An $O(n \log n)$-Space Decision Procedure for the Relevance Logic B^{+*}

Luca Viganò
Institut für Informatik
Albert-Ludwigs-Universität Freiburg
Georges-Köhler-Allee 52
D-79110 Freiburg (Germany)
luca@informatik.uni-freiburg.de
http://www.informatik.uni-freiburg.de/~luca

Abstract

In previous work we gave a new proof-theoretical method for establishing upper-bounds on the space complexity of the provability problem of modal and other propositional non-classical logics. Here we extend and refine these results to give an $O(n \log n)$-space decision procedure for the basic positive relevance logic B^+. We compute this upper-bound by first giving a sound and complete, cut-free, labelled sequent system for B^+, and then establishing bounds on the application of the rules of this system.

Keywords: Relevance Logics, Computational Complexity, Labelled Deduction Systems, Sequent Systems.

1 Introduction

Context This paper is part of ongoing research that aims at developing a framework for applying proof theory to the analysis of the complexity of families of non-classical logics, and thereby designing decision procedures for these logics [4, 5, 6, 34].

Modal, relevance and other non-classical logics are usually presented using Hilbert-style systems. However, although uniform, these systems are difficult to use in practice, especially in comparison with the more 'natural' Gentzen-style systems such as natural deduction, sequent and tableaux systems. Unfortunately, devising Gentzen-style systems for non-classical logics often requires considerable ingenuity, as well as trading uniformity for simplicity and usability.

A solution to the problem of formalizing and using non-classical logics is to employ *labelling* techniques, which provide a general framework for presenting different logics in a uniform way as Gentzen-style deduction systems. The intuition behind these techniques is that labelling (sometimes also called prefixing, annotating or subscripting) allows one to encode explicitly additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the

*Studia Logica 66(3):385-407, 2000; © Kluwer Academic Publishers.

logic one wants to formalize. So, for instance, instead of considering a modal formula A we consider the *labelled formula* $x{:}A$, where x denotes a possible world in the underlying Kripke semantics for modal logics. In other words, the label x keeps track of the possible world in which the formula A lives. We can then use labels also to specify they way in which possible worlds are related in a particular Kripke model, e.g. the binary *relational formula* $x\,Ry$ tells us that the world y is accessible from x.

Based on the works of Anderson and Belnap [1, 2], Fitting [13] and Gabbay [14], among others, labelled deduction systems have been used extensively to formalize non-classical logics, and exploited for applications that can be modelled by these logics. For example, several labelled deduction systems have been given for relevance logics, e.g. [1, 2, 5, 8, 9, 13, 16, 20], ranging from systems where the labels explicitly encode information from the Kripke or algebraic semantics, to systems where the additional information is of a proof-theoretical nature, e.g. encoding resources and their use.

Research on labelling has focused not only on the design of labelled deduction systems for specific non-classical logics, but also, more generally, on the characterization of the classes of logics that can be formalized this way, as well as on investigations of the general properties and limitations of labelling techniques [3, 5, 10, 14, 17, 34].

Labelled deduction systems can also be used to establish the decidability of logics and analyze their complexity. For example, in [34] we use labelling to devise a new proof-theoretical method for establishing upper-bounds on the space complexity of the decision problem for the propositional modal logics K, T, K4 and S4. Our method consists of three steps. In the first step, we give sound and complete, cut-free, labelled sequent systems for propositional modal logics. The absence of cut is a first step towards decidability in that it yields a subformula property bounding the number of different formulas arising in proofs. To bound proof search, however, we must also bound how often subformulas might appear. In fact, some of the structural rules (i.e. the contraction rules) of our sequent systems duplicate formulas. Since these rules are always applicable, infinite branches may arise and proof search may not terminate. Thus, in the second step of our method, we analyze our modal labelled sequent systems and show how to restrict their structural rules; in other words, we establish bounds on the applications of the rules and use these bounds to define a measure that decreases at every rule application. In the third and final step, we combine these bounds with an analysis of the accessibility relation of the corresponding Kripke models to give decision procedures for K, T, K4 and S4, and compute polynomial upper-bounds on the space requirements of these procedures.

In [6] we generalize this method to a recipe for giving decision procedures for propositional non-classical logics, which we present using cut-free labelled sequent systems, and for establishing upper-bounds on their space complexity. Our recipe consists of guidelines and examples so that, given a particular logic, the user can contribute bounds on structural reasoning and the accessibility relation of the corresponding Kripke semantics.

Contributions Our main contribution here is to give an $O(n \log n)$-space decision procedure for the basic positive relevance logic B^+ by introducing a sound and complete, cut-free, labelled sequent system $\mathrm{S}(\mathrm{B}^+)$, and investigating $\mathrm{S}(\mathrm{B}^+)$

to eliminate some of its rules or establish bounds on their applications. We view this as a two-fold contribution. First, our decision procedure and space upper-bound for B^+ are new. Second, this work constitutes not only an interesting case study but also, more concretely, a first step towards the application of our framework to the proof-theoretical investigation of the decidability and computational complexity of the relevance logics that can be obtained by extending the basic logic B^+. Indeed, our procedure extends the method of [34] and the results fit and refine the abstract recipe of [6]. It is however important to note that that recipe has a general character, while most of the results that we establish here follow by a refined analysis of $S(B^+)$; i.e. they could not have been captured simply by instantiating the guidelines of our recipe to B^+.

Related Work Our bound is new. To our knowledge, no similar investigations have been carried out on relevance logics, and, in particular, we are not aware of any subrecursive bounds for the basic relevance logic B or its fragments. In fact, the analysis of decidability and complexity of relevance logics is a subtle issue (especially in comparison with other non-classical logics such as modal logics), and positive results have been presented alongside open problems and negative results, including the undecidability of the principal relevance logics R and E, e.g. [1, 2, 25, 26, 28, 30, 31, 32].

The decidability of relevance logics is often established semantically by defining filtrations on possible worlds and proving that the logics possess the finite model property. Although effective to establish decidability, this often yields poor complexity bounds. Moreover, as remarked in, e.g., [25], the standard proofs of the finite model property fail for several relevance logics, although not for B and its fragments.

Proof-theoretical decision procedures based on Gentzen-style systems have also been given for B and other relevance logics, e.g. [1, 2, 7, 15, 16, 25, 26]. Although we are not aware of any analysis of the complexity of these procedures, some of this work is close to ours, at least in spirit.

In [16], Gochet, Gribomont and Rossetto extend the *connection method* [36] to B^+, and give a decision procedure that yields finite models for satisfiable formulas. However, this method investigates the need for duplications, via explicit and implicit contractions, of indexed formulas, rather than explicitly bounding applications of the contraction rules as we do here. (We return to this below.)

In [25], Restall gives *display calculi* for a wide range of substructural logics, and investigates their decidability by a proof-theoretical analysis. Similar to ours, this analysis focuses on controlling applications of the contraction rules. However, Restall must also control applications of other rules, which are specific rules of display calculi, such as the intensional display rules. (A similar analysis is required by the Gentzen-style systems of [1, 7, 15], which are also based on the introduction of new intensional connectives.) While the display method establishes the decidability of several substructural logics, including the logic DW, which is a simple extension of B, no complexity results are given in [25]. Investigating the connections and complementary aspects of Restall's approach and ours remains as future work.

Organization The remainder of this paper is organized as follows. In § 2 we introduce our labelled sequent system $S(B^+)$. In § 3 and § 4, the core of the paper, we bound the space complexity of proof search in $S(B^+)$. We first prove some preliminary results and then, in § 3.1 and § 3.2, show that we can bound or eliminate altogether applications of some of the rules of $S(B^+)$. In particular, Theorem 3.14 shows that we can eliminate all contractions in proofs of theorems of $S(B^+)$. Based on this analysis, in § 4 we give our decision procedure for $S(B^+)$ and establish an $O(n \log n)$ upper-bound on its space complexity (Theorem 4.2). Finally, in § 5 we draw conclusions and discuss future work.

Proofs have been omitted or considerably shortened; details can be found in [35].

2 The Sequent System $S(B^+)$

We now introduce our cut-free labelled sequent system $S(B^+)$; we assume the reader is familiar with the syntax and semantics of relevance logics, e.g. [2, 28], and with Gentzen-style systems such as sequent and natural deduction systems, e.g. [22, 29].

