ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

D. Basin and P. Müller

Formal Methods and Functional Programming

Solution 1: First Steps in Haskell and Natural Deduction

Assignment 1:

The input file `sheet1_johndo.hs` already contains the complete solution.

Assignment 2, 3:

See the file `solution1.lhs` for the solution.

Assignment 4:

(a) (i) $(A \lor B) \to (C \to ((A \land C) \lor (B \land C)))$

(ii) $(A \to (B \to C)) \to ((A \land B) \to C)$

(b) (i) Let $\Gamma \equiv A \lor B, C$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A \lor B}\,\mathit{ax} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash A}\,\mathit{ax} \quad \dfrac{}{\Gamma, A \vdash C}\,\mathit{ax}}{\Gamma, A \vdash A \land C}\,\land I}{\Gamma, A \vdash (A \land C) \lor (B \land C)}\,\lor IL \quad \dfrac{\dfrac{\dfrac{}{\Gamma, B \vdash B}\,\mathit{ax} \quad \dfrac{}{\Gamma, B \vdash C}\,\mathit{ax}}{\Gamma, B \vdash B \land C}\,\land I}{\Gamma, B \vdash (A \land C) \lor (B \land C)}\,\lor IR}{A \lor B, C \vdash (A \land C) \lor (B \land C)}\,\lor E}{A \lor B \vdash C \to ((A \land C) \lor (B \land C))}\,\to I}{\vdash (A \lor B) \to (C \to ((A \land C) \lor (B \land C)))}\,\to I$$

(ii) Let $\Gamma \equiv A \to (B \to C), A \land B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A \to (B \to C)}\,\mathit{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash A \land B}\,\mathit{ax}}{\Gamma \vdash A}\,\land EL}{\Gamma \vdash B \to C}\,\to E \quad \dfrac{\dfrac{}{\Gamma \vdash A \land B}\,\mathit{ax}}{\Gamma \vdash B}\,\land ER}{A \to (B \to C), A \land B \vdash C}\,\to E}{A \to (B \to C) \vdash (A \land B) \to C}\,\to I}{\vdash (A \to (B \to C)) \to ((A \land B) \to C)}\,\to I$$

(c) There are several sets of suitable rules for reasoning about $\leftrightarrow$. All of them are derived by first expanding the definition of $\leftrightarrow$ (once in the premise for constructing the introduction rule and once in the conclusion for construction the elimination rule) and then applying zero or more of the existing ND rules to simplify the rules obtained after expansion.

$$\frac{\Gamma \vdash (A \to B) \wedge (B \to A)}{\Gamma \vdash A \leftrightarrow B} \leftrightarrow \textit{intro} \qquad\qquad \frac{\Gamma \vdash A \leftrightarrow B}{\Gamma \vdash (A \to B) \wedge (B \to A)} \leftrightarrow \textit{elim}$$

We use the following (common) introduction and elimination rules for $\leftrightarrow$.

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash B \to A}{\Gamma \vdash A \leftrightarrow B} \leftrightarrow \textit{I} \qquad \frac{\Gamma \vdash A \leftrightarrow B}{\Gamma \vdash A \to B} \leftrightarrow \textit{EL} \qquad \frac{\Gamma \vdash A \leftrightarrow B}{\Gamma \vdash B \to A} \leftrightarrow \textit{ER}$$

They give rise to the following proof of the commutativity of $\leftrightarrow$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \leftrightarrow B \vdash A \leftrightarrow B}\,\textit{ax}}{A \leftrightarrow B \vdash B \to A} \leftrightarrow \textit{ER} \qquad \dfrac{\dfrac{}{A \leftrightarrow B \vdash A \leftrightarrow B}\,\textit{ax}}{A \leftrightarrow B \vdash A \to B} \leftrightarrow \textit{EL}}{A \leftrightarrow B \vdash B \leftrightarrow A} \leftrightarrow \textit{I}}{\vdash (A \leftrightarrow B) \to (B \leftrightarrow A)} \to \textit{I}$$

Assignment 5 (headache of the week):

Let $\Gamma \equiv (A \to B) \to A$. Then a proof of $((A \to B) \to A) \to A$ is

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A \vee \neg A}\,\textit{TND} \qquad \dfrac{}{\Gamma, A \vdash A}\,\textit{ax} \qquad [1]}{(A \to B) \to A \vdash A} \vee\textit{E}}{\vdash ((A \to B) \to A) \to A} \to \textit{I}$$

where $[1]$ corresponds to the following derivation.

$$\dfrac{\dfrac{}{\Gamma, \neg A \vdash (A \to B) \to A}\,\textit{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg A, A \vdash \neg A}\,\textit{ax} \qquad \dfrac{}{\Gamma, \neg A, A \vdash A}\,\textit{ax}}{\Gamma, \neg A, A \vdash B} \neg\textit{E}}{\Gamma, \neg A \vdash A \to B} \to \textit{I}}{\Gamma, \neg A \vdash A} \to \textit{E}$$