Exercise session 7
– Fixed point theory –

Exercise 25. Unique least upper bound

Consider a partially ordered set $(D, \sqsubseteq)$ and assume that we have a subset Y of D. Show that if Y has a least upper bound d then d is unique.

Exercise 26. State $\hookrightarrow$ State not a lattice

Construct a subset Y of State $\hookrightarrow$ State such that Y has no upper bound and hence no least upper bound on relation $\sqsubseteq$.

Exercise 27. Monotone functions

Consider the ccpo $(P(\mathbb{N}), \subseteq)$. Determine which of the following functions in $P(\mathbb{N}) \to P(\mathbb{N})$ are monotone:

- $f_1(X) = \mathbb{N} \setminus X$
- $f_2(X) = X \ \cup \ \{27\}$
- $f_3(X) = X \cap \{7, 9, 13\}$
- $f_4(X) = \{n \in X \mid n \text{ is a prime}\}$
- $f_5(X) = \{2 * n \mid n \in X\}$

Exercise 28. Monotone functionals

Determine which of the following functionals of (State $\hookrightarrow$ State) $\to$ (State $\hookrightarrow$ State) are monotone:

$$F_1(g) \quad = \quad g$$

$$F_2(g) \quad = \quad \begin{cases} h_1 & \text{if } g = h_2 \\ h_2 & \text{otherwise} \end{cases} \quad \text{where } h_1 \neq h_2$$

$$F_3(g)\sigma \quad = \quad \begin{cases} g(\sigma) & \text{if } \sigma(\mathtt{x}) \neq 0 \\ \sigma & \text{if } \sigma(\mathtt{x}) = 0 \end{cases}$$

Exercise 29. Continuity

Show that the functional of statement `while x#0 do skip end` is continuous.

Solutions

Exercise 25. Unique least upper bound

The definition of least upper bound (d) of subset Y means that

(1) $\forall\ d' \in Y\ :\ d' \sqsubseteq d$
(2) if d'' is another upper bound then $d \sqsubseteq d''$.

Assume Y has two least upper bounds d_1 and d_2.
Since d_1 is a least upper bound $d_1 \sqsubseteq d_2$.
Since d_2 is a least upper bound $d_2 \sqsubseteq d_1$.
By anti-symmetry of $\sqsubseteq$ we get $d_1 = d_2$.

Exercise 26. State $\hookrightarrow$ State not a lattice

We have to construct a subset Y of State $\hookrightarrow$ State such that there is no function g for which $\forall\ g' \in Y\ :\ g' \sqsubseteq g$ holds.
Consider the following two functions:

$$g_1(\sigma) \;=\; \begin{cases} \sigma & \text{if } \sigma(\mathtt{x}) = 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$g_2(\sigma) \;=\; \begin{cases} \sigma[\mathtt{x} \mapsto 1] & \text{if } \sigma(\mathtt{x}) = 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

We can observe that there is no function g for which $g_1 \sqsubseteq g$ and $g_2 \sqsubseteq g$, because in states where $\sigma(\mathtt{x}) = 0$, function g should give (at the same time) $\sigma[\mathtt{x} \mapsto 0]$ due to g_1 and $\sigma[\mathtt{x} \mapsto 1]$ due to g_2.

Exercise 27. Monotone functions

To prove monotonicity of function f_i, we have to show that

$$\forall\ X, Y\ :\ X \subseteq Y \Rightarrow f_i(X) \subseteq f_i(Y).$$

1. $f_1(X) = \mathbb{N} \setminus X$
We show that the function is not monotone by a counter-example. Assume $X = \{1\}$ and $Y = \{1, 2\}$, thus $X \subseteq Y$.
$f_1(X)$ contains 2, but $f_1(Y)$ does not. Thus, $f_1(X) \not\subseteq f_1(Y)$.

2. $f_2(X) = X \cup \{27\}$
We have to show that $\forall\ X, Y\ :\ X \subseteq Y \Rightarrow X \cup \{27\} \subseteq Y \cup \{27\}$.
We do it by showing that if $a \in \mathbb{N}$ is an element of $X \cup \{27\}$ then it is also an element of $Y \cup \{27\}$.
We do a case distinction:
A) $a \in X$. By the assumption we get that $a \in Y$, thus $a \in Y \cup \{27\}$ holds.
B) $a = 27$. In this case $a \in Y \cup \{27\}$ trivially holds.

3. $f_3(X) = X \cap \{7, 9, 13\}$
We have to show that $\forall\, X, Y : X \subseteq Y \Rightarrow X \cap \{7, 9, 13\} \subseteq Y \cap \{7, 9, 13\}$.
Again, we show that if $a \in \mathbb{N}$ is an element of $X \cap \{7, 9, 13\}$ then it is also an element of $Y \cap \{7, 9, 13\}$.
An element a can only be in $X \cap \{7, 9, 13\}$ if $a \in X \wedge a \in \{7, 9, 13\}$.
Using the assumption on the first conjunct, this yields $a \in Y \wedge a \in \{7, 9, 13\}$.
Thus, $a \in Y \cap \{7, 9, 13\}$.

4. $f_4(X) = \{n \in X \mid n \text{ is a prime}\}$
Proof is analogous to 3 as the function can be rewritten as
$f_4(X) = X \cap \{2, 3, 5, 7, \ldots\}$.

