# Formal Methods and Functional Programming

## Linear Temporal Logic

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The slides in this section are partly based on the course *Automata-based System Analysis* by Felix Klaedtke

# Motivation

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  - For a non-deterministic, non-terminating program $\mathtt{s}$

$$wc : \mathrm{Stm} \times \mathrm{State} \times \mathbb{N} \rightarrow \mathrm{Bool}$$

$$\begin{aligned} wc(s, \sigma, n) \Leftrightarrow\ & \sigma(o) = 0\ \vee \\ & (\text{for all } s', \sigma' : \text{if } \langle s, \sigma \rangle \rightarrow_1 \langle s', \sigma' \rangle \text{ then there exists} \\ & m \in \mathbb{N} \text{ such that } m < n \text{ and } wc(s', \sigma', m)) \end{aligned}$$

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# 6. Linear Temporal Logic

# Transition Systems Revisited

- We use a slightly different definition here (than earlier in the course)

- A finite transition system is a tuple $(\Gamma, \sigma_I, \rightarrow)$
  - $\Gamma$: a finite set of configurations
  - $\sigma_I$: an initial configuration, $\sigma_I \in \Gamma$
  - $\rightarrow$: a transition relation, $\rightarrow \subseteq \Gamma \times \Gamma$

- Difference: we have a fixed initial configuration
  - In this section, transition systems model only one program/system, not all programs of a programming language

- Difference: we omit terminal configurations from the definition
  - Simplifies theory
  - Termination can be modelled by transition to a special extra sink state (which allows further transitions only back to itself)

# Transition System of a Promela Model

- Configurations: states (see previous section)
  - Global variables, global channels
  - Per active process: local variables, local channels, location counter

- Initial configuration: initial state (see previous section)

- Transition relation: defined by operational semantics of statements
  - We keep semantics informal

- A Promela model has a finite number of states
  - Finite number of active processes (limited to 255)
  - Finite number of variables and channels
  - Finite ranges of variables
  - Finite buffers of channels
- Therefore, it is technically possible to enumerate all possible states
  - How many states are there?

# State Space of Sequential Programs

- Number of states

$$\#\text{program locations} \times \prod_{\text{variable } x} |\, dom(x) \,|$$

  - where $|\, dom(x) \,|$ denotes the number of possible values of variable $x$

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- Example: sequential program with 10 locations and 3 boolean variables

$$10 \times 2 \times 2 \times 2 = 10 \times 2^3 = 80$$

- Adding two integer variables yields $80 \times 2^{32} \times 2^{32} = 80 \times 2^{64}$

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- State space explosion

# State Space of Concurrent Programs

- The number of states of $P \equiv P_1 \| \ldots \| P_N$ is at most

$$\#\text{states of } P_1 \times \ldots \times \#\text{states of } P_N =$$

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- Number of states grows exponentially in the number of processes

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# State Space of Promela Models

- The number of states of a system with $N$ processes and $K$ channels is at most

$$\prod_{i=1}^{N}(\#\text{program locations}_i \times \prod_{\text{variable } x_i} \mid dom(x_i) \mid) \times \prod_{j=1}^{K} \mid dom(c_j) \mid^{cap(c_j)}$$

  - $\mid dom(c) \mid$ denotes the number of possible messages of channel $c$
  - $cap(c)$ is the capacity (buffer size) of channel $c$

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  - $\mid dom(c) \mid$ denotes the number of possible messages of channel $c$
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- Number of states grows exponentially in the number and capacity of channels

- State space explosion

# Limiting the Impact of State-Space Explosion

- Only examine configurations actually reachable by the transition system
- Modeling step is important (omit unimportant details)
  - Can drastically reduce state-space of the transition system
- Spin employs many techniques/heuristics for efficiency
  - Explore certain paths first (can be customized)
  - Ignore certain interleavings (local state)

# Computations

- Infinite sequences
  - $S^\omega$ is the set of infinite sequences of elements of set $S$
  - $s_{[i]}$ denotes the $i$-th element of the sequence $s \in S^\omega$
- $\gamma \in \Gamma^\omega$ is a computation of a transition system if:
  - $\gamma_{[0]} = \sigma_I$
  - $\gamma_{[i]} \to \gamma_{[i+1]}$ (for all $i \geq 0$)
  - Note: we use $\sigma$ to range over the states $\Gamma$ of a transition system
  - Note (notation above): if $\gamma = \sigma_0\sigma_1\sigma_2\sigma_3 \ldots$ then $\gamma_{[i]} = \sigma_i$
- $\mathcal{C}(TS)$ is the set of all computations of a transition system $TS$

