Correctness*

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Correctness

- Correctness is important! **What does your program do?**
- What does this mean? What **properties** should hold?
  - **Termination**: Important for many, but not all, programs. Examples?
  - **Functional behavior**: function should return "correct" value. Can be defined by another (mathematically defined) function or an input-output relation.
- Correctness is rarely obvious.

  It must be **proven**!

Correctness (example)

- The factorial function can be written as

```
fac 0 = 1
fac n = n * fac (n-1)
```

  or as

```
fac2 (0,a) = a
fac2 (n,a) = fac2 (n-1,n*a)
```

- Do these functions terminate?

  $\texttt{fac}\ 3 \leadsto 3 * \texttt{fac}\ 2 \leadsto 3 * 2 * \texttt{fac}\ 1 \leadsto 3 * 2 * 1 * 1 \leadsto 6$

  $\texttt{fac2}\ (3, 1) \leadsto \texttt{fac2}\ (2, 3 * 1) \leadsto \texttt{fac2}\ (1, 2 * 3 * 1) \leadsto$
  $\texttt{fac2}\ (0, 1 * 2 * 3 * 1) \leadsto 1 * 2 * 3 * 1 \leadsto 6$

- Are $\texttt{fac}\ n$ and $\texttt{fac2}\ n\ 1$ equivalent?

Termination

- If f is defined in terms of functions $g_1, \ldots, g_k$ ($g_i \neq f$), and each g_i terminates, then so does f.

```
g x = x * x + 15
f x = (g x + x + 2) / 13
```

- Problem is recursion, when some $g_i = f$, e.g., `fac` n.

- Sufficient condition for termination: Arguments are smaller along a well-founded order on function's domain.
  - An order $>$ on a set S is **well-founded** iff there is no infinite decreasing chain $x_1 > x_2 > x_3 > \ldots$, for $x_i \in S$.
  - Example: $>_{\mathcal{N}}$. Counter examples: $>_{\mathcal{Z}}$, $>_{\mathcal{R}}$.
    Here we write $>_S$ to denote the set S, where $>_S \subseteq S \times S$

An aside on constructing well-founded relations

- We can construct new well-founded relations from existing ones
- Let R_1 and R_2 be binary relations on a set S. The composition of R_1 and R_2 is defined as

$$R_2 \circ R_1 \equiv \{(a, c) \in S \times S \mid \exists b \in S.\, (a, b) \in R_1 \wedge (b, c) \in R_2\}$$

- Let $R \subseteq S \times S$. Define:

$$\begin{aligned} R^1 &\equiv R \\ R^{n+1} &\equiv R \circ R^n \quad \text{for } n \geq 1 \\ R^+ &\equiv \cup_{n \geq 1} R^n \end{aligned}$$

Well-founded relations (cont.)

- **Lemma:** Let $R \subseteq S \times S$. Let s_0, $s_i \in S$ and $i \geq 1$. Then $s_0 R^i s_i$ iff there are $s_1, \ldots, s_{i-1} \in S$ such that $s_0\ R\ s_1\ R \ldots R\ s_{i-1}\ R\ s_i$.
  - Proof: induction on i (easy exercise).
- **Theorem:** If $>$ is a well-founded order on the set S, then $>^+$ is also well-founded on S.

  Proof. Assume that

  $$a_1 >^+ a_2 >^+ a_3 >^+ \ldots$$

  is an infinite descending chain. Then, there exist $i_j \geq 1$ such that $a_j >^{i_j} a_{j+1}$, for all $j \geq 1$. By the above lemma, the sequence

  $$a_1 >^{i_1} a_2 >^{i_2} a_3 >^{i_3} \ldots$$

  contradicts the well-foundedness of $>$.

Termination: examples

```
fac 0 = 1
fac n = n * fac (n-1)
```

```
fac2 (0,a) = a
fac2 (n,a) = fac2 (n-1,n*a)
```

- Factorial function `fac`
  - `fac` n has only `fac` $(n-1)$ as a recursive call and $n > n-1$.
  - Here $>$ is the standard ordering over the natural numbers.
- Function `fac2`
  - `fac2` (n, a) has only `fac2` $(n-1, n*a)$ as a recursive call.
  - The first argument is always smaller under $>$.
- Do `fac` and `fac2` terminate when $n < 0$?