We use labelling to encode explicitly semantic information. To illustrate the labelling technique that we employ, consider the valuation clause for relevant implication $\to$ in a Kripke model $\mathfrak{M}$ for positive relevance logics. Let $\mathfrak{M} = (\mathfrak{W}, \mathfrak{R}, \mathfrak{V})$, where $\mathfrak{W}$ is a non-empty set of possible worlds, including the *actual world* 0, $\mathfrak{R}$ is a ternary (*compossibility*) relation on $\mathfrak{W}$, and $\mathfrak{V}$ is a function mapping elements of $\mathfrak{W}$ and propositional variables to truth values (0 or 1), based on which we define the truth relation $x \vDash A$ for $x \in \mathfrak{W}$ and A a formula of B^+. In particular, given a model $\mathfrak{M}$, the truth of $A \to B$ at a world x is evaluated in terms of the truth of A at y and of B at z where $(x, y, z) \in \mathfrak{R}$, i.e.

$$x \vDash A \to B \text{ iff for all } y, z \in \mathfrak{W}, \text{ if } (x, y, z) \in \mathfrak{R} \text{ and } y \vDash A \text{ then } z \vDash B.$$

Now let W be a denumerable set of *labels* $x, y, z, a, b, \ldots$, and R a ternary relation over W. In our labelled sequent system $S(B^+)$ we consider two kinds of formulas: *labelled well-formed formulas* (*lwffs*), written $x{:}A$, which pair a label x and a formula A of B^+, and *relational well-formed formulas* (*rwffs*), written $R\,x\,y\,z$. The lwff $x{:}A$ expresses that the formula A holds at world x, and the rwff $R\,x\,y\,z$ expresses that (x, y, z) is in the relation $\mathfrak{R}$.[1] We can then modify the above semantic definition to suit this extended language as follows:

$$\vDash x{:}A \to B \text{ iff for all } y, z \in \mathfrak{W}, \text{ if } \vDash R\,x\,y\,z \text{ and } \vDash y{:}A \text{ then } \vDash z{:}B,$$

where the relation $\vDash$ now expresses truth for labelled and relational formulas.

This semantics can be straightforwardly captured by the sequent rules

$$\frac{\Delta \vdash R\,x\,y\,z \quad \Gamma, \Delta \vdash \Gamma', y{:}A \quad z{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \to B, \Gamma, \Delta \vdash \Gamma'} \to\mathrm{L} \qquad \frac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B} \to\mathrm{R}\,,$$

where (i) Γ and Δ, possibly annotated, vary over finite multisets of lwffs and rwffs respectively; and (ii) application of $\to$R is subject to the condition that y and z are distinct and do not occur in $\Gamma, \Delta \vdash \Gamma', x{:}A \to B$.

Two remarks. First, following [29], we call the formulas that pass through the application of a rule unchanged (they appear in the premises and in the

[1] Note that to reduce notational overhead, we do not define an interpretation function mapping labels into worlds in $\mathfrak{W}$ but implicitly identify the label x with the world $x \in \mathfrak{W}$, i.e. we identify W with $\mathfrak{W}$; cf. the discussion in [34].

conclusion) the *parametric formulas* of the rule; the formula contracted or introduced in the conclusion (on the left of the sequent symbol $\vdash$, in symbols $\ldots$L, or on its right, $\ldots$R) is the *principal formula* of the rule, and the formulas from which the principal formula derives are the *active formulas* of the rule. Second, our sequents have the form $\Gamma, \Delta \vdash \Gamma'$ or $\Delta \vdash R\,x\,y\,z$, i.e. they are multi-conclusioned for lwffs and single-conclusioned for rwffs. In [5, 34] we show that when we adopt these distinct forms for sequents we are able to formalize non-classical logics whose Kripke semantics is axiomatizable in terms of Horn clauses, including many of the most common logics. Most importantly, sequents of these forms allow us to give cut-free sequent systems for these logics and provide the basis for the complexity analysis that we perform below.[2]

The other connectives of S(B$^+$) are classical ones: conjunction $\wedge$ and disjunction $\vee$. They are evaluated locally, at the same world, e.g. $\models x{:}A \vee B$ iff $\models x{:}A$ or $\models x{:}B$. To reason about $\wedge$ and $\vee$, we simply adapt traditional sequent rules [22, 29] by adding labels and parametric rwffs, i.e.

$$\frac{x{:}A, x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \wedge B, \Gamma, \Delta \vdash \Gamma'}\ \wedge\mathrm{L}\,, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad \Gamma, \Delta \vdash \Gamma', x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \wedge B}\ \wedge\mathrm{R}\,,$$

$$\frac{x{:}A, \Gamma, \Delta \vdash \Gamma' \quad x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \vee B, \Gamma, \Delta \vdash \Gamma'}\ \vee\mathrm{L}\,, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \vee B}\ \vee\mathrm{R}\,.$$

The rules for $\wedge$, $\vee$ and $\rightarrow$ define the behavior of the connectives, and we call them *logical rules*. To define the system S(B$^+$), we also need axioms and additional rules for lwffs and rwffs (in symbols $\ldots$l$\ldots$ and $\ldots$r$\ldots$). The *axioms*

$$\frac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl} \quad \text{and} \quad \frac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\ \mathrm{AXr}$$

allow us to conclude backwards proofs, and the *structural rules*

$$\frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta, R\,x\,y\,z \vdash \Gamma'}\ \mathrm{WrL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{WlL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{WlR},$$

$$\frac{\Delta, R\,a\,b\,c, R\,a\,b\,c \vdash R\,x\,y\,z}{\Delta, R\,a\,b\,c \vdash R\,x\,y\,z}\ \mathrm{CrL}, \qquad \frac{x{:}A, x{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{ClL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{ClR},$$

operate on the structure of sequents, deleting formulas (*weakening*, W$\ldots$) or duplicating them (*contraction*, C$\ldots$). Note that for rwffs we do not need structural rules that operate on the right of $\vdash$, as we work with single-conclusioned sequents $\Delta \vdash R\,x\,y\,z$.

Relevance logics require the truth relation to be monotonic with respect to a *partial order* $\sqsubseteq$ *on worlds*, which is usually defined in terms of $\mathfrak{R}$ and the actual world 0, i.e. $x \sqsubseteq y$ iff $(0, x, y) \in \mathfrak{R}$ [2, 28]. The monotony of $\models$ is expressed by the following postulates, which hold for any worlds a, x, y and z, and for any formula A: (i) if $\models a \sqsubseteq x$ and $\models R\,x\,y\,z$, then $\models R\,a\,y\,z$; (ii) if $\models a \sqsubseteq y$ and $\models R\,x\,y\,z$, then $\models R\,x\,a\,z$; (iii) if $\models z \sqsubseteq a$ and $\models R\,x\,y\,z$, then $\models R\,x\,y\,a$; (iv) if $\models x \sqsubseteq y$ and $\models x{:}A$, then $\models y{:}A$.

[2] Note also that the forms of our sequents distinguish our approach from traditional *semantic embeddings*, in which the Kripke semantics is used to translate non-classical logics into first-higher or higher-order logic; cf. the discussion in [5, 34].

In $S(B^+)$, we define $\sqsubseteq$ in terms of R and 0, i.e. $x \sqsubseteq y$ iff $R\,0\,x\,y$, and formalize monotony in $S(B^+)$ in terms of the *monotony rules*

$$\frac{\Delta \vdash R\,0\,a\,x \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,a\,y\,z}\ \mathit{monr1}\,, \qquad \frac{\Delta \vdash R\,0\,a\,y \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,a\,z}\ \mathit{monr2}\,,$$

$$\frac{\Delta \vdash R\,0\,z\,a \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,y\,a}\ \mathit{monr3}\,, \qquad \frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl}\,.$$

Transitivity of the partial order $\sqsubseteq$ is then a special case of *monr3* when x is 0. Reflexivity of $\sqsubseteq$ corresponds to postulating that the relation $\mathfrak{R}$ satisfies the *identity property*, i.e. $(0, x, x) \in \mathfrak{R}$ for all x. In $S(B^+)$, we formalize the identity of R with the rule

$$\frac{}{\vdash R\,0\,x\,x}\ \mathit{iden}\,.$$

Definition 2.1 We call *relational rule* a rule that has an rwff as its principal formula. The system $S(B^+)$ then consists of: the axioms AXl and AXr; the logical rules $\wedge$L, $\wedge$R, $\vee$L, $\vee$R, $\rightarrow$L and $\rightarrow$R; the structural rules WrL, WlL, WlR, CrL, ClL and ClR; the relational monotony rules *monr1*, *monr2* and *monr3*; the monotony rule for lwffs *monl*; and the relational rule *iden*. ■

That our cut-free system $S(B^+)$ is a sound and complete system for the logic B^+ follows from the results in [5, 34]. In particular, there we introduced a framework for giving labelled natural deduction systems for various non-classical logics, and showed that such systems are sound and complete with respect to the corresponding Kripke semantics. Moreover, for the logics whose Kripke semantics is axiomatizable in terms of Horn clauses, we showed that derivations in our natural deduction systems normalize. That is, derivations can be reduced to a structured normal form where there are no unnecessary detours.

