ETH
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Formal Methods and Functional Programming

Solution 2: Natural Deduction and Correctness

Assignment 1:

(a) Let $\Gamma \equiv (\exists x.\, P(x)) \to Q, P(x)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\exists x.\, P(x)) \to Q}\,\textsf{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash P(x)}\,\textsf{ax}}{\Gamma \vdash \exists x.\, P(x)}\,\exists\textsf{I}}{(\exists x.\, P(x)) \to Q, P(x) \vdash Q}\,\to\textsf{E}}{(\exists x.\, P(x)) \to Q \vdash P(x) \to Q}\,\to\textsf{I}}{(\exists x.\, P(x)) \to Q \vdash \forall x.\, P(x) \to Q}\,\forall\textsf{I}}{\vdash ((\exists x.\, P(x)) \to Q) \to \forall x.\, P(x) \to Q}\,\to\textsf{I}$$

(b) Let $\Gamma \equiv \exists x.\, P(x) \wedge Q(x), P(x) \wedge Q(x)$.

$$\dfrac{\dfrac{\dfrac{}{\exists x.\, P(x) \wedge Q(x) \vdash \exists x.\, P(x) \wedge Q(x)}\,\textsf{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P(x) \wedge Q(x)}\,\textsf{ax}}{\Gamma \vdash P(x)}\,\wedge\textsf{EL}}{\Gamma \vdash \exists x.\, P(x)}\,\exists\textsf{I} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash P(x) \wedge Q(x)}\,\textsf{ax}}{\Gamma \vdash Q(x)}\,\wedge\textsf{ER}}{\Gamma \vdash \exists y.\, Q(y)}\,\exists\textsf{I}}{\Gamma \vdash (\exists x.\, P(x)) \wedge (\exists y.\, Q(y))}\,\wedge\textsf{I}}{\exists x.\, P(x) \wedge Q(x) \vdash (\exists x.\, P(x)) \wedge (\exists y.\, Q(y))}\,\exists\textsf{E}}{\vdash (\exists x.\, P(x) \wedge Q(x)) \to (\exists x.\, P(x)) \wedge (\exists y.\, Q(y))}\,\to\textsf{I}$$

(c) Let $\Gamma \equiv \forall x.\, P(x) \to Q(x), \neg Q(x), P(x)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg Q(x)}\,\textsf{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x.\, P(x) \to Q(x)}\,\textsf{ax}}{\Gamma \vdash P(x) \to Q(x)}\,\forall\textsf{E} \qquad \dfrac{}{\Gamma \vdash P(x)}\,\textsf{ax}}{\Gamma \vdash Q(x)}\,\to\textsf{E}}{\Gamma \vdash \bot}\,\neg\textsf{E}}{\forall x.\, P(x) \to Q(x), \neg Q(x) \vdash \neg P(x)}\,\neg\textsf{I}}{\dfrac{\forall x.\, P(x) \to Q(x) \vdash \neg Q(x) \to \neg P(x)}{\forall x.\, P(x) \to Q(x) \vdash \forall x.\, \neg Q(x) \to \neg P(x)}\,\forall\textsf{I}}\,\to\textsf{I}$$

$$\dfrac{\forall x.\, P(x) \to Q(x) \vdash \forall x.\, \neg Q(x) \to \neg P(x)}{\vdash (\forall x.\, P(x) \to Q(x)) \to \forall x.\, \neg Q(x) \to \neg P(x)}\,\to\textsf{I}$$

Assignment 2:

(a) $\phi_1 = (\exists x.\, P(x)) \land (\exists y.\, Q(y)) \to (\exists x.\, P(x) \land Q(x))$

Here is a structure $\mathcal{A}_1 = (\mathcal{U}_{\mathcal{A}_1}, I_{\mathcal{A}_1})$ that satisfies the formula, i.e., $\mathcal{A}_1 \models \phi_1$.

$$\begin{array}{rcl} \mathcal{U}_{\mathcal{A}_1} & = & \{a,b,c\} \\ I_{\mathcal{A}_1}(P) & = & \{a,b\} \\ I_{\mathcal{A}_1}(Q) & = & \{a,c\} \end{array}$$

Here is a structure $\mathcal{A}'_1 = (\mathcal{U}_{\mathcal{A}'_1}, I_{\mathcal{A}'_1})$ that does not satisfy the formula, i.e., $\mathcal{A}'_1 \not\models \phi_1$.

$$\begin{array}{rcl} \mathcal{U}_{\mathcal{A}'_1} & = & \{a,b,c\} \\ I_{\mathcal{A}'_1}(P) & = & \{a,b\} \\ I_{\mathcal{A}'_1}(Q) & = & \{c\} \end{array}$$

(b) $\phi_2 = \forall x.\, (\exists y.\, R(x,y) \land Q(y)) \to (\forall y.\, R(x,y) \to Q(y))$

Here is a structure $\mathcal{A}_2 = (\mathcal{U}_{\mathcal{A}_2}, I_{\mathcal{A}_2})$ that satisfies the formula, i.e., $\mathcal{A}_2 \models \phi_2$.

$$\begin{array}{rcl} \mathcal{U}_{\mathcal{A}_2} & = & \{a,b,c\} \\ I_{\mathcal{A}_2}(R) & = & \{(a,b),(a,c)\} \\ I_{\mathcal{A}_2}(Q) & = & \{b,c\} \end{array}$$

Here is a structure $\mathcal{A}'_2 = (\mathcal{U}_{\mathcal{A}'_2}, I_{\mathcal{A}'_2})$ that does not satisfy the formula, i.e., $\mathcal{A}'_2 \not\models \phi_2$.

$$\begin{array}{rcl} \mathcal{U}_{\mathcal{A}'_2} & = & \{a,b,c\} \\ I_{\mathcal{A}'_2}(R) & = & \{(a,b),(a,c)\} \\ I_{\mathcal{A}'_2}(Q) & = & \{a,c\} \end{array}$$

(c) $\phi_3 = \forall x, y.\, R(x,y) \to R(y,x) \to x = y$

Here is a structure $\mathcal{A}_3 = (\mathcal{U}_{\mathcal{A}_3}, I_{\mathcal{A}_3})$ that satisfies the formula, i.e., $\mathcal{A}_3 \models \phi_3$.

$$\begin{array}{rcl} \mathcal{U}_{\mathcal{A}_3} & = & \{a,b,c\} \\ I_{\mathcal{A}_3}(R) & = & \{(a,a)\} \end{array}$$

Here is a structure $\mathcal{A}'_3 = (\mathcal{U}_{\mathcal{A}'_3}, I_{\mathcal{A}'_3})$ that does not satisfy the formula, i.e., $\mathcal{A}'_3 \not\models \phi_3$.

$$\begin{array}{rcl} \mathcal{U}_{\mathcal{A}'_3} & = & \{a,b,c\} \\ I_{\mathcal{A}'_3}(R) & = & \{(a,b),(b,a)\} \end{array}$$

Assignments 3 and 4:

See Haskell script `solution2.lhs`.