Termination (cont.)

Do the following functions terminate?

```
f(0) = 0
f(1) = 1
f(n) = f(n-1) + f(n-2)

g(0) = 1
g(1) = 1
g(n) = g(n+1) + g(n-2)

h(0,y) = y
h(x,y) = h(x-1,y+1)
```

What is the well-founded order?

What do they actually compute?

Correctness — behavior

- Do `fac` and `fac2` compute the same function in following sense?

$$\forall n \in Nat.\ \texttt{fac}\ n = \texttt{fac2}\ (n, 1)$$

- Testing can be used to find errors

$$\begin{array}{lcll} \texttt{fac}\ 0 & = & 1 & = \quad \texttt{fac2}\ (0, 1) \\ \texttt{fac}\ 1 & = & 1 & = \quad \texttt{fac2}\ (1, 1) \\ \texttt{fac}\ 2 & = & 2 & = \quad \texttt{fac2}\ (2, 1) \end{array}$$

- Correctness requires a formal proof!

Correctness — equational reasoning

- Proofs based on a simple idea: **functions are equations**

```
swap :: (Int,Int) -> (Int,Int)
swap (a,b) = (b,a)
```

Meaning: For all possible values of a and b:

$$swap\ (a, b) = (b, a)$$

More formally:

$$\forall a \in Int.\ \forall b \in Int.\ swap(a, b) = (b, a)$$

- Some properties can be shown through equational reasoning.

  E.g., $swap\ (swap\ (a, b)) = (a, b)$. Proof?

- More generally: proofs in first-order logic with equality.

Correctness — reasoning by cases

```
maxi :: Int -> Int -> Int
maxi n m
   | n >= m    = n
   | otherwise = m
```

Can we prove that $maxi\ n\ m \geq n$?

We have $n \geq m \vee \neg(n \geq m)$

Now show $maxi\ n\ m \geq n$ for both cases

Case 1: $n \geq m$,
then $maxi\ n\ m = n$ and $n \geq n$

Case 2: $\neg(n \geq m)$,
then $maxi\ n\ m = m$. But $m > n$, so $maxi\ n\ m \geq n$

Correctness (cont.)

The previous proof used two inference rules:

Excluded Middle: For all propositions P:

$$P \vee \neg P$$

Case split: Given $Q \vee R$,

to prove any P, we must prove:

1. P follows from Q and
2. P follows from R

Compare with ND-rules ($\vee$-elimination)

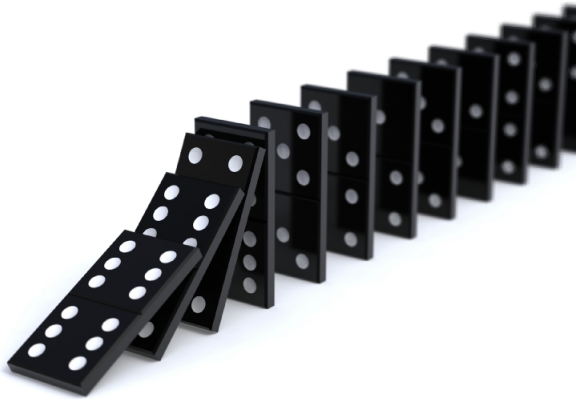
Proof by induction

- Suppose you wish to prove a formula $P(n)$, for all $n \in Nat$

$$\forall n \in Nat.\, 0 + 1 + 2 + ... + n = n \cdot (n + 1)/2$$

- Proof by cases not possible here: $P(0), P(1), P(2), P(3), \ldots$

- Use **domino principle** formulated by **Induction proof rule**



- To prove $\forall n \in Nat.\, P(n)$

  Base case: Prove $P(0)$

  Step case: For an arbitrary n, prove $P(n + 1)$ under the assumption $P(n)$.

- Exercise: formulate rule ND style and prove example in FOL.

