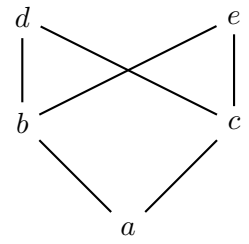
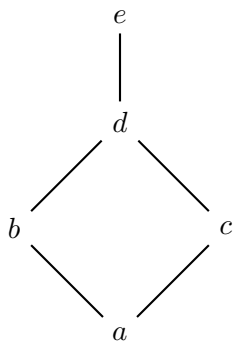
Exercise 10

Exercise 1

Are **(a)** and **(b)** complete lattices?



(a)

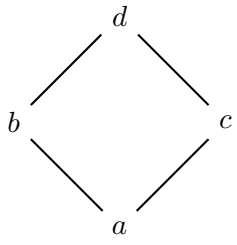


(b)

Solution:
(a) is not a complete lattice because $d \sqcup e$ does not exist.
(b) is a complete lattice.

Exercise 2

Consider the lattice $L = (A, \sqsubseteq)$, where $A = \{a, b, c, d\}$. The partial order $\sqsubseteq \subseteq A \times A$ is depicted in the Hasse diagram below.

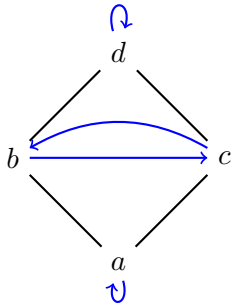


1. List the elements of $\sqsubseteq$.

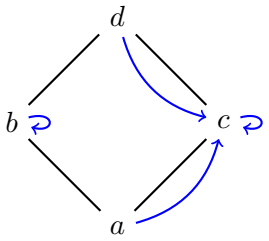
   Solution:

   $\sqsubseteq = \{(a,a), (b,b), (c,c), (d,d), (a,b), (a,c), (b,d), (c,d), (a,d)\}$

2. Consider the following functions $f, g : A \mapsto A$



Function f

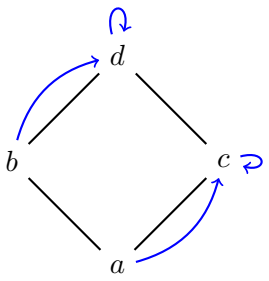


Function g

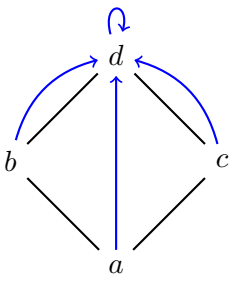
- Is f monotone? Is g monotone?
  Solution:
  f is monotone.
  g is not monotone because $b \sqsubseteq d$ but $g(b) = b \not\sqsubseteq g(d) = c$.
- List the set $Fix(f)$ of fixpoints of f, and the set $Red(f)$ of post-fixpoints of f.
  Solution:
  $Fix(f) = \{a, d\}$.
  $Red(f) = \{a, d\}$.
- List the sets of fixpoints/post-fixpoints of the function g.
  Solution:
  $Fix(g) = \{b, c\}$.
  $Red(g) = \{b, c, d\}$.

Exercise 3

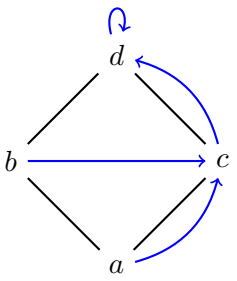
1. Consider the following three functions: $f, g, h : A \mapsto A$, defined below:



Function f



Function g



Function h

- Does g approximate f?
  Solution:
  Yes, g approximates f.
- Does h approximate f?
  Solution:
  No, h does not approximate f because $f(b) = d \not\sqsubseteq h(b) = c$.

2. Let $\mathbb{R}^\infty = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{Z}^\infty = \mathbb{Z} \cup \{-\infty, +\infty\}$, where $\mathbb{R}$ is the set of rational numbers and $\mathbb{Z}$ is the set of integers.

   $(\mathbb{R}^\infty, \leq)$ and $(\mathbb{Z}^\infty, \leq)$ are complete lattices.

