Formal Methods and Functional Programming

Part II

Peter Müller

Chair of Programming Methodology
ETH Zurich

Organization

- Most aspects do not change (web page, homework)
- Different tutors
    - Tuesday 16-18, IFW C42 (Alex Summers, English)
    - Tuesday 16-18, IFW B42 (Stefan Heule, German)
    - Wednesday 15-17, IFW A32.1 (Alex Summers, English)
    - Wednesday 15-17, IFW A34 (Pietro Ferrara, English)
    - Wednesday 15-17, IFW C33 (Yannis Kassios, English)

Please attend the same session as in the first half of the course. If you want to switch (for language reasons), please email Alex Summers (alexander.summers@inf.ethz.ch)

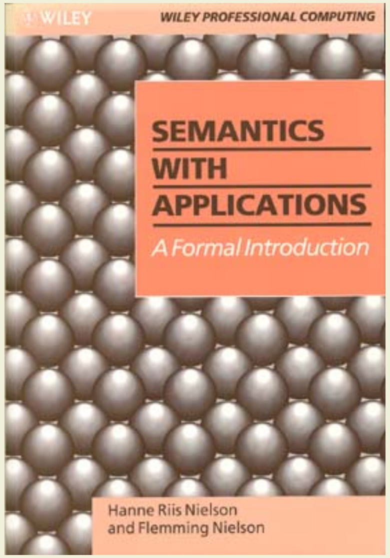
Homework and Exam

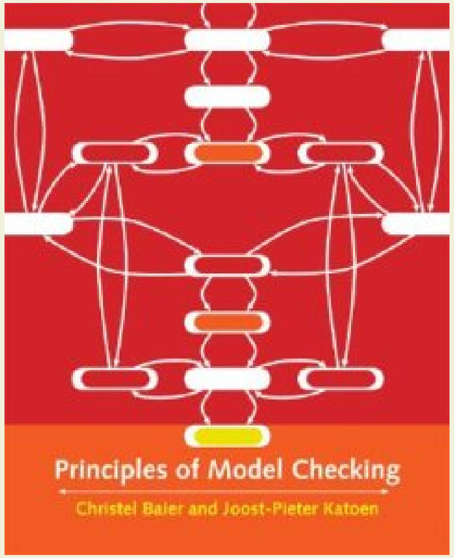
- Homework can be submitted in one of two ways:
  - By email to the appropriate tutor
  - By hand in the appropriate box outside room RZ F1

  Solutions must be received by 9:15 on the Monday after the exercise is published, in order to receive feedback.

- The final exam will take place on Tuesday, June 07, 2011, 10:00–12:00
  - See web page for details

Recommended Books



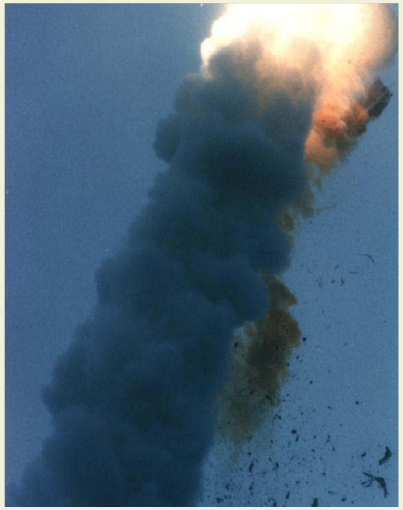

Principles of Model Checking
Christel Baier and Joost-Pieter Katoen

- Hanne Riis Nielson and Flemming Nielson: *Semantics with Applications: A Formal Introduction*
    - Available from `http://www.daimi.au.dk/~bra8130/Wiley_book/wiley.pdf`
- Christel Baier and Joost-Pieter Katoen: *Principles of Model Checking*

Software Errors Cost Large Amounts of Money

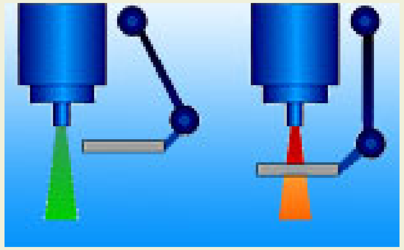
- Software errors cost US economy \$59.5 billion annually (estimate by Department of Commerce's National Institute of Standards and Technology, 2002)

- Software bugs in baggage handling system of the airport of Denver lead to damage of around \$1 million per day (for almost a year)



- Explosion of Ariane 5 destroyed satellites worth \$500 million
- In comparison: famous hardware bugs:
    - Pentium bug cost Intel \$500 million
    - Xbox bug cost Microsoft \$1 billion

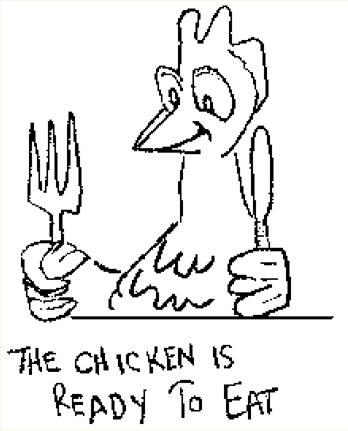
Software Errors May Cost Lives



- Software error in Therac-25 medical linear accelerator lead to overdose, which killed six people
- Rounding error caused Patriot Missile system to ignore an incoming Scud missile; 28 soldiers died
- Many other safety critical systems
  - Controllers in airplanes, cars, trains, etc.
  - Air traffic control systems
  - Nuclear reactor control systems

Traditional Software Engineering

- Describes expected behavior using natural language or semi-formal notations
  - Ambiguities
  - Contradictions
  - Incompletenesses



- Relies on testing to ensure quality
  - *Testing can show the presence of errors, but not their absence.* [E. Dijkstra]
  - Exhaustive testing possible only for trivial programs
  - Some errors are hard to find (data races, deadlocks)
  - Achieving good test coverage is difficult (rare cases)

