problems than the novice students who were administered the graphs embedded in real contexts. The present study sought to investigate the cognitive interaction between context and understanding in a design that avoids these deficiencies—a design that examines isomorphic problems embedded in real or sparse contexts. The problems were administered to groups of students who were similar in ability, domain-specific knowledge, and mathematical experience.

Method

Participants

Participants were 124 seventh-grade Israeli students (62 boys and 62 girls) who studied in two junior high-schools located in a suburb of Tel-Aviv.

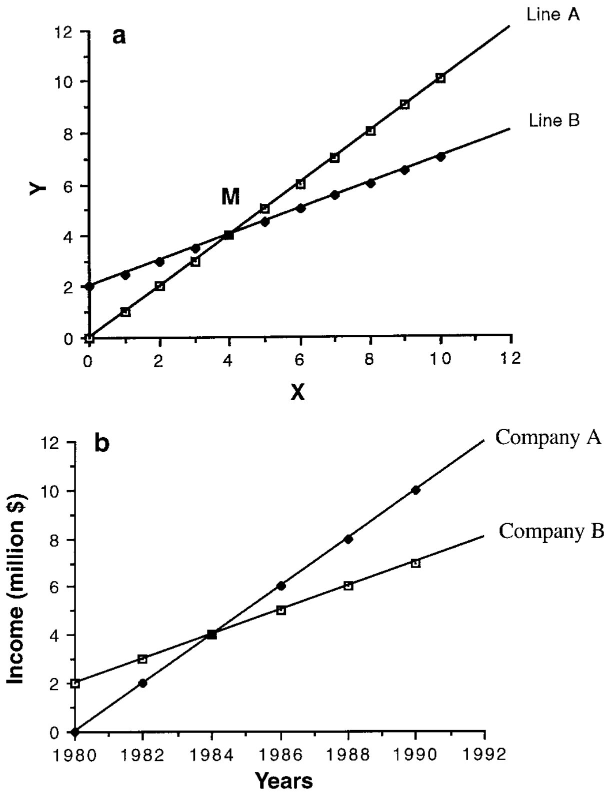

FIG. 1. (a) The graph used within the sparse context. (b) The income vs time graph. (c) The amount of water vs time graph. (d) The achievement scores vs time graph.

The subjects' mean age was 12.5 years old. Subjects studied in regular, heterogeneous classes in which mathematics was taught with no achievement trackings. The tasks were administered in mathematics classrooms before children had formally studied the topic of functions and graphs.

Tasks

Within each classroom, students were randomly assigned to one of three groups. All groups were administered an isomorphic graph presenting two intersecting lines (see Figs. 1a–1c). The tasks differed, however, in the contexts within which they were embedded. One task ($N = 47$) was embedded within a sparse context. The other two tasks were embedded within real contexts: one presenting the incomes of two companies between the years 1980 and 1990 ($N = 44$), and the other the amount of water that two pipes filled in 10 minutes ($N = 33$). The tasks were termed: sparse, income, and aquarium tasks, respectively. The tasks were adapted from the study of McDermott et al. (1987).

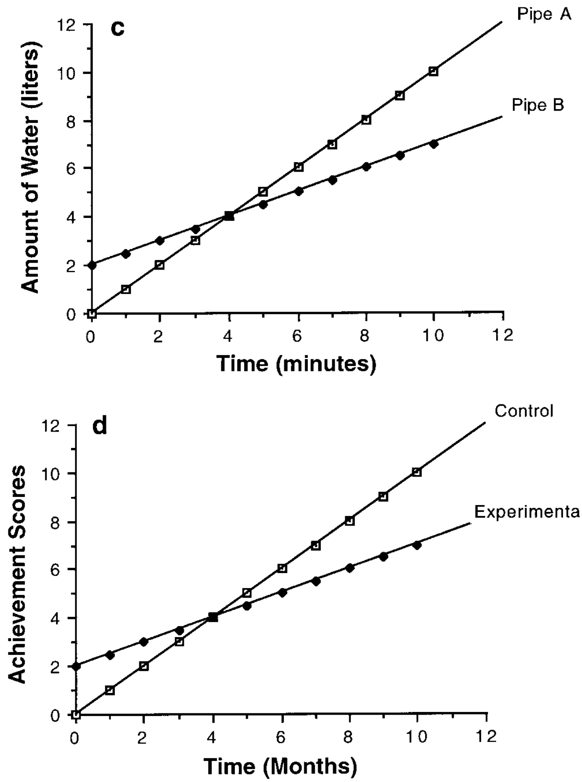

FIG. 1—*Continued*

The sparse task. Figure 1a shows a graph of two lines A and B that are intersecting at point M.

Q1: Up to point M, is the growth rate of line A greater than, less than, or equal to the growth rate of line B? Please explain your reasoning.

Q2: At point M, is there a change in the growth rate of line A? Please explain your reasoning.

Q3: From point M on, is the growth rate of line A greater than, less than, or equal to the growth rate of line B? Please explain your reasoning.

The income task. Figure 1b shows a graph representing the incomes of company A and company B between the years 1980 and 1990.

Q1: Until 1984, was the income growth rate (i.e., the growth in income per year) of company A greater than, less than, or equal to the income growth rate of company B? Please explain your reasoning.

Q2: In the year 1984, was there a change in the income growth rate (i.e., the growth in income per year) of company A? Please explain your reasoning.

Q3: From the year 1984 on, was the income growth rate (i.e., the growth in income per year) of company A greater than, less than, or equal to the income growth rate of company B? Please explain your reasoning.