Instantiating that framework to give a sound and complete, normalizing, labelled natural deduction system $N(B^+)$ for B^+ is a straightforward task, and we point the reader to [34, p. 65] without further comment. The soundness and completeness of $S(B^+)$ follows then from the general theorem of [34], which shows that normalizing labelled natural deduction systems and cut-free labelled sequent systems are intertranslatable. That is, by induction on the length of proofs of sequents in $S(B^+)$ and on the length of normal derivations in $N(B^+)$, we can show that:

Theorem 2.2 $S(B^+)$ is sound and complete with respect to the corresponding Kripke semantics. In particular, the sequent $\vdash 0{:}A$ is provable in $S(B^+)$ iff $\models 0{:}A$ (i.e. iff the formula A is valid in the Kripke semantics for B^+). ■

3 Bounding Proof Search in $S(B^+)$

As is usual in sequent systems, we build $S(B^+)$-proofs by applying the rules *backwards*, starting with the end-sequent and working towards the axioms. In order to bound the complexity of proof search in $S(B^+)$ we then need to bound which formulas can appear in proofs and with how many repetitions. To illustrate our analysis in more detail, we begin by observing that the absence of cut provides a first step towards decidability in that it implies that $S(B^+)$ has a *subformula property*.

Definition 3.1 We say that $y{:}B$ is a *labelled subformula* of $x{:}A$ iff B is a subformula of A, i.e. iff A is B, or A is $A_1 \bullet A_2$ with $\bullet \in \{\wedge, \vee, \rightarrow\}$ and B is a subformula of A_1 or A_2.

We say that a labelled subformula $y{:}B$ of $x{:}A$ *occurs positive* [*negative*] in $x{:}A$, in symbols $y{:}B \Subset_+ x{:}A$ [$y{:}B \Subset_- x{:}A$], iff B occurs positive [negative] in A, in symbols $B \Subset_+ A$ [$B \Subset_- A$]. This is inductively defined as follows: (i) if $B = A$, then $B \Subset_+ A$; (ii) if $(B \bullet C) \Subset_- A$ with $\bullet \in \{\wedge, \vee\}$, then $B \Subset_- A$ and $C \Subset_- A$; (iii) if $(B \bullet C) \Subset_+ A$ with $\bullet \in \{\wedge, \vee\}$, then $B \Subset_+ A$ and $C \Subset_+ A$; (iv) if $(B \rightarrow C) \Subset_- A$, then $B \Subset_+ A$ and $C \Subset_- A$; (v) if $(B \rightarrow C) \Subset_+ A$, then $B \Subset_- A$ and $C \Subset_+ A$.

By extension, we inductively define that an lwff $y{:}B$ *occurs positive* [*negative*] *in a multiset of lwffs* Γ, in symbols $y{:}B \Subset_+ \Gamma$ [$y{:}B \Subset_- \Gamma$], as follows: (i) if $y{:}B \in \Gamma$, then $y{:}B \Subset_+ \Gamma$; (ii) if $y{:}B \Subset_+ x{:}A$ and $x{:}A \Subset_+ \Gamma$, then $y{:}B \Subset_+ \Gamma$; (iii) if $y{:}B \Subset_- x{:}A$ and $x{:}A \Subset_- \Gamma$, then $y{:}B \Subset_+ \Gamma$; (iv) if $y{:}B \Subset_+ x{:}A$ and $x{:}A \Subset_- \Gamma$, then $y{:}B \Subset_- \Gamma$; (v) if $y{:}B \Subset_- x{:}A$ and $x{:}A \Subset_+ \Gamma$, then $y{:}B \Subset_- \Gamma$. ■

Fact 3.2 [*Subformula property*] In any S(B$^+$)-proof of a sequent $\Gamma, \Delta \vdash \Gamma'$, only labelled subformulas of Γ and Γ' occur. ■

The subformula property bounds the number of different formulas arising in proofs, but it does not bound how often the formulas might appear. In fact, some of the rules of S(B$^+$) repeat formulas (*monl*) or even duplicate them (the contractions). Since the latter rules are always applicable, infinite branches may arise and proof search may not terminate. To explain this problem further, let us define the *grade* of an lwff $x{:}A$ as the number of connectives in A. Then we can partition the rules of S(B$^+$) into two types: rules that 'simplify' sequents to sequents of smaller size (in the sense that we obtain lwffs of smaller grade and/or we weaken or delete formulas), and rules that do not simplify sequents. The contraction rules and the monotony rules are rules of the second type, while all the other rules of S(B$^+$) are of the first type.

Thus, given the subformula property, bounding proof search in S(B$^+$) reduces to the problem of bounding applications of the rules of the second type. We solve this problem by a proof-theoretical analysis of the applications of these rules, showing that we can bound applications of the monotony rules (§ 3.1) and eliminate altogether applications of the contraction rules (§ 3.2). We begin by introducing additional terminology and proving preliminary technical results.

Let us call *lwff-rule* a rule that has an lwff as its principal formula. Hence, the lwff-rules of S(B$^+$) are $\wedge$L, $\wedge$R, $\vee$L, $\vee$R, $\rightarrow$L, $\rightarrow$R, WlL, WlR, ClL, ClR and *monl*. Now suppose that, reasoning forwards, we derive a sequent S from sequents $S_1, \ldots, S_n$ by first applying the lwff-rule r_1 and then applying the lwff-rule r_2, where (i) each of the premises of r_2 results from an application of r_1; (ii) all applications of r_1 have the same principal formula $x{:}A$; and (iii) the principal formula $x{:}A$ of r_1 is parametric in the application of r_2. We then say that r_1 is *permutable* over r_2, or that r_1 *permutes* over r_2, if the original inference may be replaced by one in which the sequent S is derived from $S_1, \ldots, S_n$ by applying first r_2 and then r_1. By extension, a derivation or proof Π permutes to Π' if Π' is obtained from Π by one or more permutations of lwff-rules. For example,

$$\dfrac{\dfrac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C \quad y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}D}{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C \wedge D}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}A \rightarrow B, a{:}C \wedge D}\,\rightarrow\mathrm{R} \tag{1}$$

permutes to

$$\dfrac{\dfrac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}C} \to\mathrm{R} \quad \dfrac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}D}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}D} \to\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}C \wedge D} \wedge\mathrm{R} \tag{2}$$

since y and z are different from a by the condition on the application of $\to$R in (1). We can also reverse this permutation, i.e. (2) permutes to (1).

Lemma 3.3 Every lwff-rule of S(B$^+$) is permutable over any other lwff-rule, except for $\to$L, which is permutable over all lwff-rules but $\to$R. ■

This lemma follows immediately by inspection of the rules. The reason that $\to$L is, in general, not permutable over $\to$R is that $\to$L may have the same active rwff $R\,x\,y\,z$ as $\to$R, as is shown, for example, in the proof (3) in the discussion of Theorem 3.14. Inspection of the rules also shows that:

Fact 3.4 If the principal and active lwffs of a rule of S(B$^+$) appear on different sides of $\vdash$, then they are labelled differently. ■

Fact 3.5 Let Π be an S(B$^+$)-proof of a sequent $\vdash 0{:}Z$, and consider a sequent $S = \Gamma, \Delta \vdash \Gamma'$ appearing in Π. Then no lwff of the form $0{:}A$ may appear in the antecedent of S. ■

Using these facts, we can transform an S(B$^+$)-proof of $\vdash 0{:}Z$ to a proof that does not contain applications of *iden*; i.e. we can dispose of *iden* for theoremhood.[3]

Lemma 3.6 Every sequent $\vdash 0{:}Z$ provable in S(B$^+$) has a proof that contains no applications of the rule *iden*. ■

Corollary 3.7 The sequent $\Delta \vdash R\,0\,x\,y$ is provable in an S(B$^+$)-proof of $\vdash 0{:}Z$ if and only if $R\,0\,x\,y \in \Delta$. ■

3.1 Bounding Monotony

We consider first the relational monotony rules and then the rule *monl*.