Alternative: Formal Methods

> Formal methods are mathematical approaches to software and system development which support the rigorous specification, design, and verification of computer systems. [FME]

- Programs, programming languages, designs, etc. are mathematical objects and can be treated by mathematical methods
- Examples from Part I of the course:
  - Proving program properties

$$\forall x, y, z.(x ++ y) ++ z = x ++ (y ++ z)$$

  - Formalizing language semantics

$$(\lambda x.M)N \hookrightarrow M[x \leftarrow N]$$

  - Proving language properties

$$\text{If } e \hookrightarrow e' \text{ and } \vdash e :: \tau \text{ then } \vdash e' :: \tau$$

Example 1: Sorting Function

```
void sort(int[] input)
```

- Informal specification:
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  - `sort({2,3,1})` → `{1,2,3}` ✓

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  - `sort({2,3,1})` → `{1,2,3}` ✓
  - `sort({2,2,1})` → `{1,2,1}` ✗
  - `sort(null)` → ↯ ✗

Example 1: Sorting Function—Formal Treatment

- Specification
  - Pre and postcondition in predicate logic (contract)
  - If a is a non-null array of integers and
    in the state before a call `sort(a)`, the elements of a are $e_0 \dots e_n$,
    then the call terminates and immediately after the call,
    the elements of a, $e'_0 \dots e'_n$, are a permutation of $e_0 \dots e_n$
    and $\forall i, j \in [0, n]. i < j \Rightarrow e'_i \leq e'_j$.

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- Verification
  - Prove that `sort` satisfies its specification using a formal semantics of the programming language
- Observations
  - Specification permits duplicate elements in array: Test `sort({2,2,1})` reveals error in implementation
  - Specification excludes `null` from the valid arguments to `sort`: Test `sort(null)` is an invalid test case
  - Correctness proof covers all valid inputs, not just selected test cases

Example 2: Zune Bug

- Zune 30 did not work on Dec. 31, 2008
- Official fix: drain battery and recharge after midday on Jan. 01, 2009

```
//-----------------------------
// Split total days since
// Jan. 01, ORIGINYEAR
// into year, month and day
//-----------------------------
BOOL ConvertDays(UINT32 days, ...) {
  int year = ORIGINYEAR; /* =1980 */

  while (days > 365) {
    if (IsLeapYear(year)) {
      if (days > 366) {
        days -= 366; year += 1;
      }
    } else {
      days -= 365; year += 1;
    }
  }
  ... }
```

Example 2: Zune Bug—Formal Treatment

- Prove termination formally
- Repetition: Sufficient condition for termination of recursive functions: Arguments are smaller along a well-founded order
- Similar technique for loops

- Zune example:
    - Termination measure: variable `days`
    - Well-founded order: < with lower bound 365 (loop condition)
    - Error: measure not decreased if `IsLeapYear(year)` and `days==366`

```
while (days > 365) {
  if (IsLeapYear(year)) {
    if (days > 366) {
      days -= 366; year += 1;
    }
  } else {
    days -= 365; year += 1;
  }
}
```

Example 3: Deadlock

- Threads are synchronized via locks
- Interleaved execution of `a.transfer(b,n)` and `b.transfer(a,m)` might deadlock
- Multi-threaded programs are extremely hard to test

```
class Account {
  int balance;

  void transfer(Account to, int amount) {
    acquire this;
    acquire to;
    this.balance -= amount;
    to.balance += amount;
    release this;
    release to;
  }
}
```

Example 3: Deadlock—Formal Treatment (1)

- Prevent deadlocks by acquiring locks in ascending order
- Prove absence of deadlocks by:
  - Defining an order on locks
  - Proving for each `acquire o` that o is above all other locks held by the current thread

```
class Account {
  int balance;
  int number; // unique account number

  void transfer(Account to, int amount) {
    if (this.number < to.number) {
      acquire this;
      acquire to;
    } else {
      acquire to;
      acquire this;
    }
    this.balance -= amount;
    to.balance += amount;
    release this;
    release to;
  }
}
```

Example 3: Deadlock—Formal Treatment (2)

- Alternative approach: state space exploration
    - Enumerate all possible states of a system
    - Check properties on the states and their transitions
    - Absence of deadlock: check for each state that there is a way to reach the terminal state

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- Main problem: size of state space
- Explore abstractions of real program (here, `balance` does not matter)
- Explore state space for limited executions
  - Small number of threads (here, two are sufficient)
  - Small number of objects (here, two are sufficient)
  - Small number of context switches (here, one is sufficient)

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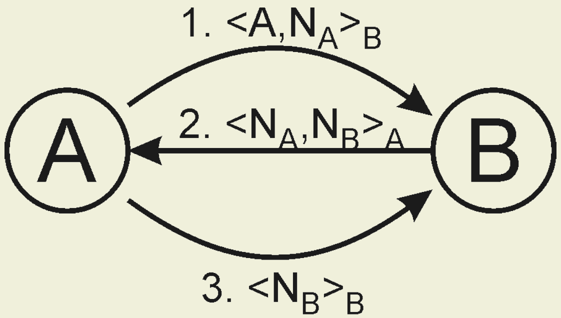
- State space exploration typically gives no correctness guarantee
  - Similar to testing
  - Very effective in practice