The aquarium task. Figure 1c shows a graph representing the amount of water that pipe A and pipe B filled aquariums within ten minutes.

Q1: Until the fourth minute, was the filling rate (i.e., the growth in the amount of water per minute) of pipe A greater than, less than, or equal to the filling rate of pipe B? Please explain your reasoning.

Q2: In the fourth minute, was there a change in the filling rate (i.e., the growth in the amount of water per minute) of pipe A? Please explain your reasoning.

Q3: From the fourth minute on, was the filling rate (i.e., the growth in the amount of water per minute) of pipe A greater than, less than, or equal to the filling rate of pipe B? Please explain your reasoning.

Scoring. For each question, students received a score of either 1 for a correct answer (i.e., the growth rate of line A/ company A/ pipe A is greater than the growth rate of line B/ company B/ pipe B) or 0 for an incorrect answer, and a total score ranging from 0 to 3. In addition, content analysis of students' explanations was carried out to gain further understanding of students' reasoning. The content analysis was performed by two independent judges. Interjudge agreement was 90%. The few cases of disagreement were discussed until consensus was achieved.

Procedure

The tasks were administered in a test-like situation. The teachers distributed the tasks in mathematics classes so that students with adjacent seating during data collection would be in different groups. Group membership was not, therefore, completely randomized, but should reflect no systematic bias. Participants were allowed a full period (45 minutes) to solve the task. No student needed more time to complete the task. Before solving the task, students were asked to read carefully the task, answer the questions, and explain their reasoning in writing. Thus, all explanations gathered from study participants were written.

To ensure that students knew the difference between total assets and rate of growth, students were asked: (a) to identify points on a linear graph; and (b) to find the rate of growth. Results showed that all students but one could identify points on a linear graph, but 20%, equally distributed across conditions, misunderstood the concept of rate of change (e.g., the growth in income per year). Ten percent of the students who responded incorrectly confused the slope with the highest point on the graph, and 10% of the students simply described the growth rate as ''fast'' or ''slow.''

Results

Table 1 shows the mean scores, standard deviations, and the summary of the content analysis. According to Table 1, the mean score on the task embed-

TABLE 1

The Mean Scores, Standard Deviations, and a Summary of Content Analysis[a] (Presented in Terms of Percents of Students Who Based Their Explanations on Each Category) by Contexts in Experiment 1

		Real tasks	
	Sparse task	Income	Aquarium
Content analysis[a]			
Logic-mathematical explanations			
Slope	47	9	3
Computation	2	2	3
y intercept and intersection	15	2	9
Specific point explanations			
y intercept	4	7	15
Intersection point	—	18	—
Other nonmathematical explanations			
Practical explanations	—	16	21
Relative positions of lines	6	7	18
Other explanations			
Verbal repetition	4	13	3
No explanations	21	27	27
Overall scores			
M	1.67	1.115	1.152
s	1.08	.83	1.09

[a] The content analysis was carried out on the basis of students' responses, not on any prior identifications.

ded within the sparse context was higher than the mean scores on the tasks embedded within the real contexts. These differences were statistically significant [$MS_e = .98$, $F(2, 121) = 3.96$, $p < .05$]. Post hoc Duncan analysis showed that the task embedded within the sparse context was significantly easier than the other two tasks, but no significant differences were found between the two tasks embedded within the real contexts.

How did students approach the isomorphic problems embedded in the different contexts? An answer is suggested by the content analysis of students' explanations. Since many students used the same line of reasoning in solving all three questions, the analysis reported here regards students' responses to Q1. Classifying students' explanations was based on students' responses, not on any prior identification. It resulted in four general categories: Logic-mathematical explanations, point explanations, other nonmathematical explanations (practical reasoning, comparison of the relative positions of the lines), and other explanations (verbal repetition, no explanations). Student's explanation was assigned to one category. The few cases which could receive more than one code, were classified according to the last argument. For example, the following explanation ''I can see that company A started from zero. But, it's line is steeper. Therefore, the growth rate of company A is bigger than that of company B'' was coded as ''logic-mathematics'' (based on the second argument), rather than specific points (based on the first argument). Below is a detailed description of each category.

The *logic-mathematical explanations* refer to the use of concepts that directly relate to the notion of rate of change. This category includes explanations referring to the qualitative properties of the *slopes* without carrying out any computations (e.g., ''line A is steeper,'' ''line A has a larger angle with the axis than line B''—the children did not learn yet about slopes and thus did not use the correct mathematical terminology), explanations based on the *computation* of the rates of change (e.g., ''pipe A poured 2 liters per minute and pipe B poured only 1 liter per minute''), and *logically inferring* the rates of change by looking at the y intercepts and the intersection point (e.g., ''line A starts from a lower point than line B, but the lines intersect at M—therefore, the growth rate of line A has to be greater than that of line B'').

The *specific point explanations* refer to points, such as: the *y intercepts* (e.g., ''the growth rate of company A is smaller because the income of company B is 2 millions and that of company A is less than 2 millions'') or the *intersection point* (e.g., ''the two pipes intersect at point M therefore their rate of growth is equal'').

Other nonmathematical explanations refer to activating *practical reasoning* (e.g., ''the initial investment of company A was zero, therefore it's income rate could not be larger than that of company B''; ''the pipes are automatic, therefore they pour equal amount of water every minute. Changing the rate can only be done manually'') or looking at the *relative positions of the lines* (e.g., ''line A is below line B, therefore the growth rate of line A is smaller'').

Other explanations regard *verbal repetition*—repeating the answer in words without providing any justifications (e.g., ''the income rate of company A is greater than the income rate of company B'') or saying something vague, such as: ''that's the way it is''; or ''I know it'') or providing *no explanations*.