Lemma 3.8 Every sequent $\vdash 0{:}Z$ provable in S(B$^+$) has a proof that contains no applications of *monr2* or *monr3*. ■

That is, *monr2* and *monr3* are eliminable for theoremhood, which follows by iterated proof-transformations, traversing proofs starting from the highest application of *monr2* or *monr3* and using Lemma 3.6. Lemma 3.8 implies that:

[3]It is also possible to dispose of *iden* by building it into special implication rules; we return to this in § 5, where we introduce the rules $\to\mathrm{L}_0$ and $\to\mathrm{R}_0$. Note however that we cannot directly eliminate *iden* when we consider *logical consequence*, i.e. proofs of sequents of the form $0{:}A \vdash 0{:}B$. The decidability of the logical consequence problem of B$^+$ has been established by a semantic filtration argument, and showing if and how our method provides an efficient decision procedure for this problem remains as future work.

Lemma 3.9 Consider an application of *monr1* in an S(B$^+$)-proof of $\vdash 0{:}Z$

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,a \vdash R\,0\,x\,a}\,\mathrm{AXr}}{\Delta, R\,0\,x\,a \vdash R\,0\,x\,a}\,\vdots\,\mathrm{WrL} \qquad \begin{matrix}\Pi_1\\ \Delta, R\,0\,x\,a \vdash R\,a\,y\,z\end{matrix}}{\begin{matrix}\Delta, R\,0\,x\,a \vdash R\,x\,y\,z\\ \Pi_0\\ \vdash 0{:}Z\end{matrix}}\,\mathit{monr1}\;.$$

Then it cannot be the case that Π_1 contains an application of *monr1* that introduces $R\,a\,y\,z$; in other words, it must be the case that $R\,a\,y\,z \in \Delta$. ■

By analogy with Lemma 3.9, for applications of *monl* we have:

Lemma 3.10 Let Π be an S(B$^+$)-proof of the sequent $\vdash 0{:}Z$. It cannot be the case that the active lwff $x{:}A$ of an application of *monl* in Π is the principal formula of another application of *monl* in Π. ■

By induction on the structure of proofs, we can further restrict applications of *monl* to atomic formulas, i.e. to lwffs of grade 0.[4]

Lemma 3.11 Every sequent provable in S(B$^+$) has a proof in which all the applications of *monl* have active formula $x{:}A$ of grade 0, i.e. A is atomic. ■

We can obtain an equivalent system by restricting also the axiom AXl to atomic lwffs.

Lemma 3.12 Every sequent provable in S(B$^+$) has a proof in which all the axioms employed contain no connectives, i.e. the formula A in each application of AXl is atomic. ■

In the remaining part of the paper we will thus restrict applications of AXl and *monl* to atomic lwffs. Note that we could consider also the other side of the spectrum and dispense with the weakening rules by building them into the axioms and the rule *iden*, i.e.

$$\dfrac{}{x{:}A, \Gamma, \Delta \vdash \Gamma', x{:}A}\;, \qquad \dfrac{}{\Delta, R\,x\,y\,z \vdash R\,x\,y\,z} \quad \text{and} \quad \dfrac{}{\Delta \vdash R\,0\,x\,x}\;.$$

This follows because, since all rules permute over WrL, WlL and WlR, we have:

Fact 3.13 Each S(B$^+$)-proof can be transformed so that all weakenings in the proof occur immediately below the axioms. ■

This also tells us that weakening does not constitute a problem in establishing the space complexity of provability in S(B$^+$). Contractions play instead a crucial role in complexity analysis and we will now focus on these rules, showing that we can eliminate all their applications in proofs of theorems of S(B$^+$).

[4]Note that this is the inverse of what is usually done when presenting the semantics of relevance logics, where a monotony property is first postulated for atomic formulas and then generalized to arbitrary ones by induction on their structure, cf. § 2 and, e.g., [2, 28].

3.2 Eliminating Contraction

Theorem 3.14 Every sequent $\vdash 0{:}Z$ provable in $\mathrm{S}(\mathrm{B}^+)$ has a proof in which there are no applications of the rules CrL, ClL and ClR. ■

A detailed proof of this result can be found in the companion report [35]. Here we sketch only the main proof ideas. The eliminability of CrL follows straightforwardly because contracted rwffs can only be introduced in the antecedent of a sequent by applications of the rule WrL. Therefore, we just need to delete both the CrL and the corresponding WrL, e.g. we transform

$$\dfrac{\dfrac{\overset{\Pi_2}{\dfrac{\overset{\Pi_1}{\Delta_1 \vdash R\,x\,y\,z}}{\Delta_1, R\,a\,b\,c \vdash R\,x\,y\,z}\;\mathrm{WrL}}}{\Delta_2, R\,a\,b\,c, R\,a\,b\,c \vdash R\,x\,y\,z}}{\Delta_2, R\,a\,b\,c \vdash R\,x\,y\,z}\;\mathrm{CrL} \qquad \text{to} \qquad \begin{array}{c} \Pi_1 \\ \Delta_1 \vdash R\,x\,y\,z \\ \Pi_2' \\ \Delta_2, R\,a\,b\,c \vdash R\,x\,y\,z \end{array}$$

where in Π_2' we apply the same sequence of rules applied in Π_2, with the same active and principal formulas but with different parametric formulas.

The eliminability of ClL and ClR follows by adapting and extending the proof for the unlabelled sequent system for propositional classical logic given by Zeman in [37]. Given a proof Π of $\vdash 0{:}Z$ in $\mathrm{S}(\mathrm{B}^+)$, we proceed by three nested inductions. The first induction is on the number of contractions in Π. Suppose therefore that Π contains $i+1$ contractions, and pick a highest (e.g. uppermost in the leftmost branch) contraction in Π, i.e. consider a subproof of Π that ends with a contraction of $x{:}A$ and such that the proof-fragment above the contraction is contraction-free. To eliminate this contraction and obtain a proof Π' of $\vdash 0{:}Z$ that contains i contractions, we proceed first by induction on the grade of $x{:}A$ and then by induction on the *rank* of the contraction (the largest number of steps immediately above the contraction and containing at least one of the two contracted lwffs). The base case, $grade(x{:}A) = 0$, is trivial; for example, if one instance of $x{:}A$ is introduced by weakening, then we conclude by deleting this weakening and the contraction. There is one step for each rule introducing $x{:}A$, and it is straightforward to eliminate contractions of lwffs with main connective other than $\rightarrow$, i.e. of lwffs of the form $x{:}B \wedge C$ or $x{:}B \vee C$. This leaves us only with left and right contractions of lwffs of the form $x{:}B \rightarrow C$. For the latter, that is for applications of ClR with principal formula $x{:}B \rightarrow C$, we exploit a 'relevant' analogue of the intuitionistic disjunction property.

One of the consequences of cut-elimination in sequent systems for propositional intuitionistic logic is the *disjunction property*, if $\vdash A \vee B$ is provable then so is $\vdash A$ or $\vdash B$, which reflects the constructive character of the logic [29, 33]. The property holds also under hypotheses,

$$\text{if } \Gamma \vdash A \vee B \text{ is provable then so is } \Gamma \vdash A \text{ or } \Gamma \vdash B,$$

provided that the hypotheses in Γ are *Harrop formulas* [33], where the class $\mathcal{H}$ of Harrop formulas is inductively defined by: (i) $p \in \mathcal{H}$ for p a propositional variable, (ii) $A \wedge B \in \mathcal{H}$ if $A, B \in \mathcal{H}$, and (iii) $A \supset B \in \mathcal{H}$ if $B \in \mathcal{H}$, where $\supset$ is intuitionistic implication.