Example 4: Needham-Schroeder Protocol

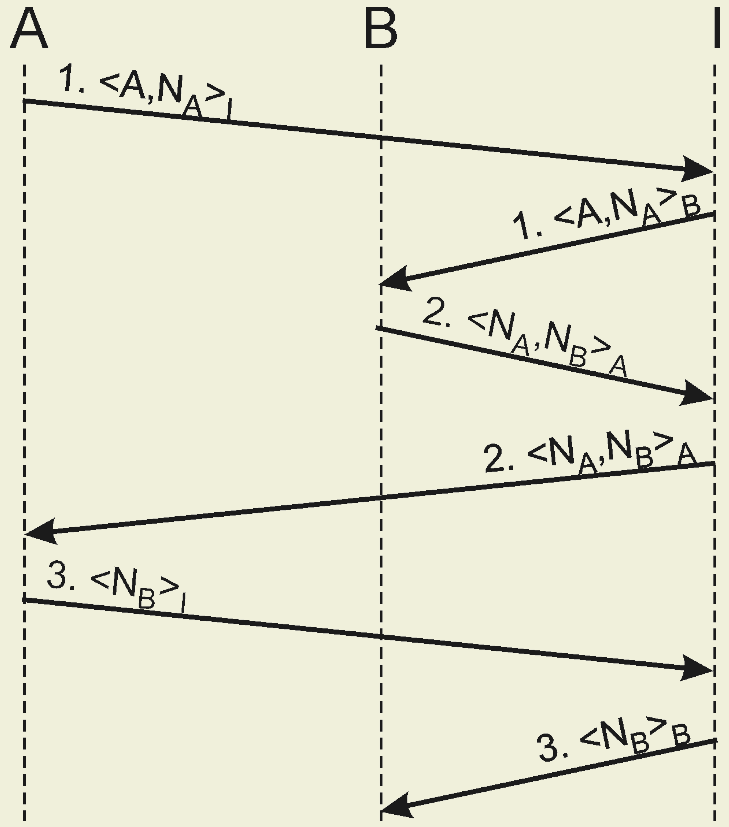
- Establish a common secret over an insecure channel
    1. Alice sends random number N_A to Bob, encrypted with Bob's public key: $\langle A, N_A \rangle_B$
    2. Bob sends random number N_B to Alive, encrypted with Alice's public key: $\langle N_A, N_B \rangle_A$
    3. Alice responds with $\langle N_B \rangle_B$



- Intruders may:
    - Intercept, store, and replay messages
    - Initiate or participate in runs of the protocol
    - Decrypt messages only if encrypted with intruder's public key
- Error: intruder can pretend to be another party

Example 4: Needham-Schroeder Protocol—Formal Treatment

- State space exploration: enumerate protocol runs
  - Develop formal model of intruder as non-deterministic program
  - Simplifications: two agents, one intruder with limited memory
  - Check whether there is a protocol run such that agent believes to talk to other agent, but in fact talks to intruder
- Error was found this way 17 years after protocol was published



Observations: Formal Specification

- Use mathematical notations to describe:
  - Assumptions about the environment (e.g., intruder model)
  - Requirements for the system (desired properties, e.g., deadlock freedom)
  - System design to accomplish these requirements (e.g., program code)

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- Requirements
  - Safety properties: Something bad will never happen
    - Functional behavior of `sort`
    - Absence of certain faults (e.g., buffer overflow)
  - Liveness properties: Something good will happen eventually
    - Termination of `ConvertDays`
    - Deadlock freedom of `transfer`
  - Non-functional requirements
    - Resource consumption, e.g., memory usage
    - Runtime, e.g., realtime guarantees

Observations: Formal Verification

- Use formal logic to:
    - Validate specifications by checking consistency
      Example: termination measure uses well-founded order
    - Prove that design satisfies requirements under given assumptions
      Example: code does not deadlock
    - Prove that a more detailed design implements a more abstract one (refinement)
      Example: protocol implementation refines protocol specification

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- Proof method
  - Deductive: proof system
    Example: prove termination in a program logic
  - Algorithmic: state space exploration (model checking)
    Example: enumerate and check protocol runs

Formal Methods: Ingredients

- Specification language
  - Modeling or programming language with precise semantics
  - Desired properties expressed as logical formulas or abstract system
  - Precise meaning of "system satisfies property"

- Proof method
  - Method to establish or refute that a system satisfies a property

- Tool support
  - For specification and verification
  - Proofs are often simple, but tedious (in contrast to mathematics)
  - Tools needed to check details
  - Main examples: theorem provers and model checkers

Benefits of Formal Methods

- Strong guarantees
  - Detect faults with greater certainty than testing
  - Guarantee absence of specific faults
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    - Programming languages (e.g., type safety proof)
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- Didactic value: Studying formal methods:
  - Leads to deep understanding of semantics of programs, design specifications, etc.
  - Increases awareness of subte issues of programs, languages, etc.
  - Makes you a better engineer!

Success Stories

- Paris driverless metro (Meteor)
  - Safety-critical system
  - Pilot software developed through stepwise refinement in B
  - Most detailed design translated automatically to 30,000 lines of Ada
  - 28,000 proofs

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  - Windows device drivers running in kernel mode should respect API
  - Third-party device drivers not respecting APIs responsible for 90% of Windows crashes
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  - Windows device drivers running in kernel mode should respect API
  - Third-party device drivers not respecting APIs responsible for 90% of Windows crashes
  - SLAM inspects C code using a combination of model checking and theorem proving
- Airbus 380 flight controller
  - Safety-critical system
  - Static analysis of 500,000 lines of C code
  - Proved absence of runtime errors (e.g., buffer overflows)

Limitations

- Incorrect specifications
    - Formal methods per se do not guarantee correctness
    - Verifying the wrong specification is useless
    - It is difficult to get specifications right

- Technical limitations
    - Almost all interesting properties are undecidable
    - Many tools quickly reach limits (scope, computing resources)

- Most formal methods require specialist users
    - Strong background in mathematics
    - Training in formal modelling

- Application of formal methods is expensive
    - But testing is expensive, too

Formal Methods and Testing

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  - Validate specifications
  - Test properties not formally proven (e.g., performance)
  - Detect errors in environment (e.g., compiler)

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- Testing still necessary
    - Validate specifications
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    - Detect errors in environment (e.g., compiler)