The Appendix presents examples of students' correct and incorrect explanations under the sparse and real conditions by contexts.

The content analysis shows that children activated different knowledge structures in solving the isomorphic problems embedded in the different contexts. To answer correctly Q1, solvers have to activate logic-mathematical schemes relevant to the concept of rate of change. In general, any constant increase or constant decrease situation, can be modeled by a linear equation of the form $y = mx + b$, where the slope (m) corresponds to the rate of change in the situation, and the y intercept (b) to the values of y when $x = 0$. To compare the rates of change in two situations solvers can: (a) look directly at the steepness of the slopes; (b) compute the rate of change by selecting two points [(x_1, y_1) and (x_2, y_2)] and dividing the change in the vertical distance $(y_2 - y_1)$ by the change in the horizontal distance $(x_2 - x_1)$; or (c) logically inferring that when two lines intersect, the increase per unit of the line that starts from the lower point is greater than the increase per unit of the other line. According to Table 1, in the sparse context most of the children activated the ''logic-mathematical schemes'' (i.e., referred to the steepness of the slope, logically inferred the properties of the slope, or calculated the slope). Furthermore, in this condition, reference to specific points, the provision of practical explanations, comparison of the relative positions of the lines, and verbal repetition were rarely found. In contrast, the income and aquarium tasks triggered subjects to provide a large variety of explanations, many of which were based on nonmathematical explanations (e.g., practical meanings or comparison of the relative positions of the lines), and reference to specific points (the *y* intercepts or the intersection point). These differences were statistically significant ($\chi^2(6) = 41.0$, $p < .001$).

Students under the real context condition rarely used terms such as ''steep'' or ''angle'' to describe the slopes. Instead, students often interpreted these tasks on the basis of their informal knowledge of company incomes and aquarium filling. In interpreting the income task, for example, some students gave a particular meaning to the fact that although the initial investment of company A was lower, it eventually made more money than company B— as one student wrote: ''Clearly, the first question (i.e., Until 1984, was the income growth rate of company A greater than, less than, equal to the growth rate of company B?) is a relative question. Company A did not invest any money, company B invested more money and each year made some money therefore its rate of growth is larger.'' In interpreting the aquarium task, a student wrote that ''the filling rate of the two pipes is equal because at a certain point the two pipes intersect and then the water is equally distributed between the two pipes.'' Obviously, similar interpretations were rarely made for the task embedded in the sparse context, although all tasks involved the same mathematical structure. Table 1 further shows that under all conditions, a few students attempted the solution by calculating the rate of change (dy/dx). This should not surprise us since students at this stage did not learn the properties of the linear function. In addition, Table 1 shows no significant differences between the contexts regarding the percents of students who did not explain their reasoning, or simply repeated their answers in words.

Discussion

Experiment 1 indicates that the task embedded within the sparse context was easier to solve than the tasks embedded within the real contexts. The difficulties associated with the solution of the task embedded within the real contexts were not due to students' misunderstanding of the terminology nor to students' inability to identify points on a linear graph. It seems that the different kinds of contexts led children to activate different kinds of knowledge structures: abstract logic-mathematics in the sparse context, vs nonmathematical (e.g., practical reasoning or comparison of line positions) and reference to specific points in the real contexts. The content analysis showed that the real contexts introduced extraneous, distracting information that might shift students' attention from the mathematical structure of the problem to its nonmathematical features. This has been observed, for instance, when children referred to the initial investment of the companies arguing that a company with no investment money cannot make profit more than a company with some investment money. These findings are in line with the study of Johnson-Laird (1983) who investigated syllogistic reasoning that underlies common-sense inference, and Kotovsky, Hayes, and Simon (1985) who examined a quite different task: The Tower of Hanoi. Interestingly, Kotovsky et al. found that the tasks embedded in a standardized, sparse context were much easier to solve than the isomorphic tasks embedded in enriched contexts involving acrobats and monsters. In fact, some form of the tasks embedded in the enriched contexts took 16 times as long to solve, on average, as other isomorphic versions embedded in sparse contexts. Kotovsky et al. explained that the additional information presented in the enriched contexts overloaded the problem solver processing capacity which in turn ''prevents even minimal amounts of necessary planning from occurring'' (p. 292).

Two explanations suggested for the difficulties associated with the real-context tasks should be rejected. First, one may argue that the propositional complexity of the tasks varies considerably and that variation might account for the better performance on the sparse task. Second, one may claim that children were unfamiliar with the real situations described in the income and aquarium tasks. Analysis of children's explanations indicates, however, that these two hypotheses should be rejected because many children did refer to the characteristics of the situations, and therefore activated different knowledge structures in attempting the tasks embedded in the sparse vs real contexts. Future research based on interviews may further investigate these hypotheses.

The findings of Experiment 1 raise the question of the extent to which the solution of a problem embedded in a sparse context facilitates the solution of a problem embedded in real contexts and vice versa. Ceci and Roazzi (1993) argue that ''sometimes subjects in psychology experiments will gain insight by answering one version of a problem that helps them answer its isomorphic'' (p. 83). To address this issue, we designed a second study in which students in seventh grade were administered the same tasks as those used in Experiment 1. About half of the students were administered initially the graph embedded within the sparse context and then the graph embedded within the real contexts, while for the second group, the order of task administration was reversed.

EXPERIMENT 2

The purpose of Experiment 2 was twofold: (a) to examine the extent to which the findings of Experiment 1 will be replicated for different subjects; and (b) to investigate the differential effects of the order of task administration on students' ability to solve isomorphic graphs embedded in various contexts.