# Linear-Time Properties

- Linear-time properties (LT-properties) can be used to specify the permitted computations of a transition system
- A linear-time property $P$ over $\Gamma$ is a subset of $\Gamma^\omega$
  - $P$ specifies a particular set of infinite sequences of configurations
- $TS$ satisfies LT-property $P$ (over $\Gamma$)

$$TS \models P \quad \text{if and only if} \quad \mathcal{C}(TS) \subseteq P$$

  - All computations of $TS$ belong to the set $P$
- By contrast: branching-time properties (not in this course) can also express the existence of a computation
  - Example: "It is always possible to return to the initial state"

# LT-Properties: Example

- All opened files must be closed eventually

$$P = \{\gamma \in \Gamma^\omega \mid \forall i \geq 0 : \gamma_{[i]}(o) = 1 \Rightarrow \exists n > 0 : \gamma_{[i+n]}(o) = 0\}$$

- LT-properties precisely express properties of computations
  - Non-termination is handled by infinite sequences
  - Non-determinism is handled by considering each computation separately

- However, the explicit representation above (defining the set of sequences) is not convenient

- Logical formalism needed to simplify specification of LT-properties

# From Configurations to (Sets of) Propositions

- For a transition system $TS$, we additionally specify a set $AP$ of atomic propositions (of our choice)
  - An atomic proposition is a proposition containing no logical connectives
  - Example: $AP = \{open, closed\}$ (for files)
  - Example: $AP = \{x > 0, y \leq x\}$

- We must provide a labeling function that maps configurations to sets of atomic propositions from $AP$
  - $L : \Gamma \to \mathcal{P}(AP)$
  - Example: $L(\sigma) = \begin{cases} \{open\} & \text{if } \sigma(o) = 1 \\ \{closed\} & \text{if } \sigma(o) = 0 \\ \{\} & \text{otherwise} \end{cases}$

- We call $L(\sigma)$ an abstract state

- From now on, we consider $AP$ and $L$ to be part of the transition system

# Traces

- A trace is an abstraction of a computation
  - Observe only the propositions of each state, not the concrete state itself
  - Infinite sequence of abstract states ($\mathcal{P}(AP)^\omega$)

- $t \in \mathcal{P}(AP)^\omega$ is a trace of a transition system $TS$ if $t = L(\gamma_{[0]})L(\gamma_{[1]})L(\gamma_{[2]}), \ldots$ and $\gamma$ is a computation of $TS$

- $\mathcal{T}(TS)$ is the set of all traces of a transition system $TS$

- LT-properties are typically specified over infinite sequences of abstract states, rather than over sequences of configurations:

$$\boxed{P = \{t \in \mathcal{P}(AP)^\omega \mid \forall i \geq 0 : open \in t_{[i]} \Rightarrow \exists n > 0 : closed \in t_{[i+n]}\}}$$

# Safety Properties

- Intuition
  - "Something bad is never allowed to happen (and can't be fixed)"

- An LT-property $P$ is a safety property if for all infinite sequences $t \in \mathcal{P}(AP)^\omega$:
  if $t \notin P$ then there is a finite prefix $\hat{t}$ of $t$ such that for every infinite sequence $t'$ with prefix $\hat{t}$, $t' \notin P$
  - $\hat{t}$ is called a bad prefix; essentially, this finite sequence of steps already violates the property (whatever happens afterwards)

- Safety properties are violated in finite time and cannot be repaired

- Examples
  - State properties, for instance, invariants

$$\boxed{P = \{t \in \mathcal{P}(AP)^\omega \mid \forall i \geq 0 : open \in t_{[i]} \vee closed \in t_{[i]}\}}$$

  - "Money can be withdrawn only after correct PIN has been entered"

# Liveness Properties

- Intuition
  - "Something good will happen eventually"
  - "If the good thing has not happened yet, it could happen in the future"

- An LT-property $P$ is a liveness property if every finite sequence $\hat{t} \in \mathcal{P}(AP)^*$ is a prefix of an infinite sequence $t \in P$
  - A liveness property does not rule out any prefix
  - Every finite prefix can be extended to an infinite sequence that is in $P$

- Liveness properties are violated in infinite time

- Examples
  - All opened files must be closed eventually

$$\boxed{P = \{t \in \mathcal{P}(AP)^\omega \mid \forall i \geq 0 : open \in t_{[i]} \Rightarrow \exists n > 0 : closed \in t_{[i+n]}\}}$$

  - "The program terminates eventually"

# 6. Linear Temporal Logic

# Linear Temporal Logic

- Linear Temporal Logic (LTL) allows us to formalize LT-properties of traces in a convenient and succinct way
- We will see syntax and semantics for LTL (no inference rules, etc.)
- Whether or not the traces of a finite transition system satisfy an LTL formula is decidable (see next section)

