Zap Q-Learning

Fastest Convergent Q-Learning

Vistas in Control

.IfA AUTOMATIC CONTROL LABORATORY

50th birthday !

September 10-11, 2018

Sean Meyn

COGNITION & CONTROL IN COMPLEX SYSTEMS

Department of Electrical and Computer Engineering — University of Florida

Based on joint research with Adithya M. Devraj and Ana Bušić + ...

Thanks to to the National Science Foundation

Background and Goals

Reinforcement Learning for Control Scientists

Approximate Dynamic Programming

Seminal paper, Watkins & Dayan, Q-learning, 1992:

compute optimal policy for an MDP

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Many papers since, and rich theory developed by Tsitsiklis et. al.

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Magic ingredient may look familiar ...

Consider the discounted cost optimal control problem:

$$\frac{d}{dt} x_t = f(x_t, u_t)\,, \qquad J^*(x) = \min_{\boldsymbol{u}} \int_0^\infty e^{-\gamma t} c(x_t, u_t)\, dt$$

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Magic: Hamiltonian and HJB equation

$$\min_u \{\underbrace{c(x,u) + f(x,u)\cdot \nabla J^*\,(x)}_{\text{Q-function}}\} = \gamma J^*(x)$$

Often easier to estimate Q^* rather than the value function J^*

Q-learning in control language

System: $\dot{x} = f(x,u)$ cost: $c(x,u)$

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$$Q^*(x,u) = c(x,u) + f(x,u) \cdot \nabla J^*(x)$$

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Denote $\underline{Q}^* = \min_u Q^*(x,u) = \gamma J^*(x)$

$\Longrightarrow$ Fixed point equation for Q-function

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Parameterization set of approximations, $\{Q^\theta(x,u) : \theta \in \mathbb{R}^d\}$

Bellman error: $\mathcal{E}^\theta(x,u) = \gamma[Q^\theta(x,u) - c(x,u)] - f(x,u)\cdot\nabla \underline{Q}^\theta(x)$

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Apply Magic:

$$
\begin{aligned}
\mathcal{E}^\theta(x,u) &= \gamma[Q^\theta(x,u) - c(x,u)] - f(x,u)\cdot\nabla \underline{Q}^\theta\,(x) \\
&= \gamma[Q^\theta(x,u) - c(x,u)] - \frac{d}{dt}\underline{Q}^\theta\,(x_t)\Big|_{x=x(t),\ u=u(t)}
\end{aligned}
$$

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Apply Magic: Model Free Error Representation

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Q-learning and Quasi Stochastic Approximation

- Find zeros of $\overline{h}(\theta) = \nabla \mathsf{E}[\mathcal{E}^\theta(x_\infty, u_\infty)^2]$ using QSA
- (x_∞, u_∞) ergodic steady-state.
- Choose input: stable feedback + mixture of sinusoids,
$$u(t) = -k(x(t)) + \omega(t)$$

[9, 8]

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Alert! Approximation based on online input-output measurements!

Q-learning in practice

Questions to be addressed

- How to select function class $\{Q^\theta\}$?

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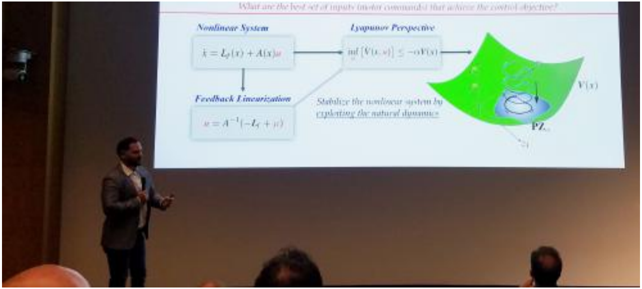

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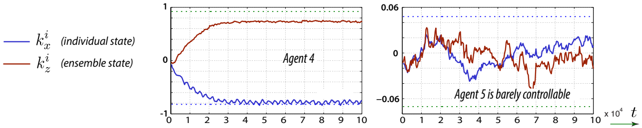
Feature selection for neuro-dynamic programming, 2011 [8]



Mean-field game used for basis construction for Q-learning
Resulting estimates are consistent with MFG solution

Q-learning in practice

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- How to make an algorithm work?

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- How to make an algorithm work?

Let's get started ...

$$\boxed{\mathsf{E}[f(\theta, W)]\Big|_{\theta=\theta^*} \quad = \quad 0}$$

Stochastic Approximation

What is Stochastic Approximation?