Method

Subjects

Participants were 36 Israeli children (20 boys and 16 girls) who studied in a kibbutz school located in the center of the country. Students were randomly selected from two seventh-grade classes. As in Experiment 1, students studied in regular mathematics classes with no groupings. One student was dropped from the analysis because she was a newcomer and had difficulties in understanding the tasks. Thus, the responses of 35 students were analyzed. Students' mean age was 12.2. Experiments 1 and 2 were conducted in different years and different schools. Therefore, none of the students who participated in Experiment 2, participated in Experiment 1.

Measurements

Three measurements were used in this experiment: (a) Interpretation Tasks; (b) Standard Progressive Matrices (Raven, 1960); and (c) Domain-Specific Mathematics Test.

Interpretation Tasks. Three tasks were adapted from McDermott et al. (1987) for assessing students skills in interpreting graphs constructed of two intersecting lines. The tasks were identical to those used in Experiment 1.

Scoring: Students received a score of either 1 for a correct answer (the growth rate of line A/ company A/ pipe A/ is greater than the growth rate of line B/ company B/ pipe B) or 0 for an incorrect answer, and a total score ranging from 0 to 3. In addition, content analysis of students' reasoning was carried out to provide further understanding of their ability to interpret graphs embedded in the two conditions (sparse vs real contexts). The scoring procedure is identical to that used in Experiment 1.

Raven's Standard Progressive Matrices (Raven, 1960). This measurement was used to assess students' general ability. The test is constructed of five sections, each includes 12 items. Items are constructed of patterns in which there is a missing part. Subjects had to select the correct answer from six or eight alternatives. Raven's examination is widely used for assessing nonverbal intelligence of children aged 8–14 and adults. Kuder–Richardson reliability coefficient for the present participants was .83.

A Domain-Specific Mathematics Test. A 25-item test was constructed to assess students' specific knowledge regarding graphing skills. The test included the following kinds of graphs: bar graph, histogram, pictogram, pie diagrams, and linear graphs. The test questions referred to three levels of graph processing: the elementary level—identifying specific points on a graph; the intermediate level—interpreting a range on the graph; and the overall level—involving ''an understanding of the deep structure of the data, usually comparing trends and seeing groupings'' (Wainer, 1992). Each question on the test was scored either 1 (a correct answer) or 0 (an incorrect answer); thus the total score ranged from 0 to 25. Kuder–Richardson reliability coefficient for the present participants equals .73.

Procedure

Participants were randomly assigned to one of two conditions. In both conditions, the tasks were administered in mathematics classes. In condition A, the graph embedded in the sparse context was initially administered to all students (N = 18). A week later, the students were randomly assigned to one of two equal groups: one was administered the income versus time graph (N = 9), and the remainder, the amount of water versus time graph (N = 9).

In condition B (N = 17), the order of task administration was reversed. On the initial administration, the students were randomly assigned to one of two groups: one was administered the income versus time graph (N = 9), and the remainder, the water versus time graph (N = 8). A week later, all students were administered the graph embedded in the sparse context. The tasks were administered in a test-like situation, as in Experiment 1.

To control for possible group differences in general ability and domain specific knowledge, two weeks prior to the beginning of the study, students were administered the Raven Standard Progressive Matrices (Raven, 1960) and the Domain-Specific Mathematics Test.

Results and Discussion

Prior to analyzing students' performance on the three isomorphic problems, students' entry characteristics under both conditions were compared. The analyses indicated that the mean scores and standard deviations of the two groups on Raven's Standard Progressive Matrices and on the Domain-Specific Mathematics Test were quite the same. The means on Raven Test = 48.3 (s = 8.3) and 49.8 (s = 6.1) for conditions A and B, respectively; the means on the Domain-Specific Mathematics Test = 16.5 (s = 3.2) and 17.2 (s = 4.6) for conditions A and B, respectively. *T*-tests indicated no significant differences between the two conditions (t-values on Raven's and Domain-Specific Mathematics Test = .59 and .51, respectively, both p values $>$ .05). In addition, no significant differences were found in the within-group comparisons of students assigned to the different tasks (the largest F value = .76; p values $>$ .05).

To address the main research questions, a 2 (context: real/sparse) × 2 (order of task administration) analysis of variance with repeated measures was performed on the overall performance scores. The between-subjects variable had two levels corresponding to the two conditions: sparse and real contexts. The within-subjects factor represented the order of task administration. This statistical model was utilized after a t-test for independent samples indicated no significant differences on students' performances within the two real contexts. The mean scores and standard deviation (in parentheses) on the

TABLE 2
The Mean Scores, Standard Deviations, and Summary of the Content Analysis (Presented in Terms of Percents of Students Who Based Their Explanations on Each Category) by Contexts and Order of Task Administration in Experiment 2

	Condition A		Condition B	
	Sparse	Real	Sparse	Real
Content analysis				
Logic-mathematical explanations				
Slope	29	24	17	—
Computation	—	—	6	—
y interc. and intersection	24	12	11	—
Specific point explanations				
y intercepts	24	41	6	11
Intersection point	—	6	—	17
Other nonmathematical explanations				
Practical explanations	—	—	—	22
Relative positions of lines	—	6	33	11
Other				
Verbal repetition	18	—	7	26
No explanations	6	12	11	11
Overall scores				
M	2.65	1.88	2.12	1.59
s	.70	.99	.93	.87

Note. Condition A refers to initial administration of the problem embedded in the sparse context. Condition B refers to initial administration of the problem embedded in the real contexts.

aquarium task and the income tasks were 1.8 (s = .9) and 1.7 (s = 1.0), respectively; t-value for independent samples = .21, $p > .05$).