Since $C \vee D \notin \mathcal{H}$, the restriction to Harrop formulas ensures that there is no disjunctive formula $C \vee D$ that occurs positive in Γ, i.e. $C \vee D \not\in_+ \Gamma$. This amounts to ensuring that $\Gamma \vdash A \vee B$ is not the conclusion of an application of the branching intuitionistic left disjunction rule ($\vee$L), e.g.

$$\frac{C, \Gamma_1 \vdash A \vee B \quad D, \Gamma_1 \vdash A \vee B}{C \vee D, \Gamma_1 \vdash A \vee B} \ (\vee\mathrm{L})\,.$$[5]

It can be easily checked semantically that in S(B$^+$) we have: if $\vdash x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2$ is provable, then so is $\vdash x{:}A_1 \rightarrow B_1$ or $\vdash x{:}A_2 \rightarrow B_2$. To generalize this property to hold under hypotheses, we first define an analogue of the branchings caused by formulas that are not Harrop: in S(B$^+$) we need to consider the branchings caused by (i) applications of $\vee$L with principal formula labelled with x or with a world 'preceding' x, or (ii) applications of $\rightarrow$L with principal formula labelled with a world 'preceding' x.

Definition 3.15 The *class* $\mathfrak{B}(x, \Delta)$ *of x-branching lwffs with respect to a multiset of rwffs* Δ *in* S(B$^+$) is inductively defined as follows: (i) $a{:}A \vee B \in \mathfrak{B}(x, \Delta)$ if either $a = x$ or there exists a label b such that $\Delta \vdash R\,a\,b\,x$ is provable; (ii) $a{:}A \rightarrow B \in \mathfrak{B}(x, \Delta)$ if there exist labels b and c such that $\Delta \vdash R\,a\,b\,c$ is provable in S(B$^+$) and $c{:}B \in \mathfrak{B}(x, \Delta)$.

By extension, given a sequent $\Gamma, \Delta \vdash \Gamma'$, we say that Γ is *x-branching with respect to* Δ *in* S(B$^+$) if $a{:}A \in \mathfrak{B}(x, \Delta)$ for some $a{:}A \Subset_+ \Gamma$. ∎

By induction on the structure of proofs we can then prove:

Proposition 3.16 [*$\rightarrow$-disjunction property*] If there is an S(B$^+$)-proof of

$$\Gamma, \Delta \vdash \Gamma', x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2, \ldots, x{:}A_n \rightarrow B_n$$

where Γ is not x-branching with respect to Δ in S(B$^+$), then there exists an i, with $1 \leq i \leq n$, such that there is an S(B$^+$)-proof of

$$\Gamma, \Delta \vdash \Gamma', x{:}A_i \rightarrow B_i\,.$$ ∎

The intuition for this is that if in a proof of $\Gamma, \Delta \vdash \Gamma', x{:}A_1 \rightarrow B_1, \ldots, x{:}A_n \rightarrow B_n$ the multiset Γ is not x-branching with respect to Δ, then only one $x{:}A_i \rightarrow B_i$ (if any) will lead to axioms, i.e. will allow us to close the proof branch.[6]

The condition that Γ is not x-branching with respect to Δ in S(B$^+$) requires that in the proof there is neither an application of $\vee$L with the principal formula $a{:}C \vee D$ where $a = x$ or $\Delta \vdash R\,a\,b\,x$ is provable for some b, nor an application of $\rightarrow$L with principal formula $a{:}C \rightarrow D$ where $\Delta \vdash R\,a\,b\,x$ is provable for some b. Thus the condition requires that in the proof there is no implicit duplication of the lwffs $x{:}A_i \rightarrow B_i$.[7]

[5]Note that ($\vee$L) is the only rule that gives rise to 'dangerous' (for the satisfiability of the disjunction property) branchings in unlabelled sequent systems for intuitionistic logic. This is because the intuitionistic left implication rule (with built-in contraction) has the form

$$\frac{C \supset D, \Gamma \vdash C \quad D, \Gamma \vdash E}{C \supset D, \Gamma \vdash E} \ (\supset\mathrm{L})$$

and it therefore suffices to require that $D \in \mathcal{H}$.

[6]Note that choosing one $x{:}A_i \rightarrow B_i$ constitutes a backtracking point: if the chosen lwff does not allow us to close the branch, we must backtrack and try the other ones.

[7]Note that the implicit duplication of $x{:}A_i \rightarrow B_i$ resulting from an application of $\wedge$R does not affect the proposition. Neither does the implicit duplication resulting from an application of $\vee$L or $\rightarrow$L with principal formula $y{:}C \vee D$ or $y{:}C \rightarrow D$, where $\Delta \vdash R\,x\,y\,z$ or $\Delta \vdash R\,x\,z\,y$ is provable; this is because in a backwards proof this application of $\vee$L or $\rightarrow$L can only occur after an application of $\rightarrow$R with some $x{:}E \rightarrow F$ or some formula in Γ or Γ' as its principal formula.

Corollary 3.17 Every sequent $\vdash 0{:}Z$ provable in $\mathrm{S}(\mathrm{B}^+)$ has a proof in which there are no applications of ClR with principal formula $x{:}B \to C$. ■

The intuition for this result, which follows by permuting x-branching rules so that we can repeatedly apply Proposition 3.16, is that in doing so we replace explicit right contractions of $x{:}B \to C$ with implicit ones, i.e. with the duplications of parametric formulas in applications of $\vee$L or $\to$L with x-branching principal formulas.[8]

Explicit left contractions of $x{:}B \to C$ can similarly be replaced by implicit ones. Indeed, the $\to$-disjunction property is also useful for eliminating such contractions, and we conclude the sketch of the proof of Theorem 3.14 with an illustrative case. Suppose that both instances of $x{:}B \to C$ are introduced by applications of $\to$L and that the contraction has rank $j+1$, e.g.

$$
\dfrac{\dfrac{
\begin{array}{c}\Pi_9\\ \Delta \vdash\\ R\,x\,y\,z\end{array}
\quad
\begin{array}{c}
\dfrac{\begin{array}{c}\Pi_1\\ \Delta_1 \vdash\\ R\,x\,b\,c\end{array}\quad \begin{array}{c}\Pi_2\\ \Gamma_1,\Delta_1\vdash\\ \Gamma_1', b{:}B'\end{array}\quad\begin{array}{c}\Pi_3\\ c{:}C,\Gamma_1,\\ \Delta_1\vdash\Gamma_1''\end{array}}{x{:}B\to C,\Gamma_1,\Delta_1\vdash\Gamma_1'}\,{\to}\mathrm{L}\\ \Pi_4\\ x{:}B\to C,\Gamma,\\ \Delta\vdash\Gamma',y{:}B
\end{array}
\quad
\begin{array}{c}
\dfrac{\begin{array}{c}\Pi_5\\ \Delta_2 \vdash\\ R\,x\,d\,e\end{array}\quad \begin{array}{c}\Pi_6\\ \Gamma_2,\Delta_2\vdash\\ \Gamma_2', d{:}B\end{array}\quad\begin{array}{c}\Pi_7\\ e{:}C,\Gamma_2,\\ \Delta_2\vdash\Gamma_2'\end{array}}{x{:}B\to C,\Gamma_2,\Delta_2\vdash\Gamma_2'}\,{\to}\mathrm{L}\\ \Pi_8\\ z{:}C, x{:}B\to C,\\ \Gamma,\Delta\vdash\Gamma'
\end{array}
}{x{:}B\to C, x{:}B\to C,\Gamma,\Delta\vdash\Gamma'}\,{\to}\mathrm{L}}{\begin{array}{c}x{:}B\to C,\Gamma,\Delta\vdash\Gamma'\\ \Pi_0\\ \vdash 0{:}Z\end{array}}\,\mathrm{ClL}
\tag{3}
$$

where we write sequents on two lines to ease readability.