- Formal methods aid testing
    - Derive test cases, test data, and test oracles from specifications
    - Increase test coverage
    - Replace (infinitely) many tests

Course Outline—Part II

- Focus: formal methods for (stateful) software
    - Imperative programs and languages
    - Software designs

1. Formal semantics of programming languages
    - Operational semantics
    - Hoare logic
2. State space explotarion
    - Temporal logic
    - Model checking

1. Introduction to Language Semantics

C: Expression Evaluation

```
int print(char* text) {
  printf("%s\n", text);
  return 5;
}
```

```
print("One")+print("Two");
```

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```c
print("One")+print("Two");
```

```
One
Two
```

```
Two
One
```

In C and C++, evaluation order of expressions is undefined

- Precedence and associativity define rules for structuring expressions
- But do not define operand evaluation order

Haskell and SML: Evaluation

Haskell

```
const :: Int -> Int
const x = 1
```

```
const ( 2 ’div’ 0 )
```

SML

```
fun const (x: int):int = 1;
```

```
const ( 2 div 0 );
```

Haskell and SML: Evaluation

Haskell

```
const :: Int -> Int
const x = 1
```

```
const ( 2 ’div’ 0 )
```

```
1
```

SML

```
fun const (x: int):int = 1;
```

```
const ( 2 div 0 );
```

```
uncaught exception divide by zero
```

- Haskell uses lazy evaluation:
  Arguments are evaluated when they are needed
- SML uses eager evaluation:
  Arguments are evaluated when function is applied

Java: Dynamic Method Binding

```
class C1 {
  int x;
  public  void inc1( )
    { this.inc2( ); }
  private void inc2( )
    { x++; }
}
```

```
class CS1 extends C1 {
  public void inc2( )
    { inc1( ); }
}
```

```
CS1 cs = new CS1(5);
cs.inc2( );
System.out.println(cs.x);
```

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```
CS1 cs = new CS1(5);
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System.out.println(cs.x);
```

```
class C2 {
  int x;
  public    void inc1( )
    { this.inc2( ); }
  protected void inc2( )
    { x++; }
}
```

```
class CS2 extends C2 {
  public void inc2( )
    { inc1( ); }
}
```

```
CS2 cs = new CS2(5);
cs.inc2( );
System.out.println(cs.x);
```

Java: Class Initialization

```
class C {
  public static int x;
}
```

```
class D {
  public static char y;
  ...

}
```

```
C.x = 0;
D.y = ’?’;
System.out.println(C.x);
```

Java: Class Initialization

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class C {
  public static int x;
}
```

```
class D {
  public static char y;

  static { C.x = C.x + 1; }
}
```

```
C.x = 0;
D.y = '?';
System.out.println(C.x);
```

```
1
```

Why Formal Semantics?

- Programming language design
    - Formal verification of language properties
    - Reveal ambiguities
    - Support for standardization

- Implementation of programming languages
    - Compilers
    - Interpreters
    - Portability

- Reasoning about programs
    - Formal verification of program properties
    - Extended static checking

Language Properties

- Type safety:
  In each execution state, a variable of type T holds a value of T or a subtype of T
- Very important question for language designers
- Example:
  If String is a subtype of Object, should `String[]` be a subtype of `Object[]`?

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- Example:
  If String is a subtype of Object, should `String[]` be a subtype of `Object[]`?

```
void m(Object[] oa) {
  oa[0]=new Integer(5);
}
```

```
String[] sa=new String[10];
m(sa);
String s = sa[0];
```

Compiler Optimization

- Common subexpression elimination

```
d = a * Math.sqrt(c);
e = b * Math.sqrt(c);
```

```
double tmp=Math.sqrt(c);
d = a * tmp;
e = b * tmp;
```

Compiler Optimization

- Common subexpression elimination

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d = a * Math.sqrt(c);
e = b * Math.sqrt(c);
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```
double tmp=Math.sqrt(c);
d = a * tmp;
e = b * tmp;
```

- Optimization works only for side-effect free expressions

```
d = a * c++;
e = b * c++;
```

```
double tmp = c++;
d = a * tmp;
e = b * tmp;
```

Formal Verification

```
/* returns the
   factorial of n */
int fac(int n) {
  if (n>1)
    return n*fac(n-1);
  else
    return 1;
}
```

Formal Verification

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```
fac(17);
```

```
-288522240
```

Formal Verification

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/* returns the
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}
```

`fac(17);`

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- Verification could run by induction
- Induction hypothesis: $n \geq 0 \Rightarrow fac(n) = n!$
- Induction base is trivial
- Induction step requires to prove $n \times (n-1)! = n!$ which is not the case in computer arithmetic