The analysis of variance with repeated measures indicated that although no significant differences were found between and within groups on entry characteristics, a significant main effect was found for the contexts [MS_e = 7.11, $F(1, 64) = 9.197$ ($p < .001$] and marginally significant for the order of task administration [MS_e = 2.88, $F(1, 64) = 3.724$, $p < .06$], but the interaction between context and order was not significant [MS_e = 2.35, $F(1, 64) < 1.00$, $p > .001$]. Table 2 shows the overall mean scores, standard deviations, and the summary of the content analysis by contexts and order of task administration.

According to Table 2, in both conditions, the task embedded in the sparse context was easier than the task embedded within the real contexts. Furthermore, students who were exposed initially to the sparse context, tended to perform better than their counterparts not only in the sparse context condition (first administration), but also in the real context condition (second administration). The differences were largely observed on Q1 and Q2. They may be due to the fact that for both questions confusion between slope and height or failure to realize that rate of growth cannot be extracted from the relative positions of the lines result in an incorrect answer. This is in contrast to Q3 for which simply looking at the relative positions of the lines may result in a correct answer. In fact, almost all students who correctly interpreted the trends seen at the interval between (0,0) and the intersection point (Q1) also correctly answered the question which referred to the intersection point itself (Q2).

These findings replicate the findings of Experiment 1. Both experiments show that compared to sparse contexts, a real context does not always facilitate students' understanding of abstract mathematical concepts. Similar findings were reported also by Stern and Mevarech (1996). In their study, fourth graders who were first administered successive division problems embedded in a sparse context performed better on this problem and were better able to solve transfer problems embedded in contextual stories than their counterparts who were first given the contextual problems for solution.

To gain a deeper understanding of the interaction between context and knowledge, students' explanations for Q1 were analyzed exactly as in Experiment 1. Similar to Experiment 1, also in Experiment 2 about half of the students (53%) in condition A (i.e., students who were initially exposed to the sparse context) interpreted the task embedded in the sparse context in a logic-mathematical way by referring to the slope (e.g., ''line A is steeper than line B''). Others (24% of the subjects) referred to the *y* intercept (''line A starts from zero and line B starts from a higher point''), and the rest (24% of the subject) either did not provide any explanation or said something vague, such as: ''That's the way it is'' or ''I know it.'' Interestingly, without using the correct mathematical terminology, students nonetheless provided correct mathematical arguments. When these same students were administered later the isomorphic tasks embedded in the real contexts, about one third of them activated the logic-mathematical schemes, and about 40% of them referred to the *y* intercepts, but none used practical reasoning in interpreting the graph.

Students in condition B, however, approached the task differently. These students who were initially exposed to the real contexts, interpreted the income and aquarium tasks by activating various knowledge structures, including: practical reasoning, reference to specific points, and visual justifications regarding the relative positions of the lines. In this case, none of the students interpreted the graphs by referring directly to the slopes. When these students were administered later the task embedded within the sparse context, about one third of these students still based their interpretation on visual justifications (e.g., ''line A is below line B'') and about one third based their explanations on logic-mathematical reasoning (one student compared the values of the slopes by saying that ''line A increased two points in one unit whereas line B increased one point''). The rest did not provide any explanations or verbally repeated their answers. Obviously, none of the students based their explanations on practical reasoning.

There are at least two possible explanations for the pattern of results observed in Experiment 2: (a) students saw the similarities between the problems presented in the first and second administrations and thus solved the tasks in a similar way (about 40% of the students used the same line of reasoning under both task administrations); (b) although the subjects in the two conditions did not differ on the two tests that were given, they might differ in some other (unknown) way. Future research based on clinical interviews may investigate these two hypotheses and provide further insight on the effects of administration order on performance.

The findings of Experiments 1 and 2 raise the question of ''what develops'': Would adults behave differently than children in solving isomorphic problems embedded in different contexts? Several previous studies showed that even undergraduate students encounter difficulties in solving isomorphic problems (e.g., Wollman, 1983; Hall et al., 1989). Yet, these studies did not compare students ability to solve isomorphic problems embedded in real vs sparse contexts. Experiments 3 and 4 were designed to address this issue.

EXPERIMENT 3

The primary purpose of Experiment 3 was to examine the effects of a sparse vs a real context on adults.

Method

Participants and Procedure

Participants were 165 Israeli undergraduate freshmen who majored in education. Approximately 85% of the participants were women ($N = 140$). Student age ranged from 19 years old to 59 years old with a mean age of 32 years old. All participants had learned the topic of linear functions and graphs in high school as part of the Israeli mathematics matriculation examination. We chose to study university students who had studied the topic of linear functions and graphs in high school and thus had learned the algebraic formalism, but who were not recent recipients of algebra-based instruction that relates to this topic. Furthermore, there is reason to suppose that compared to seventh graders, adults had more experience in interpreting graphs not only in school, but also out of school.

Participants were randomly assigned to tasks, as in Experiment 1. About half of the students ($N = 82$) were administered the graph of the two intersecting lines embedded in a sparse context, while the other half ($N = 83$) was administered the isomorphic graph embedded in a real context. The tasks were administered to all subjects in the Introduction to Statistics course, prior to introducing the notions of distribution, frequency graphs, and normal curves. Introduction to Statistics is a compulsory course for all students who major in education.

Measures

The sparse task was the same as the sparse task described above, and asked the same three questions: (a) Up to point M, is the growth rate of line A greater than, less than, equal to the growth rate of line B? (b) At point M, is there a change in the growth rate of line A? and (c) From point M, is the growth rate of line A greater than, less than, equal to the growth rate of line B? Students were asked to explain their reasoning for each response.

