Exercise 10

Exercise 1

Theorem (Rice). *A property P of the computable partial functions (c.p.f.) is decidable iff it is* trivial, *i.e., either no c.p.f. has P or all c.p.f. have P.*

The theorem speaks about a property of functions, simply because it is not true for arbitrary properties of programs. For example, the property "the program κ has length of 13 characters" is non-trivial and decidable.

Let P be a decidable and non-trivial property of computable partial functions. We shall give an informal proof of Rice's theorem by reducing the halting problem to the problem of deciding P. Let `P` be any program that decides P, that is for all programs κ we have that

$$\mathtt{P}(\kappa) = \begin{cases} \mathtt{true} & \text{if } P(\kappa) \\ \mathtt{false} & \text{otherwise.} \end{cases}$$

Our goal is to define an algorithm `halts(k, n)` that, given a program κ and an input n, decides whether $\kappa(n)$ halts:

$$\mathtt{halts}(\kappa, n) = \begin{cases} \mathtt{true} & \text{if } \kappa(n) \text{ halts} \\ \mathtt{false} & \text{otherwise.} \end{cases}$$

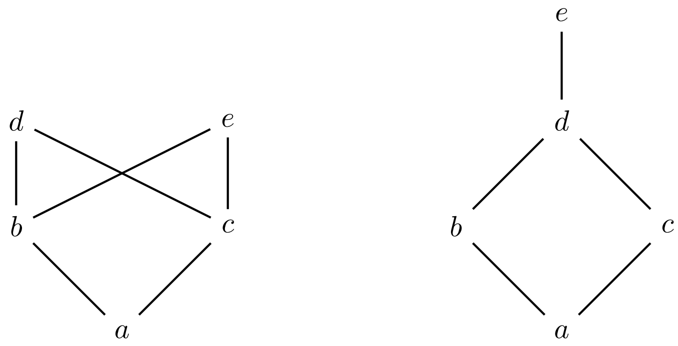
Now, observe that a never terminating program implements the nowhere defined partial function. Without loss of generality, we can assume that the property P does not hold for all such programs, for otherwise we could choose its complement $\neg P$ instead, which is again decidable and non-trivial. Let `ok` be any program for which P holds. Define `halts` as:

```
def halts(k, n):
    def test(m):
        k(n)
        return ok(m)
    return P(test)
```

If $\kappa(n)$ halts, then `test` behaves like `ok` for all inputs m. If $\kappa(n)$ loops forever, then `test` loops forever for all inputs. Consequently, `halts` uses the algorithm `P` to distinguish between these two cases.

Exercise 2

Are **(a)** and **(b)** complete lattices?

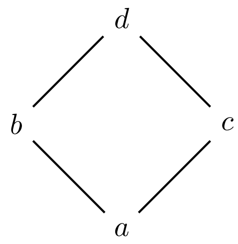


(a) **(b)**

Solution:
(a) is not a complete lattice because $d \sqcup e$ does not exist.
(b) is a complete lattice.

Exercise 3

Consider the lattice $L = (A, \sqsubseteq)$, where $A = \{a, b, c, d\}$. The partial order $\sqsubseteq \subseteq A \times A$ is depicted in the Hasse diagram below.

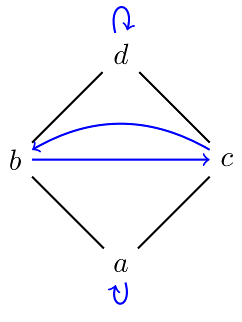


1. List the elements of $\sqsubseteq$.

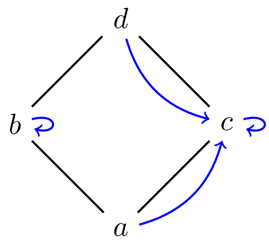
   Solution:

   $\sqsubseteq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(b,d),(c,d),(a,d)\}$

2. Consider the following functions $f, g : A \mapsto A$



Function f



Function g

- Is f monotone? Is g monotone?

  Solution:

  f is monotone.

  g is not monotone because $b \sqsubseteq d$ but $g(b) = b \not\sqsubseteq g(d) = c$.

- List the set $\mathit{Fix}(f)$ of fixpoints of f, and the set $\mathit{Red}(f)$ of post-fixpoints of f.

  Solution:

  $\mathit{Fix}(f) = \{a, d\}$.

  $\mathit{Red}(f) = \{a, d\}$.

- List the sets of fixpoints/post-fixpoints of the function g.

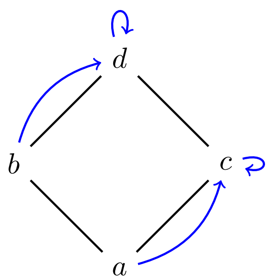
  Solution:

  $\mathit{Fix}(g) = \{b, c\}$.

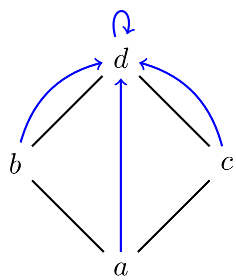
  $\mathit{Red}(g) = \{b, c, d\}$.

Exercise 4

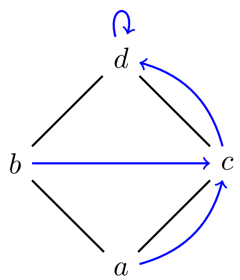
1. Consider the following three functions: $f, g, h : A \mapsto A$, defined below:



Function f



Function g



Function h

- Does g approximate f?
  Solution:
  Yes, g approximates f.
- Does h approximate f?
  Solution:
  No, h does not approximate f because $f(b) = d \not\sqsubseteq h(b) = c$.

2. Let $\mathbb{R}^{\infty} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{Z}^{\infty} = \mathbb{Z} \cup \{-\infty, +\infty\}$, where $\mathbb{R}$ is the set of rational numbers and $\mathbb{Z}$ is the set of integers.

   $(\mathbb{R}^{\infty}, \leq)$ and $(\mathbb{Z}^{\infty}, \leq)$ are complete lattices.

   Let $\alpha : \mathbb{R}^{\infty} \mapsto \mathbb{Z}^{\infty}$ as $\alpha(x) = \lceil x \rceil$. (Here $\lceil x \rceil$ rounds-up x to the nearest integer.)

   Let $\gamma : \mathbb{Z}^{\infty} \mapsto \mathbb{R}^{\infty}$ as $\gamma(x) = x$.

   Consider the function $f : \mathbb{R}^{\infty} \mapsto \mathbb{R}^{\infty}$ defined as $f(x) = x^2$.

   - Give two functions $g, h : \mathbb{Z}^{\infty} \mapsto \mathbb{Z}^{\infty}$ that approximate f. Which one is more precise?
     Solution:

     $$g(x) = (|x| + 1)^2$$

     $$h(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ (|x| + 1)^2 & \text{otherwise} \end{cases}$$

     h is more precise.
   - Give a function $k : \mathbb{Z}^{\infty} \mapsto \mathbb{Z}^{\infty}$ that approximates any function $f : R \mapsto R$.
     Solution:
     $k(x) = +\infty$.