As we remarked for Lemma 3.3, we cannot, in general, reduce the rank of this ClL by permuting the uppermost applications of $\to$L over the rules below them. Indeed, the only difficult subcase of this case of the proof of Theorem 3.14 occurs when in Π_4 and/or in Π_8 there is an application of $\to$R whose active rwff is also active in one of the two uppermost applications of $\to$L displayed in (3). As an example, suppose that the highest rule application in Π_4 is an application of $\to$R with active rwff $R\,x\,b\,c$, i.e. with the same active rwff as the $\to$L immediately above Π_4. Then the middle branch of (3) is

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Delta_3, R\,x\,b\,c \vdash R\,x\,b\,c\end{array}\quad \begin{array}{c}\Pi_2\\ b{:}P_1,\Gamma_3,\Delta_3,R\,x\,b\,c\\ \vdash \Gamma_3', c{:}Q_1, b{:}B\end{array}\quad\begin{array}{c}\Pi_3\\ c{:}C, b{:}P_1,\Gamma_3,\Delta_3,R\,x\,b\,c\\ \vdash\Gamma_3', c{:}Q_1\end{array}}{x{:}B\to C, b{:}P_1,\Gamma_3,\Delta_3,R\,x\,b\,c\vdash\Gamma_3', c{:}Q_1}\,{\to}\mathrm{L}}{x{:}B\to C,\Gamma_3,\Delta_3\vdash\Gamma_3', x{:}P_1\to Q_1}\,{\to}\mathrm{R}\\ \Pi_{10}\\ x{:}B\to C,\Gamma,\Delta\vdash\Gamma',y{:}B
\end{array}
\tag{4}
$$

with $\Gamma_1 = \{b{:}P_1\} \cup \Gamma_3$, $\Delta_1 = \Delta_3 \cup \{R\,x\,b\,c\}$ and $\Gamma_1' = \Gamma_3' \cup \{c{:}Q_1\}$, and the application of $\to$L does not permute over the application of $\to$R.

An analogous observation holds for the rightmost branch. If the highest rule application in Π_8 is one of $\to$R with the same active rwff $R\,x\,d\,e$ as the $\to$L immediately above Π_8, then the rightmost branch of (3) is

---

[8] As we remarked in the introduction, the clear distinction between explicit and implicit contractions is one of the differences between our decision procedure for B^+ and that given in [16] based on the connection method.

$$
\cfrac{\begin{array}{c}\Pi_5\\ \Delta_4, R\,x\,d\,e \vdash R\,x\,d\,e\end{array} \quad \begin{array}{c}\Pi_6\\ d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e\\ \vdash \Gamma_4', e{:}Q_2, d{:}B\end{array} \quad \begin{array}{c}\Pi_7\\ e{:}C, d{:}P_2, \Gamma_4, \Delta_4,\\ R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2\end{array}}{\cfrac{x{:}B \to C, d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2}{\begin{array}{c}x{:}B \to C, \Gamma_4, \Delta_4 \vdash \Gamma_4', x{:}P_2 \to Q_2\\ \Pi_{11}\\ z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'\end{array}} \to\mathrm{R}} \to\mathrm{L} \tag{5}
$$

with $\Gamma_2 = \{d{:}P_2\} \cup \Gamma_4$, $\Delta_2 = \Delta_4 \cup \{R\,x\,d\,e\}$, and $\Gamma_2' = \Gamma_4' \cup \{e{:}Q_2\}$, and we cannot permute the application of $\to$L over the application of $\to$R.

Suppose that these two situations occur simultaneously (which must not necessarily be the case, although we would then conclude analogously). That is, suppose that the proof has the form shown in (3) with (4) and (5) substituted for Π_4 and Π_8. Although we cannot permute the uppermost applications of $\to$L over the applications of $\to$R below them, we can permute these applications of $\to$R over the second, lowest, application of $\to$L in (3). This is possible because $x{:}P_1 \to Q_1$ and $y{:}B$, and $x{:}P_2 \to Q_2$ and $z{:}C$ are 'independent', in the sense that reasoning backwards it cannot be that $x{:}P_1 \to Q_1$ follows from (is a subformula of) $y{:}B$, or that $x{:}P_2 \to Q_2$ follows from $z{:}C$. In the middle branch, $x{:}P_1 \to Q_1$ must then follow from some formula in Γ or Γ', and Π_{10} can be divided into two separate subproofs, Π_{10}' and Π_{10}''; the first yields $y{:}B$, and $x{:}P_1 \to Q_1$ is active in the second. (If $y{:}B$ is already contained in Γ_3, then Π_{10}' is empty and $\Pi_{10}'' = \Pi_{10}$.) Similarly, in the rightmost branch, $x{:}P_2 \to Q_2$ must follow from some formula in Γ or Γ', and Π_{11} can be divided into two separate subproofs, Π_{11}' and Π_{11}''; the first yields $z{:}C$, and $x{:}P_2 \to Q_2$ is active in the second. (If $z{:}C$ is already contained in Γ_4, then Π_{11}' is empty and $\Pi_{11}'' = \Pi_{11}$.)

Given these separations, we can transform the proof (3) by (possibly) renaming some labels and permuting the applications of $\to$R downwards to obtain a proof with $rank(x{:}B \to C) \leq j$, and then conclude by the induction hypothesis. (A detailed example of this transformation can be found in [35].) Reasoning similarly and iterating these proof-transformations, we can eliminate all applications of ClL and thus conclude the proof of Theorem 3.14.

4 Complexity of Proof Search

We show the decidability of S(B$^+$) by combining bounds on contraction and monotony rules, which bound the number of labelled formulas that can appear in an S(B$^+$)-proof Π of $\vdash 0{:}Z$, with bounds on the number of labels and relational formulas that can be generated in Π. These bounds, combined with the soundness and completeness of S(B$^+$), tell us that the provability (validity) problem for the logic B$^+$ is decidable in $O(n \log n)$-space.

Let us begin by summarizing the above results, from which it follows that to prove $\vdash 0{:}Z$ in S(B$^+$) it suffices to search for a proof in the restriction of S(B$^+$) that consists of: the axiom AXl restricted to atomic lwffs and the axiom AXr; the logical rules for $\wedge$, $\vee$ and $\to$; the weakening rules; and the monotony rules

$$
\frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl} \qquad \text{and} \qquad \frac{\Delta \vdash R\,0\,x\,a \quad \Delta \vdash R\,a\,y\,z}{\Delta \vdash R\,x\,y\,z}\ \mathit{monr1},
$$

where by Lemmas 3.10 and 3.11 *monl* is such that A is atomic, $x{:}A$ is not introduced by *monl* and $R\,0\,x\,y \in \Delta$, and by Lemmas 3.8 and 3.9 *monr1* is such that $R\,0\,a\,x \in \Delta$ and $R\,x\,y\,z \in \Delta$.

When we embarked on our proof-theoretical analysis of $\mathrm{S(B^+)}$, we did so aiming at bounding the rules that do not simplify sequents, so that no infinite branches may arise and proof search terminates. The bounds that we established in § 3 allow us to define a measure that decreases at *almost* every rule application. Let us namely define the *degree of an lwff* $x{:}A$ to be twice the number of symbols in the string representation of A, i.e. $2 \times |A|$, and the *degree of a sequent* S to be the sum of the degrees of the lwffs in S plus the number of rwffs in S. Then we have:

Fact 4.1 In each rule of (the restriction of) $\mathrm{S(B^+)}$, other than *monl* and *monr1*, each premise has smaller degree than the conclusion. ■

The rules *monl* and *monr1* are exceptions because they do not reduce the number of connectives or rwffs. For these rules we must then reason differently. Although applications of *monr1* generate sequents of the same size as the conclusion, they do not generate new rwffs since only labels and rwffs currently in existence can be employed. Moreover, Lemma 3.9 tells us that there need not be two consecutive applications of *monr1*. Thus, the provability of $\Delta \vdash R\,x\,y\,z$ in $\mathrm{S(B^+)}$ reduces to checking either if $R\,x\,y\,z \in \Delta$, or, in the case of an application of *monr1*, if there is a label a such that $R\,0\,x\,a \in \Delta$ and $R\,a\,y\,z \in \Delta$. Both problems are solvable in space linear in the *size* of the end-sequent $S = \vdash 0{:}Z$, which we denote by $|S|$.

Applications of *monl* are similarly bounded, in that *monl* need not be applied twice to the same (atomic) formula, even though labelled differently, in the same branch.

As a consequence, in each branch of a proof of $\vdash 0{:}Z$, leading from the conclusion to an axiom, there can be at most one application of *monl* or of *monr1*, depending on whether the axiom is AXl or AXr. This, together with Fact 4.1, implies that every proof of a sequent $S = \vdash 0{:}Z$ is bounded in length by the degree of S, which is in turn linearly bounded in terms of its size $n = |S|$, plus the linear space required for applications of *monl* and *monr1*. That is, any proof of $S = \vdash 0{:}Z$ will have depth $O(n)$, where $n = |S|$.