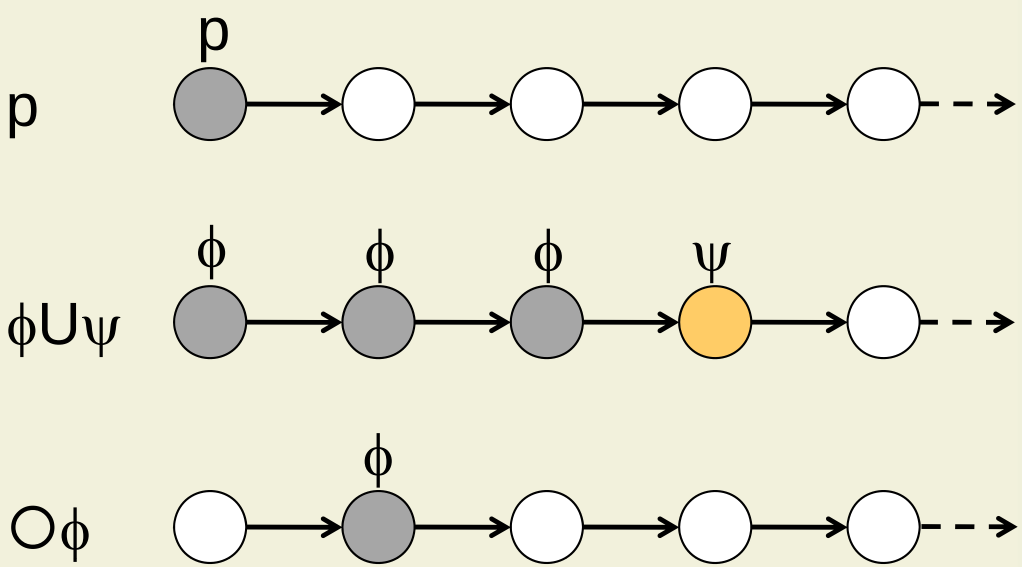
# LTL: Basic Operators

- Syntax

$$\phi = p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \, \mathsf{U} \, \phi \mid \bigcirc\phi$$

  - where $p$ is a proposition from a chosen set of atomic propositions $AP \neq \emptyset$

- Intuitive meaning of temporal logic formulas

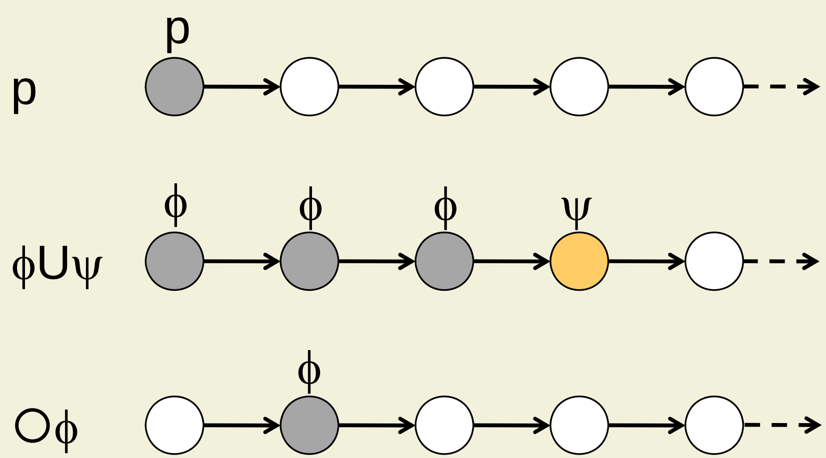


# LTL: Semantics

- $t \vDash \phi$ expresses that trace $t \in \mathcal{P}(AP)^\omega$ satisfies LTL formula $\phi$

| | | |
|---|---|---|
| $t \vDash p$ | iff | $p \in t_{[0]}$ |
| $t \vDash \neg\phi$ | iff | not $t \vDash \phi$ |
| $t \vDash \phi \wedge \psi$ | iff | $t \vDash \phi$ and $t \vDash \psi$ |
| $t \vDash \phi \,\mathsf{U}\, \psi$ | iff | there is a $k \geq 0$ with $t_{(\geq k)} \vDash \psi$ and $t_{(\geq j)} \vDash \phi$ for $0 \leq j < k$ |
| $t \vDash \bigcirc\phi$ | iff | $t_{(\geq 1)} \vDash \phi$ |

