Handout # 1

Sample Natural-Language Proof

2.2.8 Given R a relation on a set S and P a predicate on S preserved by R. We prove that P is also preserved by R^*, the reflexive and transitive closure of R in three steps.

1. Let R^+ be the transitive closure of R as defined in 2.2.7, and define R'

   $$R' = R^+ \cup \{(s, s) \mid s \in S\}$$

   Then we prove that R' is the reflexive transitive closure of R, that is $R' = R^*$.

   Now R' is reflexive since $(s, s) \in R'$ for all $s \in S$ by definition of R'. Similarly R' is transitive because if we pick $(s_1, s_2), (s_2, s_3) \in R'$ then either both pairs are in R^+ which is already proved transitive (2.2.7), or one of them at least is a reflexive pair and thus (s_1, s_3) is either the same as (s_1, s_2) or the same as (s_2, s_3) (or both), and in either case is therefore in R'. As a result, since R^* is the smallest reflexive and transitive relation including R, we have $R^* \subseteq R'$.

   Now since R^+ is proved to be the transitive closure of R and thus the smallest transitive relation including R, we have $R^+ \subseteq R^*$. Since R^* is reflexive, it must also include the identity relation on S, that is $\{(s, s) \mid s \in S\}$, thus we have $R' \subseteq R^*$.

   Hence $R' = R^*$.

2. Next we prove by induction that P is preserved by R_i (from 2.2.7). First we prove that P is preserved by $R_0 = R$ which is immediate by assumption.

   Next, we assume P is preserved by R_i and prove that P is preserved by R_{i+1}. Let $(s, u) \in R_{i+1}$ where $P(s)$, and we must prove that $P(u)$. By definition of R_{i+1}, either $(s, u) \in R_i$ in which case we are done because our assumption that $R - i$ preserves P, or else there is some $t \in S$ where

   $$(s, t) \in R_i \text{ and } (t, u) \in R_i$$

   Now R_i preserves P and thus $P(t)$ since $(s, t) \in R_i$ and likewise $P(u)$ since $(t, u) \in R_i$, which is what we needed to prove for our inductive case.

   Thus P is preserved by every R_i.

3. Now we prove that R^* preserves P. Suppose $P(s)$ and $(s, t) \in R^*$. We must show $P(t)$.

   Since $R^* = R'$ and R' is the union of two sets, either $(s, t) \in R^+$ or else $s = t$. In the latter case we have $P(t)$ immediately. Otherwise, by definition of R^+, (s, t) must be in some set R_i. And as just proved, each R_i preserves P, and thus $P(t)$.

QED