

Solutions Assignment 8

Exercise 1

Theorem (Rice). *A property P of the computable partial functions (c.p.f.) is decidable iff it is trivial, i.e., either no c.p.f. has P or all c.p.f. have P .*

The theorem speaks about a property of functions, simply because it is not true for arbitrary properties of programs. For example, the property “the program κ has length of 13 characters” is non-trivial and decidable.

Let P be a decidable and non-trivial property of computable partial functions. We shall give an informal proof of Rice’s theorem by reducing the halting problem to the problem of deciding P . Let P be any program that decides P , that is for all programs κ we have that

$$P(\kappa) = \begin{cases} \text{true} & \text{if } P(\kappa) \\ \text{false} & \text{otherwise.} \end{cases}$$

Our goal is to define an algorithm `halts(k, n)` that, given a program κ and an input n , decides whether $\kappa(n)$ halts:

$$\text{halts}(\kappa, n) = \begin{cases} \text{true} & \text{if } \kappa(n) \text{ halts} \\ \text{false} & \text{otherwise.} \end{cases}$$

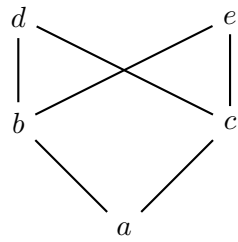
Now, observe that a never terminating program implements the nowhere defined partial function. Without loss of generality, we can assume that the property P does not hold for all such programs, for otherwise we could choose its complement $\neg P$ instead, which is again decidable and non-trivial. Let `ok` be any program for which P holds. Define `halts` as:

```
def halts(k, n):  
    def test(m):  
        k(n)  
        return ok(m)  
    return P(test)
```

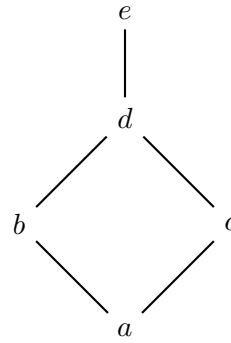
If $\kappa(n)$ halts, then `test` behaves like `ok` for all inputs m . If $\kappa(n)$ loops forever, then `test` loops forever for all inputs. Consequently, `halts` uses the algorithm P to distinguish between these two cases.

Exercise 2

Are (a) and (b) complete lattices?



(a)



(b)

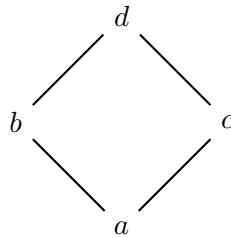
Solution:

(a) is not a complete lattice because $d \sqcup e$ does not exist.

(b) is a complete lattice.

Exercise 3

Consider the lattice $L = (A, \sqsubseteq)$, where $A = \{a, b, c, d\}$. The partial order $\sqsubseteq \subseteq A \times A$ is depicted in the Hasse diagram below.

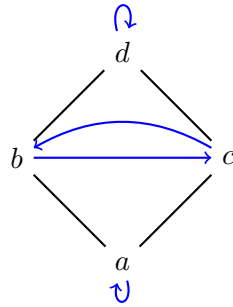


1. List the elements of \sqsubseteq .

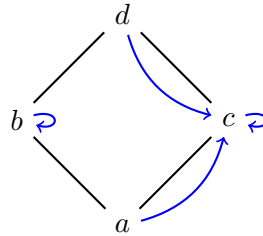
Solution:

$$\sqsubseteq = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (b, d), (c, d), (a, d)\}$$

2. Consider the following functions $f, g : A \mapsto A$



Function f



Function g

- Is f monotone? Is g monotone?

Solution:

f is monotone.

g is not monotone because $b \sqsubseteq d$ but $g(b) = b \not\sqsubseteq g(d) = c$.

- List the set $Fix(f)$ of fixpoints of f , and the set $Red(f)$ of post-fixpoints of f .

Solution:

$Fix(f) = \{a, d\}$.

$Red(f) = \{a, d\}$.

- List the sets of fixpoints/post-fixpoints of the function g .

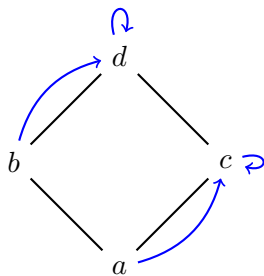
Solution:

$Fix(g) = \{b, c\}$.

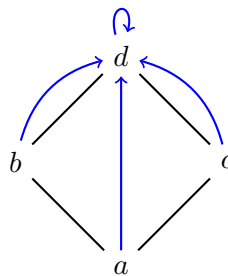
$Red(g) = \{b, c, d\}$.

Exercise 4

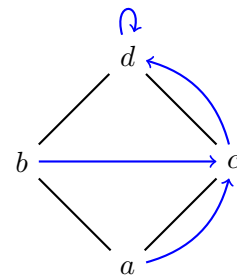
1. Consider the following three functions: $f, g, h : A \mapsto A$, defined below:



Function f



Function g



Function h

- Does g approximate f ?

Solution:

Yes, g approximates f .

- Does h approximate f ?

Solution:

No, h does not approximate f because $f(b) = d \not\sqsubseteq h(b) = c$.

2. Let $\mathbb{R}^\infty = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{Z}^\infty = \mathbb{Z} \cup \{-\infty, +\infty\}$, where \mathbb{R} is the set of rational numbers and \mathbb{Z} is the set of integers.

$(\mathbb{R}^\infty, \leq)$ and $(\mathbb{Z}^\infty, \leq)$ are complete lattices.

Let $\alpha : \mathbb{R}^\infty \mapsto \mathbb{Z}^\infty$ as $\alpha(x) = \lceil x \rceil$. (Here $\lceil x \rceil$ rounds-up x to the nearest greater or equal integer.)

Let $\gamma : \mathbb{Z}^\infty \mapsto \mathbb{R}^\infty$ as $\gamma(x) = x$.

Consider the function $f : \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ defined as $f(x) = \frac{x^2}{3}$.

- Give two functions $g, h : \mathbb{Z}^\infty \mapsto \mathbb{Z}^\infty$ that approximate f . Which one is more precise?

Solution:

$$g(x) = x^2$$

$$h(x) = \lceil \frac{x^2}{3} \rceil$$

h is more precise.

- Give a function $k : \mathbb{Z}^\infty \mapsto \mathbb{Z}^\infty$ that approximates any function $f : R \mapsto R$.

Solution:

$$k(x) = +\infty.$$