   Let $\alpha : \mathbb{R}^\infty \mapsto \mathbb{Z}^\infty$ as $\alpha(x) = \lceil x \rceil$. (Here $\lceil x \rceil$ rounds-up x to the nearest integer.)

   Let $\gamma : \mathbb{Z}^\infty \mapsto \mathbb{R}^\infty$ as $\gamma(x) = x$.

   Consider the function $f : \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ defined as $f(x) = x^2$.

   - Give two functions $g, h : \mathbb{Z}^\infty \mapsto \mathbb{Z}^\infty$ that approximate f. Which one is more precise?
     Solution:

     $$g(x) = (|x| + 1)^2$$

     $$h(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ (|x| + 1)^2 & \text{otherwise} \end{cases}$$

     h is more precise.
   - Give a function $k : \mathbb{Z}^\infty \mapsto \mathbb{Z}^\infty$ that approximates any function $f : R \mapsto R$.
     Solution:
     $k(x) = +\infty$.

Exercise 4

Recall that an interval transformer for an action a is defined as:

$$[\![a]\!]_i : (Var \mapsto L^i) \mapsto (Var \mapsto L^i)$$

where $L^i = \{[x, y] \mid x, y \in \mathbb{Z}^\infty, x \leq y\} \cup \{\bot_i\}$ are the interval domain's elements (see slide 19 from the lecture).

1. Consider the interval maps:

   $m_1 = x \mapsto [-3, 8], y \mapsto [0, 5]$

   $m_2 = x \mapsto [-3, 8], y \mapsto \bot_i$

   The interval transformer for $\leq$ is defined on slide 36. Apply the transformer to compute the result of:

   $[\![x \leq y]\!](m_1) =$ $\quad$ $[\![x \leq y]\!](m_2) =$

   $[\![3 \leq 5]\!](m_1) =$ $\quad$ $[\![3 \leq 5]\!](m_2) =$

   $[\![5 \leq 3]\!](m_1) =$ $\quad$ $[\![5 \leq 3]\!](m_2) =$

   Solution:

   $[\![x \leq y]\!](m_1) = \ y \mapsto [-3, 5], y \mapsto [0, 5]$

   $[\![3 \leq 5]\!](m_1) = \ x \mapsto [-3, 8], y \mapsto [0, 5]$

   $[\![5 \leq 3]\!](m_1) = \ x \mapsto \bot_i, y \mapsto \bot_i$

   $[\![x \leq y]\!](m_2) = \ x \mapsto \bot_i, y \mapsto \bot_i$

   $[\![3 \leq 5]\!](m_2) = \ x \mapsto \bot_i, y \mapsto \bot_i$

   $[\![5 \leq 3]\!](m_2) = \ x \mapsto \bot_i, y \mapsto \bot_i$

2. Define the interval transformer for assignment:

   Solution:

   $[\![x := a]\!](m) = \ m[x \mapsto [p, q]]$ where $\langle a, m\rangle \Downarrow_i [p, q]$

3. Define the multiplication expression for interval elements:

   Solution:

   Let

   $\langle a_1, m\rangle \Downarrow_i [p, q]$

   $\langle a_2, m\rangle \Downarrow_i [r, s]$

   $A = \{p * r, p * s, q * r, q * s\}$

   Then,

   $\langle a_1 * a_2, m\rangle \Downarrow_i [min(A), max(A)]$

4. Define the interval transformer for equality:

   Solution:

   Let

   $[\![x = y]\!](m) = m[x \mapsto m(x) \sqcap_i m(y), y \mapsto m(x) \sqcap_i m(y)]$

   Recall that if $m(x) = [p, q]$ and $m(y) = [r, s]$ then

   $$[p, q] \sqcap_i [r, s] = \begin{cases} [max(p, r), min(q, s)] & \text{if } max(p, r) \leq min(q, s) \\ \bot_i & \text{otherwise} \end{cases}$$

Exercise 5

Consider the following program:

```
foo (int x) {
1       y := 2
2       if (x <= y)
3           z := 3 * x
else
4           z := y
5       z := y * z
6 }
```

- Give two concrete traces t_1 and t_2 of the program.