5. $f_5(X) = \{2 * n \mid n \in X\}$
f_5 is an injective (or one-to-one) function, which preserves monotonicity on $\subseteq$.

Exercise 28. Monotone functionals

Monotonicity of F means that for any functions g_1, g_2, if $g_1 \sqsubseteq g_2$ then $F(g_1) \sqsubseteq F(g_2)$.

1. By the definition of F_1, we get $F_1(g_1) = g_1$ and $F_1(g_2) = g_2$. Applying these on $F_1(g_1) \sqsubseteq F_1(g_2)$ we get $g_1 \sqsubseteq g_2$ which holds by assumption.

2. We show that the functional is not monotone by contradiction. Assume that F_2 is monotone, we continue by case distinction:

A) $h_2 = \perp$. There is a function a, where $h_2 \sqsubseteq a \wedge h_2 \neq a$. By the monotonicity of F_2, we get $F_2(h_2) = h_1 \sqsubseteq F_2(a) = h_2$. From $h_1 \sqsubseteq h_2$ and monotonicity, we get $F_2(h_1) = h_2 \sqsubseteq F_2(h_2) = h_1$. Now we have $h_1 \sqsubseteq h_2$ and $h_2 \sqsubseteq h_1$, which implies by anti-symmetry that $h_1 = h_2$, which is a contradiction to the definition of F_2.

B) $h_2 \neq \perp$. There is a function b, where $b \sqsubseteq h_2 \wedge b \neq h_2$. By the monotonicity of F_2, we get $F_2(b) = h_2 \sqsubseteq F_2(h_2) = h_1$. From $h_2 \sqsubseteq h_1$ and monotonicity, we get $F_2(h_2) = h_1 \sqsubseteq F_2(h_1) = h_2$. Now we have $h_2 \sqsubseteq h_1$ and $h_1 \sqsubseteq h_2$, which implies by anti-symmetry that $h_1 = h_2$, which is a contradiction to the definition of F_2.

The fact that we got a contradiction in both cases means that our assumption that F_2 is monotone was wrong.

3. We make a case distinction according to the definition of F_3:
A) $\sigma(\mathtt{x}) \neq 0$. Applying the definition of F_3 we get $F_3(g_1)\sigma = g_1(\sigma)$ and $F_3(g_2)\sigma = g_2(\sigma)$. Applying these on $F_3(g_1)\sigma \sqsubseteq F_3(g_2)\sigma$ we get $g_1(\sigma) \sqsubseteq g_2(\sigma)$ which holds by assumption.

B) $\sigma(\mathtt{x}) = 0$. The definition can be seen as $F_3(g)\sigma = id(\sigma)$ for any function g. Applying these on $F_3(g_1)\sigma \sqsubseteq F_3(g_2)\sigma$ we get $id(\sigma) \sqsubseteq id(\sigma)$ which holds by reflexivity of $\sqsubseteq$.

Exercise 29. Continuity

The functional F of the statement is:

$$F(g)\sigma \quad = \begin{cases} g(\sigma) & \text{if } \sigma(\mathtt{x}) \neq 0 \\ \sigma & \text{otherwise} \end{cases}$$

To show that it is continuous, we have to prove that:
1) It is monotone,
2) For all non-empty chains Y, $\sqcup\{F(g) \mid g \in Y\} = F(\sqcup Y)$ holds.

1. We have proved that F is monotone in the previous exercise (see F_3).
2. From Lemma 3.4 we know that $(\text{State} \hookrightarrow \text{State}, \sqsubseteq)$ is a ccpo and the least upper bound $\sqcup Y$ of a chain Y is given by

$$\sqcup Y(\sigma) = \begin{cases} \sigma' & \text{if } \exists g \in Y : g(\sigma) = \sigma' \\ \text{undefined} & \text{otherwise} \end{cases}$$

Since F is monotone and Y is a chain, by Lemma 3.6 we get that $\{F(g)|g \in Y\}$ is also a chain, and $\sqcup\{F(g)|g \in Y\} \sqsubseteq F(\sqcup Y)$.
By anti-symmetry of $\sqsubseteq$, it remains to prove $F(\sqcup Y) \sqsubseteq \sqcup\{F(g)|g \in Y\}$. This can be rewritten as $F(\sqcup Y)\sigma = \sigma' \Rightarrow \sqcup\{F(g)|g \in Y\}\sigma = \sigma'$

We make a case split on the value of $\sigma(\mathtt{x})$ (according to the definition of F):
A) $\sigma(\mathtt{x}) \neq 0$.
– By the definition of F, we get $F(\sqcup Y)\sigma = \sqcup Y(\sigma) = \sigma'$ for some state σ'. By Lemma 3.4 we know that there is a $g' \in Y$ such that $g'(\sigma) = \sigma'$.
– By the definition of F, we get $\sqcup\{F(g)|g \in Y\}\sigma = \sqcup\{g|g \in Y\}\sigma$.
– $g'(\sigma) = \sigma'$ implies $\sqcup\{g|g \in Y\}\sigma = \sigma'$ since the least upper bound summarizes all information.

B) $\sigma(\mathtt{x}) = 0$.
– By the definition of F, we get $F(\sqcup Y)\sigma = \sigma$.
– By the definition of F, we get $(\sqcup\{F(g)|g \in Y\})\sigma = (\sqcup\{id\})\sigma = id(\sigma) = \sigma$.