The real task presented the isomorphic graph, but it was embedded in a real context, closely related to the area in which students were majoring, as follows: Two groups of children learned mathematics under different treatments. The experimental group was exposed to a Computer Assisted Learning Program. The control group learned in a ''conventional classroom'' with no computers. Each month, the children's performance in mathematics was assessed. Figure 1d presents children's scores at each month by treatment. The students were asked to answer the above three questions which were modified to the contextual story by: (a) referring to the experimental and control groups rather than to lines A and B; and (b) specifying the ''fourth month'' rather than point M.

Results and Discussion

The findings of Experiment 3 show that university students were better able to interpret the graph when embedded in the sparse context than when embedded in the real context. The mean performance on the task embedded in the sparse context was 2.6 (s = .60), while the mean score on the task embedded in the real context was 2.4 (s = .79) (the maximum score = 3.0). This difference was statistically significant ($t(163) = 3.05$, $p < .001$). The differences were observed mainly on Q1 (92% vs 67% correct in the sparse vs real contexts, respectively). No significant differences were found on Q3 (87% vs 94% correct in the sparse vs real contexts, respectively).

Content analysis of students' reasoning indicated that even at the university level students activated different knowledge structures in the sparse vs the real context. Under the sparse condition, approximately 40% of the students referred to the slope and explained the relationship between the slope and the ''rate of change,'' and about 30% computed the rate of change dy/dx by selecting two points [(x_1, y_1) and (x_2, y_2)], and dividing $(y_2 - y_1)$ by $(x_2 - x_1)$. Under the real context condition, however, while only 26% of the students referred to the slope, about 60% computed the rate of change of each treatment group. A small number of students (about 8%) tended to explain the differences in the groups' rate of change in achievement scores by pointing toward the advantages or disadvantages of learning with computers. Naturally, students under the sparse context condition could not bring parallel arguments. This finding is in line with the microgenetic study of Schoenfeld et al. (1993) who showed the difficulties in interpreting linear graphs encountered by a student who had studied calculus.

Although the findings for Experiment 3 are similar to those reported for Experiments 1 and 2, one may argue that in order to examine the effects of the contexts on students' ability to interpret graphs, more than one real context has to be used. Thus, to gain further information on the interaction between knowledge and context, we designed an additional experiment in which university students were asked to interpret isomorphic graphs embedded in various types of real contexts.

EXPERIMENT 4

The main purpose of Experiment 4 is to characterize the ways adults interpret linear graphs embedded in different contexts. When compared with seventh graders, the contrast (if any) should give a rough image of how life experience and additional years of schooling affect the solution processes. Similarly to Experiment 3, also in Experiment 4 subjects had studied the topic of linear functions and graphs in high school, but were not recent recipients of algebra-based instruction that relates to this topic.

Method

Participants

Participants were 169 Israeli university students who majored in education. All students but seven were female. Student age ranged from 19 to 50 with a mean age of 33.1 ($s = 8.49$) and mode of 24 years old. The students were enrolled in either an introductory course on research methodology or a statistics course. Participation in the study was part of students' classroom activities. Experiments 3 and 4 were conducted in the same university, but in two different academic years and none of the students who participated in Experiment 3 participated also in Experiment 4.

Measurements and Procedure

The tasks used in Experiment 4 were identical to those used in Experiments 1 and 3 (see Figures 1a–1d). Thus, four tasks were used in Experiment 4: the sparse (N = 38), income ($N = 48$), aquarium ($N = 44$), and achievement ($N = 39$). Participants were randomly assigned to tasks as in Experiment 1.

Scoring: Each question was scored separately as either 1 (correct answer: the growth rate of line A/ company A/ pipe A/ group A was greater than the rate growth of line B/ company B/ pipe B/ group B) or 0 (incorrect answer). A total score, ranged from 0 to 3, was calculated by adding the scores assigned to each question. In addition, as in Experiment 1, content analysis of students' responses was carried out to gain further understanding of students' reasoning.

Results and Discussion

Table 3 presents the mean scores, standard deviations, and the summary of the content analysis by context. As can be seen from Table 3, university

TABLE 3

The Mean Scores, Standard Deviations, and Summary of the Content Analysis[a] (Presented in Terms of Percents of Students Who Based Their Explanations on Each Category) by Contexts in Experiment 4

		Real contexts		
	Sparse context	Income	Aquarium	Achievement
Content analysis[a]				
Logic-mathematical explanations				
Slope	44	15	11	16
Computation	9	7	15	8
y interc. and intersection	11	33	12	24
Specific point explanations				
y intercepts	20	4	20	5
Intersection point	—	7	20	5
Highest point	4	2	—	—
Other nonmathematical explanations				
Practical explanations	—	4	—	22
Relative positions of lines	—	—	—	—
Other explanations				
Verbal repetition	4	6	7	5
No explanations	7	22	13	16
Overall scores				
M	2.5	2.0	2.1	2.1
s	.7	.9	1.0	.9

[a] The content analysis was carried out on the basis of students' responses, not on any prior identifications.

students performed significantly better on the task embedded within the sparse context ($M = 2.5$; $s = .7$) than on the isomorphic task embedded in the real contexts ($M = 2.1$, 2.2, and 2.0; $s = .9$, 1.0, and .9 on the income, aquarium, and achievement tasks, respectively). One-way analysis of variance showed significant differences between conditions ($F(3,165) = 2.76$, $p < .05$; $MS_e = 2.04$). Contrast analysis showed significant differences between the sparse and the real context tasks, but no significant differences within the real contexts. Further analysis indicates large differences on the solution rate for Q1 (89 vs 69% correct in the sparse vs real contexts, respectively) and Q2 (76 vs 60% correct in the sparse vs real contexts, respectively), but not on Q3 (84 vs 81% in the sparse vs real contexts, respectively).