  Solution:

  Trace t_1 for `foo(1)`:

  $(y := 2;\ (\text{if } (x \leq y)\ z := 3 * x;\ \text{else } z := y;); z := y * z, \{x \mapsto 1, y \mapsto 0, z \mapsto 0\})$

  $\rightarrow ((\text{if } (x \leq y)\ z := 3 * x;\ \text{else } z := y;); z := y * z, \{x \mapsto 1, y \mapsto 2, z \mapsto 0\})$

  $\rightarrow (z := 3 * x; z := y * z, \{x \mapsto 1, y \mapsto 2, z \mapsto 0\})$

  $\rightarrow (z := y * z, \{x \mapsto 1, y \mapsto 2, z \mapsto 3\})$

  $\rightarrow \{x \mapsto 1, y \mapsto 2, z \mapsto 6\}$

  Trace t_2 for `foo(5)`:

  $(y := 2;\ (\text{if } (x \leq y)\ z := 3 * x;\ \text{else } z := y;); z := y * z, \{x \mapsto 5, y \mapsto 0, z \mapsto 0\})$

  $\rightarrow ((\text{if } (x \leq y)\ z := 3 * x;\ \text{else } z := y;); z := y * z, \{x \mapsto 5, y \mapsto 2, z \mapsto 0\})$

  $\rightarrow (z := y; z := y * z, \{x \mapsto 5, y \mapsto 2, z \mapsto 0\})$

  $\rightarrow (z := y * z, \{x \mapsto 5, y \mapsto 2, z \mapsto 2\})$

  $\rightarrow \{x \mapsto 5, y \mapsto 2, z \mapsto 4\}$

- Apply the interval abstraction function α^i given on slide 21 from the lecture on the set $\{t_1, t_2\}$.

  Solution:

  $\{1 \mapsto \{x \mapsto [1,5], y \mapsto [0,0], z \mapsto [0,0]\},$

  $\{2 \mapsto \{x \mapsto [1,5], y \mapsto [2,2], z \mapsto [0,0]\},$

  $\{3 \mapsto \{x \mapsto [1,1], y \mapsto [2,2], z \mapsto [0,0]\},$

  $\{4 \mapsto \{x \mapsto [5,5], y \mapsto [2,2], z \mapsto [0,0]\},$

  $\{5 \mapsto \{x \mapsto [1,5], y \mapsto [2,2], z \mapsto [2,3]\},$

  $\{6 \mapsto \{x \mapsto [1,5], y \mapsto [2,2], z \mapsto [4,6]\},$

- Compute the least fixpoint $\mathsf{lfp}F^i$ of the program using the interval domain abstraction.

  Solution:

  $\{1 \mapsto \{x \mapsto [-\infty, +\infty], y \mapsto [-\infty, +\infty], z \mapsto [-\infty, +\infty]\},$

  $\{2 \mapsto \{x \mapsto [-\infty, +\infty], y \mapsto [2, 2], z \mapsto [-\infty, +\infty]\},$

  $\{3 \mapsto \{x \mapsto [-\infty, 2], y \mapsto [2, 2], z \mapsto [-\infty, +\infty]\},$

  $\{4 \mapsto \{x \mapsto [3, +\infty], y \mapsto [2, 2], z \mapsto [-\infty, +\infty]\},$

  $\{5 \mapsto \{x \mapsto [-\infty, +\infty], y \mapsto [2, 2], z \mapsto [-\infty, 6]\},$

  $\{6 \mapsto \{x \mapsto [-\infty, +\infty], y \mapsto [2, 2], z \mapsto [-\infty, 12]\},$

- Give a concrete trace $t \in \gamma^i(\mathsf{lfp}F^i)$ that is not a valid trace. Here γ^i is the concretization function; see slides 21-22 from the lecture.

  Solution:

  $(y := 2;\ (\text{if } (x \leq y)\ z := 3 * x;\ \text{else } z := y;); z := y * z, \{x \mapsto 0, y \mapsto 0, z \mapsto 0\})$

  $\rightarrow ((\text{if } (x \leq y)\ z := 3 * x;\ \text{else } z := y;); z := y * z, \{x \mapsto 5, y \mapsto 2, z \mapsto 0\})$

  $\rightarrow \star$