Induction as the dual of recursion

```
fac 0     = 1
fac n     = n * fac (n-1)
```

1. **Base case**, like in induction

2. **Step case**

   - States how to compute $fac\ n$, given value for $fac\ (n-1)$.
   - Corresponds to proving $P(n)$ follows from $P(n-1)$, for $n > 0$
     or equivalently: proving $P(n+1)$ follows from $P(n)$, for $n \geq 0$

3. Correspondence can be precisely formulated

Induction is not more complicated or more circular than (terminating) recursive definitions!

Example: powers

```
power2 :: Int -> Int        -- computes 2^r as 2 * 2 * ... * 1
power2 0 = 1
power2 r = 2 * power2 (r-1)

sumPowers :: Int -> Int     -- computes 1 + 2 + 4 + ... + 2^r
sumPowers 0 = 1
sumPowers r = sumPowers (r-1) + power2 r
```

Examples:

$$\begin{aligned} sumPowers\ 3 &= 15 \\ power2\ 4 &= 2^4 = 16 \end{aligned}$$

$$\begin{aligned} sumPowers\ 5 &= 63 \\ power2\ 6 &= 2^6 = 64 \end{aligned}$$

How are these functions related?

Powers

- **Conjecture:** $\forall n \in Nat.\, (sumPowers\ n) + 1 = power2\ (n + 1)$
- **Proof**: Let $P(n) \equiv (sumPowers\ n) + 1 = power2\ (n + 1)$.
  We show for $\forall n.\, P(n)$ by induction on n.

  Base case: Show $P(0)$

  $(sumPowers\ 0) + 1 = 1 + 1 = 2$

  and

  $power2\ (0 + 1) = 2 \cdot power2\ 0 = 2 \cdot 1 = 2$

  Step case: Assume $P(n)$ for an arbitrary n, i.e.,

  $$(sumPowers\ n) + 1 = power2\ (n + 1)$$

  and prove $P(n + 1)$

  $$(sumPowers\ (n + 1)) + 1 = power2\ (n + 2)$$

Powers proof (cont.)

$$
\begin{aligned}
(sumPowers\ (n+1))+1 &= sumPowers\ ((n+1)-1)+power2\ (n+1)+1 \text{ (def.)} \\
&= sumPowers\ (n)+1+power2\ (n+1) \text{ (arithmetic)} \\
&= power2\ (n+1)+power2\ (n+1) \text{ (ind. hypothesis)} \\
&= 2 \cdot power2\ (n+1) \text{ (arithmetic)} \\
&= power2\ (n+2) \text{ (def.)}
\end{aligned}
$$

- We have proven $(sumPowers\ n)+1 = power2\ (n+1)$

  Equivalently: $sumPowers\ n = (power2\ (n+1))-1$

- Equation above yields a better definition of $sumPowers$:
  - $power2$: $O(n)$ iterations
  - $sumPowers = O(n)$ iterations + with each iteration a call to $power2$. Overall: $O(n^2)$ calls of $power2$.

Noetherian induction

- Induction schema

  Induction: To prove $P(n)$, for all natural numbers n.

  To prove: $P(n)$ for an arbitrary n under the assumption that $P(m)$ holds, for all $m < n$

- In general, we can use any well-founded ordering $<$.

  Here, the transitive closure of the predecessor relation on Nat

- Same principle applies to any set, not just Nat.

Deriving 1-step (structural, weak) induction from Noetherian induction

To derive 1-step induction we must derive $P(n)$ for an arbitrary $n \in Nat$, from the assumptions

$$P(0) \tag{1}$$

$$\forall n \in Nat. P(n) \to P(n+1) \tag{2}$$

By Noetherian induction, we may assume that

$$\forall m \in Nat. (m < n) \to P(m) \tag{3}$$

We establish $P(n)$ by a case split on $n = 0 \vee n > 0$.

Case 1: n $= 0$, and $P(0)$ follows from (1).

Case 2: $n > 0$, so $n = k + 1$ for some $k \in Nat$, i.e., we must prove $P(k+1)$.

By (3), $k < k + 1 \to P(k)$. Hence $P(k)$ holds and $P(k+1)$ follows from (2).