Content analysis of students' responses for Q1 revealed similar findings to those obtained in Experiment 1. According to Table 3, also at the university level, a larger percent of students used the logic-mathematical explanation in interpreting the sparse task than the real tasks. In particular, whereas in the sparse context 44% of the students directly referred to the slopes, in the real contexts only 14% of the students provided similar explanations. Evidently, under all conditions, students rarely computed the rate of change or verbally repeated their answers. The real contexts triggered students to base their explanations on logical inference, practical justifications, or reference to specific points. Interestingly, practical explanations were not provided by university students to any task but the achievement (20%) and income (4%) task. Overall the differences in the number of students who activated different kinds of knowledge structures under the different contexts were statistically significant (χ^2 (9) = 59.9, $p < .001$).

These findings led us to examine what develops between seventh grade and university with regard to linear graph interpretation. Analyzing students' explanations under each condition revealed the following major improvements. First, many university students acquired the proper mathematical terminology and used it spontaneously in interpreting linear graphs. Second, university students rarely interpreted the graph by looking at the relative positions of the lines. Third, not providing any explanations or verbally repeating the answers were more often observed for seventh graders than for university students. Fourth, whereas at the university level about half of the students (47%) used the logic-mathematical schemes (across all tasks), at seventh grade only 14% of the students responded in a similar way. Finally, in interpreting the sparse task, in both age groups many students used the logic-mathematical schemes (64% of the subjects in each group) and none provided practical justifications (see Tables 1 and 3).

GENERAL DISCUSSION

The present research investigates differences in students' ability to interpret linear graphs embedded in sparse vs real contexts. Taken together, all four experiments show that even small changes in the surface structure of mathematics problems can cause considerable differences in the way students approach the problems: interpretation of linear graphs embedded in sparse contexts involved more often the activation of abstract logic-mathematical knowledge structures, which in this case refer to the properties of the slopes, whereas interpretation of the isomorphic graphs embedded in real contexts involved the application of everyday knowledge or an elementary level of graph processing that in the lack of fine grained mathematical structures (the Cartesian connection) resulted in low mathematical performance.

Why did students perform better on problems embedded in sparse than real contexts? At this point we can only speculate. According to theories in cognitive psychology (e.g., Kintsch, 1988), the successful problem solver reads the problem, constructs a meaningful representation of the problem from its verbal statement, and uses this representation to solve the problem. The context thus serves as a filter for subsequent information processing. When the context is enriched with surface cues, the solver has to understand not only the particular mathematical characteristics of the problem, but also to find the relationship between the surface and the structural cues. The latter is not trivial and may be associated with considerable difficulties. Sometimes it may divert the student's attention from the mathematical features of the problem (see Experiment 1). In other cases, it may result in the activation of a simplistic mathematical model that involves unconnected elements, as seen when students referred to specific points without considering the pattern of changes in an interval or in the graph as a whole.

The findings of Experiment 2 replicate the findings of Experiment 1 and further indicate that being exposed initially to problems embedded in a sparse context influences the way students approach the task embedded within a real context and vice versa. It seems, as Smith et al. (1992) indicated, that ''performance on a rule-governed item is facilitated when preceded by another item based on the same rule'' (p. 7). These findings suggest that teachers should be extremely cautious in selecting problems for practicing and introducing new topics. On the one hand, introducing problems embedded in sparse contexts may be too abstract, not interesting to many students, and detached from their everyday experience. On the other hand, initial examples embedded in real contexts, may exert debilitating effects on the solution of problems embedded in a sparse context, particularly if the real contexts are unfamiliar or non-meaningful to students. Yet, the present study did not focus on transfer effects of teaching new procedures, but rather on the different effects of initial exposure to problems embedded in different kinds of contexts. Future research may address the transfer issue in a systematic way.

The findings of the present study illustrate that some of the results obtained for the school children were similar to those obtained for adults. Performance differences between older and younger students may be due to quantitative differences in domain-specific knowledge rather than to qualitative developmental differences. Both adults and children activated pragmatic reasoning schemes in the real contexts and both groups performed better in the sparse context. Adults knew the correct mathematical terminology and often used the correct formula even in the real context, but children, without having the formal knowledge, nonetheless gain a sense for the linear graphs when presented in a sparse context and often base their argumentation on correct intuition and mathematical competence. These findings incorporate with a growing number of studies showing that children gain understanding of abstract mathematical concepts much before these concepts are formally introduced in school (Resnick, 1989). For example, kindergarten children can discover strategies without being taught how to do it (Siegler & Jenkins, 1989), and elementary school children understand concepts of successive divisions and limit much before they learn calculus in high school (Stern & Mevarech, 1996). Yet, in order to generalize the findings to other populations, there is a need to continue investigating how children of different age groups and different mathematical backgrounds solve isomorphic problems embedded in different contexts. Undoubtedly, the issue of cognitive interactions between knowledge and contexts on children's ability to understand abstract mathematical concepts merits future research.

