Exercise session 1
– Induction –

Exercise 1. Power statement

Assume that the initial value of the variable `x` is n and that the initial value of `y` is m. Write a statement in **IMP** that assigns `z` the value of n to the power of m, that is

$$\underbrace{n * \cdots * n}_{m\ times}$$

Give the abstract syntax tree.

Exercise 2. Using $\mathcal{A}$ and $\mathcal{B}$

Assume that $\sigma(x) = 3$. Determine $\mathcal{B}[\![\texttt{not(x=1)}]\!]\sigma$ and $\mathcal{A}[\![\texttt{x+1}]\!]\sigma$.

Induction

There are two main types of induction

- when using **weak induction** we first prove the base case (most often for $n = 0$ or $n = 1$) and for the step case ($n+1$) we assume that the statement holds for case n.
- when using **strong induction** we first prove the base case (most often for $n = 0$ or $n = 1$) and for the step case ($n + 1$) we assume that the statement holds for all cases $m < n + 1$.

The assumption is called *induction hypothesis*.

Exercise 3. The winning strategy

There are two piles of cards, two players take turn:

- in each turn: one player removes any positive number of cards from one pile (any of the two),
- the player who removes the last card wins.

Using strong induction show that if, in the beginning, the two piles contain the same number of cards, then the second player can always win.

Can you use weak induction?

Exercise 4. x*4 + y*5

Prove that any number greater or equal 12 can be formed by adding multiples of 4 and/or 5.
Try both strong and weak induction!

Exercise 5. Math...

Let f and g be the following functions:
$f(n) = 0^2 + 1^2 + 2^2 + \ldots + n^2$
$g(n) = \frac{1}{6}n(n+1)(2n+1)$.

Show that $\forall n \in N : f(n) = g(n)$.

Solutions

Exercise 1. Power statement

IMP statement: `z:=1; while y>0 do z:=z*x; y:=y-1 end`

Exercise 2. Using $\mathcal{A}$ and $\mathcal{B}$

The definition of $\mathcal{B}$ gives

$$\mathcal{B}[\![\texttt{not(x=1)}]\!]\sigma \quad = \quad \begin{cases} tt & \text{if } \mathcal{B}[\![\texttt{x=1}]\!]\sigma = \mathit{ff} \\ \mathit{ff} & \text{otherwise} \end{cases}$$

We have $\mathcal{B}[\![\texttt{x=1}]\!]\sigma = (\mathcal{A}[\![\texttt{x}]\!]\sigma = \mathcal{A}[\![1]\!]\sigma) = (\sigma(\texttt{x}) = 1) = (3 = 1) = \mathit{ff}$.
Thus $\mathcal{B}[\![\texttt{not(x=1)}]\!]\sigma = tt$.

$\mathcal{A}[\![\texttt{x+1}]\!]\sigma = \mathcal{A}[\![\texttt{x}]\!]\sigma + \mathcal{A}[\![1]\!]\sigma = \sigma(\texttt{x}) + 1 = 3 + 1 = 4$

Exercise 3. The winning strategy

First we prove a lemma which is slightly stronger than the theorem we would like to prove.

Lemma. If it is the first player's turn and the two piles contain the same number of cards then the second player can always win.

Proof of the lemma.
Let n be the number of cards in each pile.
1. Base case: when $n = 1$, the first player can only remove one card from one pile. There is no other choice for him. So the second player can remove the one remaining card in the other pile and win.
2. Induction hypothesis: in a state when it is the first player's turn and there are m $(1 \leq m < n+1)$ cards in both pile, the second player can always win.
3. Step case: We must show that the second player can win when there are $n+1$ cards in each pile.
Let's say the first player removes j cards from one pile, leaving $n+1-j$ cards in the pile. The second player can remove j cards from the other pile, leaving the same amount on both pile.
If $j = n+1$ (first player takes a whole pile) then the second player takes the other pile and wins. If $j < n+1$ then it can be seen that $1 \leq n+1-j < n+1$, hence the second player can win by the induction hypothesis.

Proof of the original theorem. The theorem is a special case of the lemma, namely when the first player starts the game.

The lemma is needed because the induction hypothesis of the original theorem would state that: "If we start the game with m $(1 \leq m < n+1)$ cards in both pile, the second player can always win".
This hypothesis could not be used in 3. as we would not be in the starting state

of the game.

Weak induction could only be used if j was restricted to be 1.

Exercise 4. x*4 + y*5

Using weak induction
1. Base case: $n = 12$, this can be formed as $4 + 4 + 4$.
2. Induction hypothesis: n is multiples of 4 and/or 5.
3. Step case: We must show that $n + 1$ is multiples of 4 and/or 5.
For $n + 1$

- if at least one 4 is used for case n, then replace this 4 by 5 and thus we get $n + 1$ from additions of 4 and/or 5.
- if there is no 4, then only 5 is used and since $n + 1 \geq 12$ at least three times. This $5 + 5 + 5$ can be replaced by $4 + 4 + 4 + 4$ and we get $n + 1$ from additions of 4 and/or 5.

Using strong induction
1. Base case: is divided into the following cases:

- 12 = 4 + 4 + 4,
- 13 = 4 + 4 + 5,
- 14 = 4 + 5 + 5,
- 15 = 5 + 5 + 5.

2. Induction hypothesis: assume that all numbers from 12 to n is the result of adding 4 and/or 5 where $n \geq 15$.
3. Step case: we must show that $n + 1$ is multiples of 4 and/or 5.
For $n + 1$ we use the result of $n + 1 - 3$ (which satisfies the hypothesis) and add 4 to it.

Note that the subcases for the base cases are needed otherwise the induction would not work with 13, 14 and 15.

Exercise 5. Math...

We use weak induction.
1. Base case: when $n = 0$, $f(0) = g(0) = 0$.
2. Induction hypothesis: we assume that for n, $f(n) = g(n)$.
3. Step case: we must show that $f(n + 1) = g(n + 1)$, that is:

$$0^2 + 1^2 + 2^2 + ... + n^2 + (n + 1)^2 = \frac{1}{6}(n + 1)(n + 1 + 1)(2(n + 1) + 1)$$

On the left side we can use the hypothesis, i.e.

$$0^2 + 1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

On the right side we can do arithmetic simplifications. Thus we get:

$$\frac{1}{6}n(n+1)(2n+1)+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3)$$

Now we multiply both sides by $\frac{6}{n+1}$ and get:

$$n(2n+1)+6(n+1)=(n+2)(2n+3)$$

After simple arithmetic steps we get:

$$2n^2+n+6n+6=2n^2+3n+4n+6$$

Now we can easily see that both sides equal $2n^2+7n+6$.