In Fact 3.2 we observed that $\mathrm{S(B^+)}$ has a subformula property, so that when searching for a proof of $\vdash 0{:}Z$ we only need to look for sequents that contain subformulas of Z. However, subformulas may be labelled differently. While new rwffs can be generated by applications of $\rightarrow$ L or $\rightarrow$ R, new labels can only be generated by applications of $\rightarrow$R, since applications of $\rightarrow$L and *monl* label subformulas with different (currently in existence) labels. (Similarly, the active rwffs of *monr1* must already exist.) In particular, each application of $\rightarrow$R generates two new labels, so that the number of labels in a proof is linearly bounded above in any branch in terms of the number of $\rightarrow$ operators appearing in the end-sequent S, i.e. $O(n)$. Similarly, the number of lwffs and rwffs in any generated sequent is also $O(n)$.

These results give rise to a proof procedure that uses $O(n \log n)$-space. To illustrate this, we begin by distinguishing two different kinds of branching in the search space for proofs.

- *Conjunctive branching:* applying rules with multiple premises builds a branching tree, where all branches must be proved.
- *Disjunctive branching:* more than one rule may be applicable and a given rule may be applicable in different ways.

Conjunctive branching, caused by rules like $\wedge$R or $\rightarrow$L, leads to proofs that, if stored in their entirety, are exponential in size. Disjunctive branching arises when more than one rule may be applied to a sequent or when a single rule can be applied in different ways (e.g. a weakening). This is reflected by a branching point in the search space for proofs, rather than in proofs themselves.[9]

We base our decision procedure on the adaptation and extension of standard ideas, e.g. [18, 19]. Rather than storing entire proofs (branches), we store a sequent and a stack that maintains information sufficient to reconstruct branching points. Each stack entry is a triple: (i) the name of the rule applied, (ii) the principal formula of the sequent, and (iii) an index that records sufficient information such that on return to the branching point another branch, if one remains, can be generated. In other words, the index allows us to reconstruct the sequent associated with a branching point by replaying the stack entries to that point. For example, for $\rightarrow$L the index records which premise we choose, the first, second or third premise.

A proof begins with the end-sequent $S = \vdash 0{:}Z$ and the empty stack. Each rule application (we assume rules are ordered somehow and we apply them in order) generates a new sequent and appropriately extends the stack. There are two possibilities.

If the generated sequent is an axiom, and the stack contains no conjunctive branching points that still need to be explored, then S is provable. Otherwise we pop entries off the stack until we find a conjunctive branching point that must be further explored and then generate the next branch (first incrementing the index on the stack to record this).

Alternatively, if we arrive at a sequent that is not an axiom and no rule applies, then we pop stack entries and continue at the first available disjunctive branching point with the choices that remain, e.g. apply a different weakening, determined by the index, or apply the next available rule. If no such branching point remains, then S is not provable.

Note that $\rightarrow$L introduces, as well as conjunctive branching, also disjunctive branching: if the principal formula is $x{:}B \rightarrow C$, any labels x_1 and x_2 currently in existence are possible candidates for producing $x_1{:}B$ and $x_2{:}C$ in the generated sequents. We use the index to record which labels we choose so that on backtracking we can generate the next possible candidates; we can determine what the possible candidates are by looking into the stack and examining previous applications of $\rightarrow$R, WrL and *monr1*. Moreover, since $\rightarrow$L is not invertible, it might be that $\rightarrow$L simply does not lead to axioms. In this case, rules other than $\rightarrow$L must be tried on backtracking.

[9] Note that disjunctive branching is related to the invertibility of the rules, where we call a rule *invertible* when the provability of the conclusion implies the provability of the premise(s). By induction on the lengths of proofs it follows that all the rules of (the restricted system) $S(B^+)$ other than $\rightarrow$L, WrL, WlL, WlR, *monl* and *monr1* are invertible. However, if we are interested only in proofs of theorems $\vdash 0{:}Z$, then all rules other than $\rightarrow$L, WrL, WlL and WlR are invertible.

This procedure terminates and is complete. Termination follows because the stack depth (i.e. the proof depth) is $O(n)$ and the branching is bounded (by n, the number of rules, and the maximum number of premises). Completeness follows straightforwardly from the soundness and completeness of S(B^+) and the fact that the entire search space is navigated, if need be, on backtracking.

To compute the space requirement of the procedure, we simply need to sum the requirements of the stack and of generated sequents. (Strictly speaking, we should also add the space required to solve relational queries of the form $\Delta \vdash R\,x\,y\,z$, but we argued above that these queries reduce to a checking problem that is solvable in linear space.)

Storing the stack requires $O(n \log n)$-space. This is obtained by multiplying the stack depth, $O(n)$, with the space required by each stack entry, which is $O(log\,n)$-space. The latter bound results as follows. (i) The names of the rules require constant space. (ii) We can store each of the $O(n)$ labels using $O(log\,n)$ bits. Similarly, we can represent each of the $O(n)$ subformulas by an index into the end-sequent S, which requires only $O(log\,n)$ bits. Thus, storing the principal formula of each rule application requires $O(log\,n)$ bits. (iii) The index recording branching points requires $O(log\,n)$-space as well.

Above we argued that the bounds on the number of labels, lwffs and rwffs in any generated sequent are $O(n)$, like the bound on the depth of proofs. Since we can represent each of the $O(n)$ labels using $O(log\,n)$ bits, each of the $O(n)$ rwffs can be represented using $O(log\,n)$ bits as well. Since also each of the $O(n)$ lwffs can be represented using $O(log\,n)$ bits, it follows that any generated sequent requires $O(n \log n)$-space.

Thus, our decision procedure requires $O(n \log n)$-space, and since a formula Z of B^+ is provable iff the sequent $\vdash 0{:}Z$ is provable in S(B^+), we have:

Theorem 4.2 Provability for the basic positive relevance logic B^+ is decidable in $O(n \log n)$-space. ■

5 Conclusions and Future Work

We have given an $O(n \log n)$-space decision procedure for the basic positive relevance logic B^+ by extending the proof-theoretical method of [34] and showing how the results fit and refine the abstract recipe of [6]. More specifically, we first introduced a sound and complete, cut-free, labelled sequent system S(B^+), and then investigated and optimized S(B^+) to eliminate some of its rules or establish bounds on their applications.

This paper is part of ongoing research that aims at developing a framework for applying proof theory to the analysis of complexity of families of non-classical logics, and thereby designing decision procedures for these logics [4, 5, 6, 34]. As future work, we plan to use our proof-theoretical method to investigate the complexity of other relevance logics, and thereby try to solve some of the decidability problems that are still open in this area. Current work includes extending our approach to the basic relevance logic B, which we obtain by extending B^+ with negation, whose addition raises a number of interesting questions.

To begin with, since there are different ways of dealing with negation in relevance logics, e.g. [11, 12, 26, 27], we must first choose the one that best suits our needs. For example, in [5, 34] we gave natural deduction systems for relevance logics with negation by defining a unary function $*$ on worlds such that $\models a{:}\neg A$ iff $\not\models a^*{:}A$. (Informally, a^* is the world that does not deny what a asserts, i.e. a and a^* are 'compatible' worlds.) Assuming that we follow this choice, we are then faced with other decisions. For example, we could formalize the logic B in terms of the cut-free labelled sequent system that we obtain by extending S(B$^+$) with the logical rules for negation

$$\frac{\Gamma, \Delta \vdash \Gamma', x^*{:}A}{x{:}\neg A, \Gamma, \Delta \vdash \Gamma'}\ \neg\mathrm{L} \qquad \text{and} \qquad \frac{x^*{:}A, \Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}\neg A}\ \neg\mathrm{R}\,,$$

and with the relational rules

$$\frac{\Delta \vdash R\,0\,x\,y}{\Delta \vdash R\,0\,y^*\,x^*}\ \textit{antitonic}\,, \quad \frac{}{\vdash R\,0\,x\,x^{**}}\ {**}\textit{intuitionistic}\,, \quad \frac{}{\vdash R\,0\,x^{**}\,x}\ {**}\textit{classical}\,.$$

Or, alternatively, we could follow Priest and Sylvan [23] and Restall [24], and develop a cut-free labelled sequent system for B in terms of the *reduced semantics* they propose. This would, for example, require us to distinguish two pairs of logical rules for relevant implication, depending on whether the lwff $x{:}A \to B$ they introduce has label $x \neq 0$ or $x = 0$. That is,

$$\frac{\Delta \vdash R\,x\,y\,z \quad \Gamma, \Delta \vdash \Gamma', y{:}A \quad z{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \to B, \Gamma, \Delta \vdash \Gamma'}\ {\to}\mathrm{L}\ (x \neq 0)\,,$$

$$\frac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ {\to}\mathrm{R}\ (x \neq 0)\,,$$

$$\frac{\Gamma, \Delta \vdash \Gamma', a{:}A \quad a{:}B, \Gamma, \Delta \vdash \Gamma'}{0{:}A \to B, \Gamma, \Delta \vdash \Gamma'}\ {\to}\mathrm{L}_0\,, \qquad \frac{a{:}A, \Gamma, \Delta \vdash \Gamma', a{:}B}{\Gamma, \Delta \vdash \Gamma', 0{:}A \to B}\ {\to}\mathrm{R}_0\,,$$

where, in $\to$R, y and z are distinct and do not occur in $\Gamma, \Delta \vdash \Gamma', x{:}A \to B$, while $\to\mathrm{R}_0$ has the side condition that a does not occur in $\Gamma, \Delta \vdash \Gamma', 0{:}A \to B$.