  - where $t_{(\geq i)}$ is the suffix of $t$ starting at $t_i$



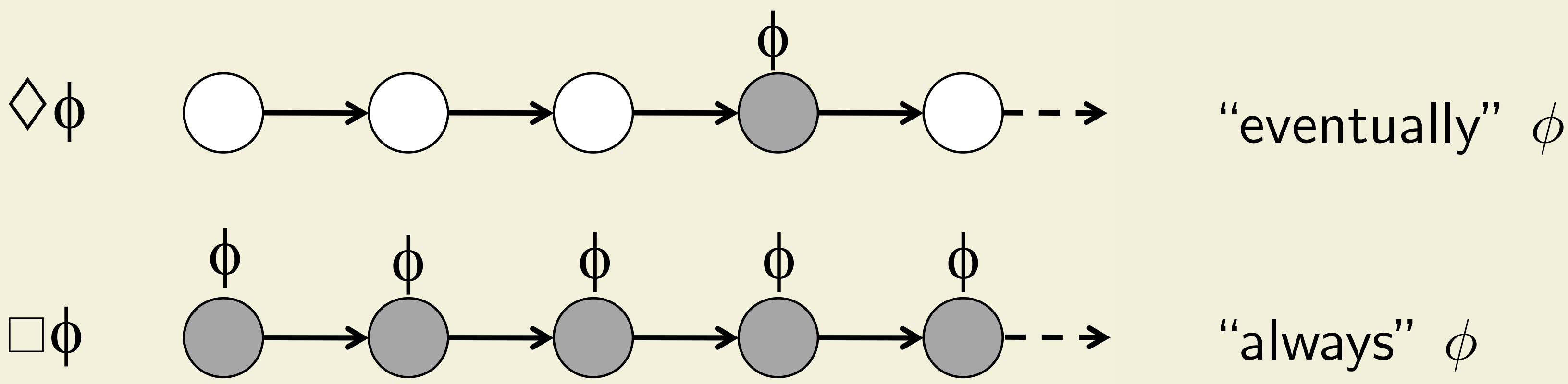

$p$ true "now"

$\phi$ "until" $\psi$

"next" $\phi$

# Derived Operators

- *true*, *false*, $\vee$, $\Rightarrow$, $\Leftrightarrow$ defined as usual
- Eventually: $\Diamond\phi \equiv (\mathit{true}\,\mathsf{U}\,\phi)$
- Always (from now): $\Box\phi \equiv \neg\Diamond\neg\phi$




- Precedence: unary operators always have highest precedence. So, $\Diamond\phi \Rightarrow \psi$ means $(\Diamond\phi) \Rightarrow \psi$. We will usually use parentheses to explicitly clarify other ambiguities.

# Useful Specification Patterns

- Strong invariant: $\Box\psi$
  - $\psi$ always holds
  - A file is always open or closed: $\Box(\mathit{open} \vee \mathit{closed})$
  - Safety property

- Monotone invariant: $\Box(\psi \Rightarrow \Box\psi)$
  - Once $\psi$ is true, then $\psi$ is always true
  - For example, once information is public, it can never become secret again (but it may always stay secret): $\Box(\mathit{public} \Rightarrow \Box\mathit{public})$
  - Safety property

- Establishing an invariant: $\Diamond \Box \psi$
  - Eventually $\psi$ will always hold
  - For example, system initialization starts server: $\Diamond \Box \mathit{serverRunning}$
  - Liveness property

# Useful Specification Patterns (cont'd)

- Responsiveness: $\Box(\psi \Rightarrow \Diamond\phi)$
  - E very time that $\psi$ holds, $\phi$ will eventually hold
  - For example, all opened files must be closed eventually: $\Box(\mathit{open} \Rightarrow \Diamond \mathit{closed})$
  - Liveness property
- Fairness: $\Box \Diamond \psi$
  - $\psi$ holds infinitely often
  - For example, producer does not wait infinitely long before entering the critical section: $\Box \Diamond \mathit{critical}$
  - Liveness property

# Needham-Schroeder Protocol

- If Alice and Bob have completed their protocol runs then Alice should believe her partner to be Bob if and only if Bob believes to talk to Alice

$$\Box(statusA = 1 \wedge statusB = 1 \Rightarrow \\ (partnerA = agentB \Leftrightarrow partnerB = agentA))$$

- If Alice completed her protocol run with Bob, the intruder should not have learned Alice's nonce

$$\Box(statusA = 1 \wedge partnerA = agentB \Rightarrow knows\_nonceA = 0)$$

- If Bob completed his protocol run with Alice, the intruder should not have learned Bob's nonce

$$\Box(statusB = 1 \wedge partnerB = agentA \Rightarrow knows\_nonceB = 0)$$