1. Introduction to Language Semantics

Language Definition

Dynamic Semantics

- State of a program execution
- Transformation of states

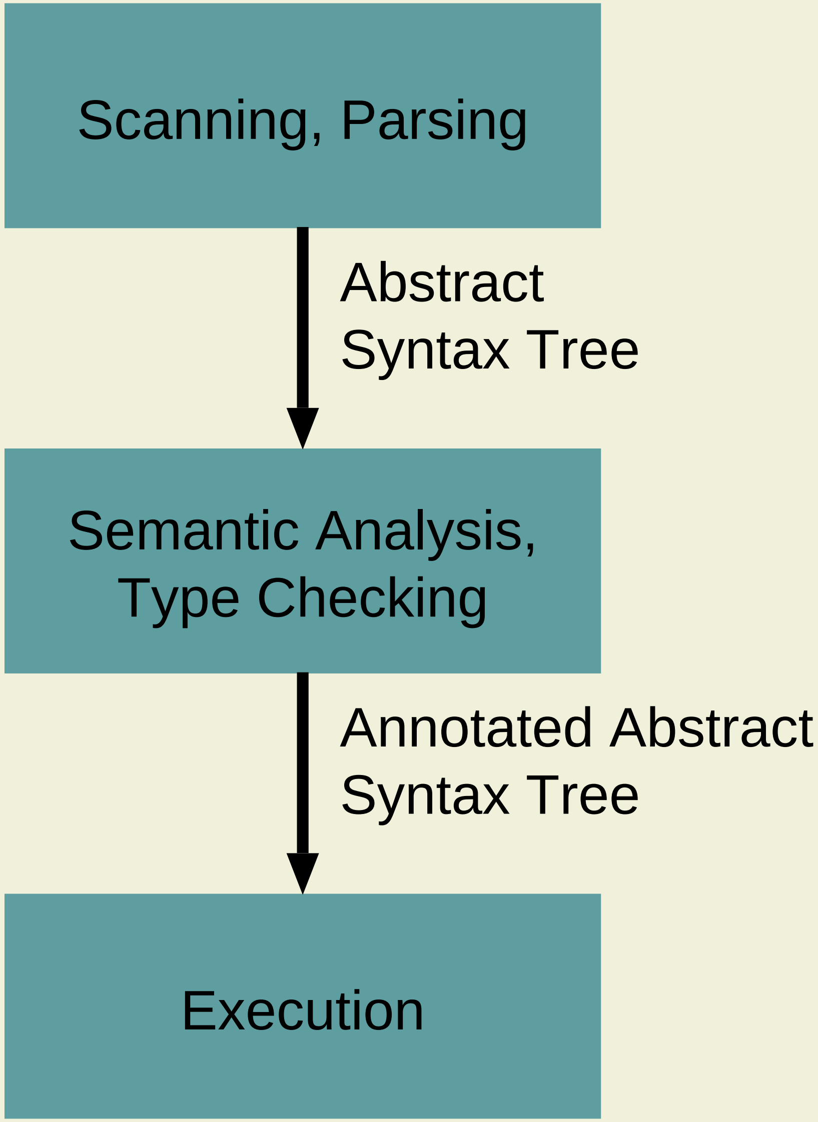
Static Semantics

- Type rules
- Name resolution

Syntax

- Syntax rules, defined by grammar

Compilation and Execution


Scanning, Parsing

Abstract
Syntax Tree

Semantic Analysis,
Type Checking

Annotated Abstract
Syntax Tree

Execution

Three Kinds of Semantics

- Operational semantics
  - Describes execution on an abstract machine
  - Describes how the effect is achieved
- Denotational semantics
  - Programs are regarded as functions in a mathematical domain
  - Describes only the effect, not how it is obtained
- Axiomatic semantics
  - Specific properties of the effect of executing a program are expressed
  - Some aspects of the computation may be ignored

Operational Semantics

```
y := 1;
while not(x=1) do ( y := x*y; x := x-1 )
```

- "First we assign 1 to `y`, then we test whether `x` is 1 or not. If it is then we stop and otherwise we update `y` to be the product of `x` and the previous value of `y` and then we decrement `x` by 1. Now we test whether the new value of `x` is 1 or not…"
- Two kinds of operational semantics
    - Natural Semantics
    - Structural Operational Semantics

Denotational Semantics

```
y := 1;
while not(x=1) do ( y := x*y; x := x-1 )
```

- "For input values of `x` greater than 0, the program computes a partial function from states to states: the final state will be equal to the initial state except that the value of `x` will be 1 and the value of `y` will be equal to the factorial of the value of `x` in the initial state"
- Two kinds of denotational semantics
  - Direct Style Semantics
  - Continuation Style Semantics

Axiomatic Semantics

```
y := 1;
while not(x=1) do ( y := x*y; x := x-1 )
```

- "If `x`= *n* holds before the program is executed then `y`= *n*! will hold when the execution terminates (if it terminates)"
- Two kinds of axiomatic semantics
  - Partial correctness
  - Total correctness

Abstraction

Concrete language implementation

Operational semantics

Denotational semantics

Axiomatic semantics

Abstract description

Selection Criteria

Constructs of the language	Application of the semantics
Imperative	Understanding the language
Functional	Program verification
Concurrent	Prototyping
Object-oriented	Compiler construction
Non-deterministic	Program analysis
Etc.	Etc.

Focus of this Course

- We discuss the major approaches to semantics for a small imperative language IMP
    - Similarities and differences
    - Important theoretical results
- Operational Semantics
    - Natural and structural operational semantics of IMP
    - Equivalence
- Axiomatic Semantics
    - Axiomatic semantics of IMP
    - Soundness and completeness

1. Introduction to Language Semantics

The Language IMP

- Expressions
  - Boolean and arithmetic expressions
  - No side-effects in expressions
- Variables
  - All variables range over integers
  - All variables are initialized
- IMP does not include
  - Heap allocation and pointers
  - Variable declarations
  - Procedures
  - Concurrency

Syntax of IMP: Characters and Tokens

Characters

Letter = 'A' | . . . | 'Z' | 'a' | . . . | 'z'
Digit = '0' | '1' | '2' | '3' | '4' | '5' | '6' | '7' | '8' | '9'

Tokens

Ident = Letter { Letter | Digit }
Numeral = Digit | Numeral Digit
Var = Ident

Syntax of IMP: Expressions

Arithmetic expressions

Aexp = '(' Aexp Op Aexp ')' | Var | Numeral
Op = '+' | '-' | '*'

Boolean expressions

Bexp = '(' Bexp 'or' Bexp ')' | '(' Bexp 'and' Bexp ')'
| 'not' Bexp | Aexp RelOp Aexp
RelOp = '=' | '#' | '<' | '<=' | '>' | '>='

We omit parentheses if permitted by the usual operator precedence

Syntax of IMP: Statemens

```
Stm   = 'skip'
      | Var ':=' Aexp
      | Stm ';' Stm
      | 'if' Bexp 'then' Stm 'else' Stm 'end'
      | 'while' Bexp 'do' Stm 'end'
```