This distinction on $x{:}A \to B$ has the effect of eliminating rwffs of the form $R\,0\,x\,y$, and thus also applications of *iden*. This has the advantage that it simplifies the solution of relational queries $\Delta \vdash R\,a\,b\,c$. However, it also has the disadvantage that it does not allow us anymore to define the partial order $\sqsubseteq$ in terms or R and 0, but requires us to introduce $\sqsubseteq$ explicitly, and reason explicitly about its properties. For example, *monl* becomes

$$\frac{\Delta \vdash x \sqsubseteq y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \textit{monl}\,,$$

and $\sqsubseteq$ is formalized by the rules

$$\frac{}{\vdash x \sqsubseteq x} \qquad \text{and} \qquad \frac{\Delta \vdash x \sqsubseteq y \quad \Delta \vdash y \sqsubseteq z}{\Delta \vdash x \sqsubseteq z}\ .$$

We are currently investigating all these possibilities for dealing with negation.

Future work will also involve the implementation of our decision procedure and the empirical evaluation of its time performance. To this end, observe that in [34] we show that our framework is such that the resulting labelled natural deduction systems can be straightforwardly implemented in the generic theorem prover Isabelle [21]; a similar observation applies for labelled sequent systems. Our Isabelle encodings provide a simple and natural environment for interactive proof development that supports hierarchical structuring, and we plan to implement our decision procedures to automate theorem proving for B$^+$ and other non-classical logics.

Acknowledgements

I thank David Basin, Wilfried Sieg and the anonymous referees for their helpful comments.

References

[1] A. R. Anderson and N. D. Belnap, Jr. *Entailment, The Logic of Relevance and Necessity* I. Princeton University Press, 1975.

[2] A. R. Anderson, N. D. Belnap, Jr., and J. M. Dunn. *Entailment, The Logic of Relevance and Necessity* II. Princeton University Press, 1992.

[3] D. Basin, M. D'Agostino, D. M. Gabbay, S. Matthews, and L. Viganò, eds. *Labelled Deduction*. Kluwer, 2000.

[4] D. Basin, S. Matthews, and L. Viganò. A new method for bounding the complexity of modal logics. In G. Gottlob, A. Leitsch, and D. Mundici, eds., *Proceedings of KGC'97*, LNCS 1289, pages 89–102. Springer, 1997.

[5] D. Basin, S. Matthews, and L. Viganò. Natural deduction for non-classical logics. *Studia Logica*, 60(1):119–160, 1998.

[6] D. Basin and L. Viganò. A recipe for the complexity analysis of non-classical logics. In D. M. Gabbay and M. de Rijke, eds., *Frontiers of Combining Systems 2*, pages 57–75. Research Studies Press/Wiley, 2000.

[7] R. T. Brady. Gentzenization and decidability of some contraction-less relevant logics. *Journal of Philosophical Logic*, 20:97–117, 1991.

[8] M. D'Agostino and D. M. Gabbay. A generalization of analytic deduction via labelled deductive systems. Part I : basic substructural logics. *Journal of Automated Reasoning*, 13:243–281, 1994.

[9] M. D'Agostino, D. M. Gabbay, and K. Broda. Tableau methods for substructural logics. In D'Agostino et al. [10], pages 397–467.

[10] M. D'Agostino, D. M. Gabbay, R. Hähnle, and J. Posegga, editors. *Handbook of Tableau Methods*. Kluwer, 1999.

[11] K. Došen. Negation as a modal operator. *Reports on Mathematical Logic*, 20:15–27, 1986.

[12] J. M. Dunn. Star and perp: Two treatments of negation. In J. Tomberlin, ed., *Philosophical Perspectives* VII, pages 331–357. Ridgeview, 1994.

[13] M. Fitting. *Proof methods for modal and intuitionistic logics*. Kluwer, 1983.

[14] D. M. Gabbay. *Labelled Deductive Systems* I. Clarendon Press, 1996.

[15] S. Giambrone. TW_+ and RW_+ are decidable. *Journal of Philosophical Logic*, 14:235–254, 1985.

[16] P. Gochet, P. Gribomont, and D. Rossetto. Algorithms for relevant logic. *Logique et Analyse*, 150-151-152:329–346, 1995.

[17] R. P. Goré. Tableau methods for modal and temporal logics. In D'Agostino et al. [10], pages 297–396.

[18] J. Hudelmaier. A $O(n\ log\ n)$-space decision procedure for intuitionistic propositional logic. *Journal of Logic and Computation*, 3(1):63–75, 1993.

[19] J. Hudelmaier. Improved decision procedures for the modal logics K, T and S4. In H. Kleine Büning, ed., *Proceedings of CSL'95*, LNCS 1092, pages 320–334. Springer, 1996.

[20] E. Orlowska. Relational proof system for relevant logics. *Journal of Symbolic Logic*, 57(4):1425–1440, 1992.

[21] L. C. Paulson. *Isabelle: a generic theorem prover.* LNCS 828. Springer, 1994.

[22] D. Prawitz. *Natural deduction.* Almqvist and Wiksell, 1965.

[23] G. Priest and R. Sylvan. Simplified semantics for basic relevant logics. *Journal of Philosophical Logic*, 21:217–232, 1992.

[24] G. Restall. Simplified semantics for relevant logics (and some of their rivals). *Journal of Philosophical Logic*, 22:481–511, 1993.

[25] G. Restall. Displaying and deciding substructural logics 1: Logics with contraposition. *Journal of Philosophical Logic*, 27(2):179–216, 1998.

[26] G. Restall. *An Introduction to Substructural Logics.* Routledge, 1999.

[27] G. Restall. Negation in relevant logics (how I stopped worrying and learned to love the Routley star). In D. M. Gabbay and H. Wansing, eds., *What is negation?*, pages 53–76. Kluwer, 1999.

[28] R. Routley, V. Plumwood, R. K. Meyer, and R. T. Brady. *Relevant logics and their rivals.* Ridgeview, 1982.

[29] A. S. Troelstra and H. Schwichtenberg. *Basic proof theory.* Cambridge University Press, 1996.

[30] A. Urquhart. The undecidability of entailment and relevant implication. *Journal of Symbolic Logic*, 49(4):1059–1073, 1984.

[31] A. Urquhart. The complexity of decision procedures in relevance logic. In J. M. Dunn and A. Gupta, eds., *Truth or Consequences*, pages 61–76. Kluwer, 1990.

[32] A. Urquhart. The complexity of decision procedures in relevance logic II. Manuscript, 1997.

[33] D. van Dalen. Intuitionistic logic. In D. M. Gabbay and F. Guenthner, eds., *Handbook of Philosophical Logic* III, pages 225–339. Reidel, 1986.

[34] L. Viganò. *Labelled Non-Classical Logics.* Kluwer, 2000.

[35] L. Viganò. An $O(n \log n)$-space decision procedure for the relevance logic B^+ (extended version). Technical Report No. 140, May 2000, Institut für Informatik, Albert-Ludwigs-Universität Freiburg, Germany. Available at http://www.informatik.uni-freiburg.de/tr/.

[36] L. Wallen. *Automated deduction in non-classical logics.* MIT Press, 1990.

[37] J. J. Zeman. *Modal logic.* Oxford University Press, 1973.