These findings have important educational implications that learning does not occur in a vacuum. It seems reasonable and promising to introduce new mathematical topics on the basis of students' sense for abstract mathematical concepts and to embed these concepts in meaningful contexts and constructive activities. Teachers should focus on working through a broad variety of real contexts in order to encourage children to do some form of reflective abstraction. The work of Kotovsky et al. (1985), as well as the series of experiments reported here, demonstrates the limitation of enriched, real contexts that in some cases merely serve to introduce extraneous distractive information and argues for the necessity of a more complete analysis of the role of sparse contexts in facilitating understanding of abstract concepts. Since research on the interaction between knowledge and context on abstract mathematical understanding is only at its beginning, little is known about possible theoretical links between different kinds of mathematical concepts and the appropriate types of contexts. In particular, empirical research based on a variety of different graphs with different numbers and line configurations may provide evidence on what people (children and adults) are doing that leads to their worse performance in real contexts. This question merits future research. Collaboration between cognitive psychologists and educational researchers holds promise for designing educational environments appropriate for teaching abstract mathematical concepts.

APPENDIX

Examples of Students' Explanations under the Sparse Context

Q1: Up to point M, is the growth rate of line A (company A, pipe A) greater than, less than, or equal to the growth rate of line B (company B, pipe B)? Please explain your reasoning.

Correct explanations for the sparse task. ''Line A is steeper''; ''Line A starts from a lower point, but intersects with line B at point M'' (or ''Line A starts with line B at point M, but goes further up than line B''); ''The angle that line A constructs with the axis is larger than that constructed by line B''; ''Line A is closer to the ''vertical axis'' than line B''; ''Because line A starts from a lower point (than line B), it has to be more ''vertical'' in order to pass line B''; ''Because line A starts from a lower point than line B, but at the end passes line B, its growth rate must be larger''; ''Although line A starts from 0 and line B starts from 2, they intersect at point M—therefore, the growth rate of line A is larger than that of line B''; ''Line A moves two units and line B moves only one unit on the *y* for every unit on the *x*.''

Correct explanations for the income task. ''The line of company A is steeper''; ''Company A started from a lower point and came to the same point as company B''; ''Company A started from zero and company B started with some money, but in 1984 their incomes were the same, therefore the growth rate of company A has to be greater''; ''To make the same (income) in 1984, the growth rate of company A had to be greater because it started from a lower point than company B (or because it started from zero).''

Correct explanations for the aquarium task. ''Coming from one point (M), pipe A reached an upper point than pipe B''; ''The line of pipe A is steeper''; ''Line A pumped 2 liters per minute and pipe B about 1 liter per minute''; ''Eventually, pipe A reaches an upper point than pipe B''; ''Pipe A started from 0 and pipe B started from an upper point, but in order to intersect, the rate of change of pipe A has to be greater than pipe B.''

Incorrect explanations for the sparse task. ''Line A is below/above line B''; ''Line A starts at a lower point than line B''; ''Line A starts at zero''; ''Lines A and B form an angle (or how its called), like a corner, B is more toward the center—it starts at 2 and arrives (intersects) at 4, therefore the growth rate of line B is bigger''; ''Line A is closer to M therefore its growth rate is bigger''; ''Both starts at *y*, but B starts from 2 not like A, and 2 is closer to 4 (the intersection point) more than 0, therefore the growth rate of line B is bigger''; ''Line A moves diagonally at 45° and line B at 65° from where they start.''

Incorrect explanations for the income task. ''Company A changed the manager therefore it could make more money''; ''Every year the rate of growth (of company A) is 2 (nothing was said about company B)''; ''Company A started from 0. It means that the company did not have any money at the beginning''; ''The income of company B is 2 millions and that of company A is less than 2 millions''; ''They intersect, therefore they made the same amount''; ''Company A sold more than company B''; ''Company A started from zero but developed more than company B''; ''Probably, company A sold more (less) than company B.''

Incorrect explanations for the aquarium task. ''The question is not about quantities but about the growth rate; therefore we have to find the rate—when we calculate it, they (the rates) are the same''; ''Pipes A and B started at different points''; ''Pipe A started from a lower point (nothing said about pipe B)''; ''Pipe A started from 0, but pipe B started from 2''; ''Eventually, pipe A came to an upper point''; ''Pipe A reached 10''; ''The line of pipe A is above the line of pipe B''; ''The two pipes are automatic—to get a change, a person has to change the rates of filling''; ''At $t = 4$ the aquarium was almost full, therefore there must be a change in the rate of filling of pipe A''; ''The question is about rates, not about amount of water. The rates are equal''; ''The amount of water in the two pipes is equal but the time is different''; ''The pipes are apart from each other.''

Q2: At point M, is there a change in the growth rate of line A? Please explain your reasoning.

Correct explanations for the sparse task. ''Line A does not ''break'' at point M, therefore its growth rate continues to be larger''; ''Both lines continue with the same angle all the time—because line A starts at a lower point and intersects with line B, the growth rate of line A is larger than that of line B all the time''; ''Point M does not make any difference—the lines continue as before.''

Correct explanations for the income task. ''Company A reaches a higher point and thus its growth rate does not change in 1984.'' Other explanations as above.

Correct explanations for the aquarium task. ''It is all the time the same, it does not change.'' Other explanations as above.

Incorrect explanations for the sparse task. ''Both lines continue to move, but now each is different''; ''The angles are different at point M''; ''Line A becomes steeper''; ''Line A starts to be above line B''; ''Line A breaks at point M''; ''At M, line A does not raise so much.'' Incorrect explanations for the real tasks: ''Company A (pipe A) makes the same as company B (pipe B) therefore, their rate of growth is the same.'' Other explanations as above.

Q3: From point M on (1984, the fourth minute), is the growth rate of line A (company A, pipe A) greater than, less than, equal to the growth rate of line B (company B, pipe B)? Please explain your reasoning. Explanations as above.

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