Notation

Meta-variables (written in *italic* font)

x, y, z	for variables (Var)
e, e', e_1, e_2	for arithmetic expressions (Aexp)
b, b_1, b_2	for boolean expressions (Bexp)
s, s', s_1, s_2	for statements (Stm)

Keywords are written in `typewriter` font

Syntax of IMP: Example

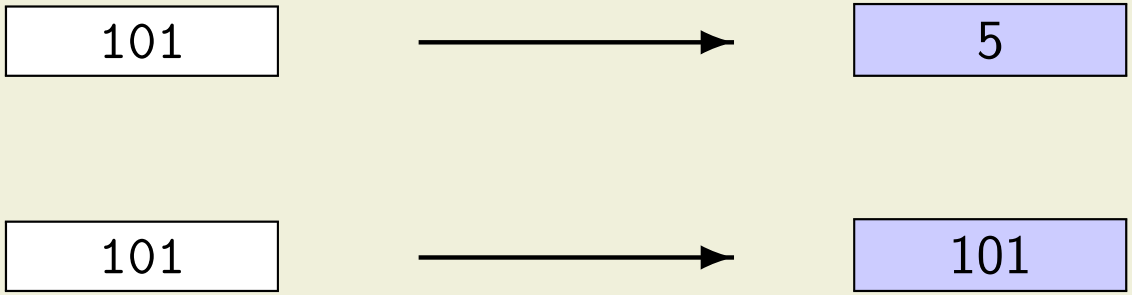
```
res := 1;
while n > 1 do
  res := res * n;
  n := n - 1
end
```

1. Introduction to Language Semantics

Semantic Categories

Syntactic category: Numeral

Semantic category: Val = $\mathbb{Z}$



`101` → 5

`101` → 101

- Semantic functions map elements of syntactic categories to elements of semantic categories
- To define the semantics of IMP, we need semantic functions for
  - Numerals (syntactic category Numeral)
  - Arithmetic expressions (syntactic category Aexp)
  - Boolean expressions (syntactic category Bexp)
  - Statements (syntactic category Stm)

Semantics of Numerals

The semantic function

$$\boxed{\mathcal{N} : \text{Numeral} \rightarrow \text{Val}}$$

maps a numeral n to an integer value $\mathcal{N}[\![n]\!]$

$$\begin{array}{llll}
\mathcal{N}[\![\mathtt{0}]\!] & = 0 & \mathcal{N}[\![\mathtt{1}]\!] & = 1 \\
\ldots & & & \\
\mathcal{N}[\![\mathtt{8}]\!] & = 8 & \mathcal{N}[\![\mathtt{9}]\!] & = 9 \\
 & & & \\
\mathcal{N}[\![n\ \mathtt{0}]\!] & = \mathcal{N}[\![n]\!] \times 10 + 0 & \mathcal{N}[\![n\ \mathtt{1}]\!] & = \mathcal{N}[\![n]\!] \times 10 + 1 \\
\ldots & & & \\
\mathcal{N}[\![n\ \mathtt{8}]\!] & = \mathcal{N}[\![n]\!] \times 10 + 8 & \mathcal{N}[\![n\ \mathtt{9}]\!] & = \mathcal{N}[\![n]\!] \times 10 + 9
\end{array}$$

States

$$\boxed{\texttt{x+1}} \longrightarrow \boxed{??}$$

- The meaning of an expression depends on the values bound to the variables that occur in it
- A state associates a value to each variable

$$\boxed{\mathsf{State} : \mathsf{Var} \rightarrow \mathsf{Val}}$$

- We represent a state σ as a finite function

$$\boxed{\sigma = \{x_1 \mapsto v_1, x_2 \mapsto v_2, \ldots, x_n \mapsto v_n\}}$$

where $x_1, x_2, \ldots, x_n$ are different elements of Var and $v_1, v_2, \ldots, v_n$ are elements of Val.

Semantics of Arithmetic Expressions

The semantic function

$$\mathcal{A} : \mathsf{Aexp} \rightarrow \mathsf{State} \rightarrow \mathsf{Val}$$

maps an arithmetic expression e and a state σ to a value $\mathcal{A}[\![e]\!]\sigma$

$$\begin{array}{ll} \mathcal{A}[\![x]\!]\sigma & = \sigma(x) \\ \mathcal{A}[\![n]\!]\sigma & = \mathcal{N}[\![n]\!] \\ \mathcal{A}[\![e_1\ op\ e_2]\!]\sigma & = \mathcal{A}[\![e_1]\!]\sigma\ \overline{op}\ \mathcal{A}[\![e_2]\!]\sigma \quad \text{for } op \in \mathsf{Op} \end{array}$$

$\overline{op}$ is the operation $\mathsf{Val} \times \mathsf{Val} \rightarrow \mathsf{Val}$ corresponding to op

Semantics of Boolean Expressions

The semantic function

$$\mathcal{B} : \mathrm{Bexp} \to \mathrm{State} \to \mathrm{Bool}$$

maps a boolean expression b and a state σ to a truth value $\mathcal{B}[\![\, b \,]\!]\sigma$

$$\mathcal{B}[\![\, e_1 \ op \ e_2 \,]\!]\sigma = \begin{cases} tt & \text{if } \mathcal{A}[\![\, e_1 \,]\!]\sigma \ \overline{op}\ \mathcal{A}[\![\, e_2 \,]\!]\sigma \\ ff & \text{otherwise} \end{cases}$$

$op \in \mathrm{RelOp}$ and $\overline{op}$ is the relation $\mathrm{Val} \times \mathrm{Val}$ corresponding to op

Boolean Expressions (cont'd)

$$\mathcal{B}[\![b_1 \ \texttt{or} \ b_2]\!]\sigma = \begin{cases} \mathit{tt} & \text{if } \mathcal{B}[\![b_1]\!]\sigma = \mathit{tt} \text{ or } \mathcal{B}[\![b_2]\!]\sigma = \mathit{tt} \\ \mathit{ff} & \text{otherwise} \end{cases}$$

$$\mathcal{B}[\![b_1 \ \texttt{and} \ b_2]\!]\sigma = \begin{cases} \mathit{tt} & \text{if } \mathcal{B}[\![b_1]\!]\sigma = \mathit{tt} \text{ and } \mathcal{B}[\![b_2]\!]\sigma = \mathit{tt} \\ \mathit{ff} & \text{otherwise} \end{cases}$$

$$\mathcal{B}[\![\texttt{not} \ b]\!]\sigma = \begin{cases} \mathit{tt} & \text{if } \mathcal{B}[\![b]\!]\sigma = \mathit{ff} \\ \mathit{ff} & \text{otherwise} \end{cases}$$

1. Introduction to Language Semantics

Well-Founded Relations

- Definition

  A binary relation $\prec$ on a set A is *well-founded* iff there are no infinite descending chains

  $$\ldots \prec a_i \prec \ldots \prec a_1 \prec a_0$$

- Examples

  $<$ is a well-founded relation on $\mathbb{N}$
  $<$ is not well-founded on $\mathbb{Z}$
  $\leq$ is not well-founded on $\mathbb{N}$

- Well-founded relations are also called Noetherian orders

Well-Founded Induction

- Principle of well-founded induction

> Let $\prec$ be a well-founded relation on a set A. Let P be a property. Then the following equivalence holds.
>
> $$(\forall a \in A : ((\forall b \in A : b \prec a \Rightarrow P(b)) \Rightarrow P(a)))$$
> $$\Leftrightarrow \forall a \in A : P(a)$$

- Mathematical induction is a special case of well-founded induction
  - Set: $\mathbb{N}$
  - Relation: $n \prec m$ iff $m = n + 1$
- Structural induction is a special case of well-founded induction
  - Set: the set of terms of an algebraic data type
  - Relation: $n \prec m$ iff n is a (proper) sub-term of m

Structural Induction: Example

- Syntax defined as algebraic data type

$$\text{Aexp} \quad = \text{'('}\ \text{Aexp}\ \text{Op}\ \text{Aexp}\ \text{')'}\ \mid\ \text{Var}\ \mid\ \text{Numeral}$$

  - Constructors are left implicit

- Structural induction for arithmetic expressions

$$\begin{array}{l}(\forall n \in \text{Numeral} : P(n)) \land \\ (\forall x \in \text{Var} : P(x)) \land \\ (\forall e_1, e_2 \in \text{Aexp} : P(e_1) \land P(e_2) \Rightarrow P(e_1\ op\ e_2)) \\ \Leftrightarrow \\ \forall e \in \text{Aexp} : P(e)\end{array}$$

Inductive Definitions

The semantics is given by recursive definitions of functions

- The values for the basis elements are defined directly
- The values for composite elements are defined inductively in terms of the immediate constituents

$$\begin{array}{ll}
\mathcal{A}[\![x]\!]\sigma & = \sigma(x) \\
\mathcal{A}[\![n]\!]\sigma & = \mathcal{N}[\![n]\!] \\
\mathcal{A}[\![e_1\ op\ e_2]\!]\sigma & = \mathcal{A}[\![e_1]\!]\sigma\ \overline{op}\ \mathcal{A}[\![e_2]\!]\sigma \quad \text{for } op \in \text{Op}
\end{array}$$

- Since the decomposition of the elements is unique this means that the semantics is well-defined
- Inductive definitions enable proofs by structural induction

Using Structural Induction

- Lemma: The equations for $\mathcal{N}$ define a total function $\mathcal{N} : \text{Numeral} \rightarrow \text{Val}$
- To prove the lemma, we show that for each $n \in \text{Numeral}$ there is exactly one $v \in \text{Val}$ such that $\mathcal{N}[\![n]\!] = v$
- $\mathcal{N}$ is defined inductively:

$$\begin{array}{llll}
\mathcal{N}[\![0]\!] & = 0 & \mathcal{N}[\![1]\!] & = 1 \\
\ldots & & & \\
\mathcal{N}[\![8]\!] & = 8 & \mathcal{N}[\![9]\!] & = 9 \\
\mathcal{N}[\![n\ 0]\!] & = \mathcal{N}[\![n]\!] \times 10 + 0 & \mathcal{N}[\![n\ 1]\!] & = \mathcal{N}[\![n]\!] \times 10 + 1 \\
\ldots & & & \\
\mathcal{N}[\![n\ 8]\!] & = \mathcal{N}[\![n]\!] \times 10 + 8 & \mathcal{N}[\![n\ 9]\!] & = \mathcal{N}[\![n]\!] \times 10 + 9
\end{array}$$

- Therefore, we can prove the lemma by structural induction on n

Proof: $\mathcal{N}$ is a Total Function

1. Induction base: all terms of height 1 (digits)

   There are ten cases for the induction base for the ten different digits.

   $\mathcal{N}$ maps each digit to exactly one value in Val.

2. Induction step: $n \equiv n_1\ d$ for some digit d

   There are ten cases for the induction step. Here, we show the case for a digit d

   - n_1 is a sub-term of n
   - By applying the induction hypothesis to n_1, we get:
     (a) there is exactly one $v_1 \in \mathrm{Val}$ such that $\mathcal{N}[\![n_1]\!] = v_1$
   - The equations for $\mathcal{N}$ define $\mathcal{N}[\![n_1\ d]\!] = \mathcal{N}[\![n_1]\!] \times 10 + v_d$ where v_d is the integer value for digit d. (b) There is exactly one such v_d
   - By using (a) and (b), we get that there is exactly one pair of values $v_1, v_d \in \mathrm{Val}$ such that $\mathcal{N}[\![n_1\ d]\!] = v_1 \times 10 + v_d$
   - Since multiplication and addition are total functions, we can conclude that there is exactly one value for $v_1 \times 10 + v_d$ and, thus, for $\mathcal{N}[\![n_1\ d]\!]$

$\mathcal{A}$ is a Total Function

- Lemma: The equations for $\mathcal{A}$ define a total function $\mathcal{A} : \text{Aexp} \to \text{State} \to \text{Val}$

- To prove the lemma, we show that for each $e \in \text{Aexp}$ and $\sigma \in \text{State}$ there is exactly one $v \in \text{Val}$ such that $\mathcal{A}[\![e]\!]\sigma = v$

- $\mathcal{N}$ is defined inductively:

$$\begin{array}{ll} \mathcal{A}[\![x]\!]\sigma & = \sigma(x) \\ \mathcal{A}[\![n]\!]\sigma & = \mathcal{N}[\![n]\!] \\ \mathcal{A}[\![e_1 \ op \ e_2]\!]\sigma & = \mathcal{A}[\![e_1]\!]\sigma \ \overline{op} \ \mathcal{A}[\![e_2]\!]\sigma \quad \text{for } op \in \text{Op} \end{array}$$

- Therefore, we can prove the lemma by structural induction on e

Proof: $\mathcal{A}$ is a Total Function

1. Induction base: all terms of height 1
   - Case 1: $e \equiv n$
     The equations define $\mathcal{A}[\![n]\!]\sigma = \mathcal{N}[\![n]\!]$. According to the previous lemma, $\mathcal{N}$ is a total function and, thus, $\mathcal{N}[\![n]\!]$ yields exactly one value in Val
   - Case 2: $e \equiv x$
     The equations define $\mathcal{A}[\![x]\!]\sigma = \sigma(x)$. σ is a total function, $\sigma(x) \in$ Val
2. Induction step: $e \equiv e_1 \; op \; e_2$
   - e_1 and e_2 are sub-terms of e
   - By applying the induction hypothesis to e_1 and e_2, we get:
     (a) there is exactly one $v_1 \in$ Val such that $\mathcal{A}[\![e_1]\!]\sigma = v_1$ and
     (b) there is exactly one $v_2 \in$ Val such that $\mathcal{A}[\![e_2]\!]\sigma = v_2$ and
   - The equations for $\mathcal{A}$ define $\mathcal{A}[\![e_1 \; op \; e_2]\!]\sigma = \mathcal{A}[\![e_1]\!]\sigma \; \overline{op} \; \mathcal{A}[\![e_2]\!]\sigma$
   - By using (a) and (b), we get that there is exactly one pair of values $v_1, v_2 \in$ Val such that $\mathcal{A}[\![e_1 \; op \; e_2]\!]\sigma = v_1 \; \overline{op} \; v_2$
   - Since $\overline{op}$ is a total function (addition, subtraction, or multiplication), we can conclude that there is exactly one value for $v_1 \; \overline{op} \; v_2$ and, thus, for $\mathcal{A}[\![e_1 \; op \; e_2]\!]\sigma$

Inductive Definitions: Example

New arithmetic expression: $-e$

- Inductive definition of $\mathcal{A}[\![-e]\!]\sigma$

$$\boxed{\mathcal{A}[\![-e]\!]\sigma = 0 - \mathcal{A}[\![e]\!]\sigma}$$

- e is a subterm of $-e$
- For the induction step we may assume the induction hypothesis for e

- Non-inductive definition of $\mathcal{A}[\![-e]\!]\sigma$

$$\boxed{\mathcal{A}[\![-e]\!]\sigma = \mathcal{A}[\![0-e]\!]\sigma}$$

- $0-e$ is not a subterm of $-e$
- For the induction step we may not assume the induction hypothesis for $0-e$

Free Variables

Arithmetic expressions

$$\begin{aligned}
FV(e_1 \ op \ e_2) &= FV(e_1) \cup FV(e_2) \\
FV(n) &= \emptyset \\
FV(x) &= \{x\}
\end{aligned}$$

Boolean expressions

$$\begin{aligned}
FV(b_1 \ op \ b_2) &= FV(b_1) \cup FV(b_2), op \in \mathrm{RelOp} \\
FV(\texttt{not} \ b) &= FV(b) \\
FV(b_1 \ \texttt{or} \ b_2) &= FV(b_1) \cup FV(b_2) \\
FV(b_1 \ \texttt{and} \ b_2) &= FV(b_1) \cup FV(b_2)
\end{aligned}$$

Statements

$$\begin{aligned}
FV(\texttt{skip}) &= \emptyset \\
FV(x \texttt{:=} e) &= \{x\} \cup FV(e) \\
FV(s_1 \texttt{;} s_2) &= FV(s_1) \cup FV(s_2) \\
FV(\texttt{if} \ b \ \texttt{then} \ s_1 \ \texttt{else} \ s_2 \ \texttt{end}) &= FV(b) \cup FV(s_1) \cup FV(s_2) \\
FV(\texttt{while} \ b \ \texttt{do} \ s \ \texttt{end}) &= FV(b) \cup FV(s)
\end{aligned}$$

Syntactic Abbreviations

`if` b `then` s `end`	`if` b `then` s `else skip end`
`repeat` s `until` b	s`;` `while not` b `do` s `end`

`for` x `:=` e_1 `to` e_2 `do` s `end`

$x \notin FV(e_2), y \notin FV(s)$

```
$x$ := $e_1$;
var $y$ := $e_2$ in
   while $x$ <= $y$ do
      $s$; $x$ := $x$ + 1
   end
end
```

`true`	`1=1`