A simple goal: Find the solution θ^* to

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1. The function f and the distribution of the random vector W may not be known
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1. The function f and the distribution of the random vector W may not be known
2. Even if everything is known, computation of the expectation may be expensive. For root finding, we may need to compute the expectation for many values of θ
3. Motivates stochastic approximation: $\theta(n+1) = \theta(n) + \alpha_n f(\theta(n), W(n))$
   The recursive algorithms we come up with are often slow, and their variance is often **infinite**

Algorithm and Convergence Analysis

Algorithm:

$$\theta(n+1) = \theta(n) + \alpha_n f(\theta(n), W(n))$$

Goal:

$$\bar{f}(\theta^*) := \mathsf{E}[f(\theta, W)]\Big|_{\theta=\theta^*} = 0$$

Interpretation: $\theta^* \equiv$ *stationary point* of the ODE

$$\frac{d}{dt}\theta(t) = \bar{f}(\theta(t))$$

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Analysis: Stability of the ODE $\oplus$ (See Borkar's monograph) $\Longrightarrow$

$$\lim_{n\to\infty} \theta(n) = \theta^*$$

Stochastic Approximation Example

Example: Monte-Carlo

Monte-Carlo Estimation

Estimate the mean $\eta = \mathsf{E}[c(X)]$, where random variable X has density ϱ:

$$\eta = \int c(x)\, \varrho(x)\, dx$$

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Estimate the mean $\eta = \mathsf{E}[c(X)]$

SA interpretation: Find θ^* solving $0 = \mathsf{E}[f(\theta, X)] = \mathsf{E}[c(X) - \theta]$

$$\textit{Algorithm:} \quad \theta(n) = \frac{1}{n}\sum_{i=1}^{n} c(X(i))$$

Stochastic Approximation Example

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$\sum \alpha_n = \infty, \sum \alpha_n^2 < \infty$

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$$\textit{Algorithm:} \quad \theta(n) = \frac{1}{n}\sum_{i=1}^{n} c(X(i))$$

$$\Longrightarrow \qquad (n+1)\theta(n+1) = \sum_{i=1}^{n+1} c(X(i)) = n\theta(n) + c(X(n+1))$$

$$\Longrightarrow \qquad (n+1)\theta(n+1) = (n+1)\theta(n) + [c(X(n+1)) - \theta(n)]$$

SA Recursion: $\qquad \theta(n+1) = \theta(n) + \alpha_n f(\theta(n), X(n+1))$

Performance Criteria

Two standard approaches to evaluate performance, $\tilde{\theta}(n) := \theta(n) - \theta^*$:

1. Finite-n bound:

$$\mathsf{P}\{\|\tilde{\theta}(n)\| \geq \varepsilon\} \leq \exp(-I(\varepsilon, n)), \qquad I(\varepsilon, n) = O(n\varepsilon^2)$$

2. Asymptotic covariance:

$$\Sigma = \lim_{n\to\infty} n\mathsf{E}\Big[\tilde{\theta}(n)\tilde{\theta}(n)^{\intercal}\Big], \qquad \sqrt{n}\tilde{\theta}(n) \approx N(0, \Sigma)$$

Asymptotic Covariance

$$\Sigma = \lim_{n\to\infty} \Sigma_n = \lim_{n\to\infty} n\mathsf{E}\big[\tilde\theta(n)\tilde\theta(n)^{\intercal}\big]\,, \qquad \sqrt{n}\tilde\theta(n) \approx N(0,\Sigma)$$

SA recursion for covariance:

$$\Sigma_{n+1} \approx \Sigma_n + \tfrac{1}{n}\Big\{(A+\tfrac{1}{2}I)\Sigma_n + \Sigma_n(A+\tfrac{1}{2}I)^{\intercal} + \Sigma_\Delta\Big\}$$

$$A = \tfrac{d}{d\theta}\bar f\,(\theta^*)$$

Conclusions

1. If $\mathrm{Re}\,\lambda(A) \ge -\frac{1}{2}$ for some eigenvalue then Σ is (typically) infinite
2. If $\mathrm{Re}\,\lambda(A) < -\frac{1}{2}$ for all, then $\Sigma = \lim_{n\to\infty}\Sigma_n$ is the unique solution to the Lyapunov equation:

$$\boxed{0 = (A+\tfrac{1}{2}I)\Sigma + \Sigma(A+\tfrac{1}{2}I)^{\intercal} + \Sigma_\Delta}$$

Optimal Asymptotic Covariance

Introduce a $d \times d$ matrix gain sequence $\{G_n\}$:

$$\theta(n+1) = \theta(n) + \frac{1}{n+1} G_n f(\theta(n), X(n))$$

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If $G = G^* := -A^{-1}$ then

- Resembles Monte-Carlo estimate
- Resembles Newton-Rapshon
- It is optimal: $\Sigma^* = G^* \Sigma_\Delta G^{*\,T} \leq \Sigma^G$ *any other* G

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Polyak-Ruppert averaging is also optimal, but first two bullets are missing.

Optimal Variance

Example: return to Monte-Carlo

$$\theta(n+1) = \theta(n) + \frac{g}{n+1}\Big(-\theta(n) + X(n+1)\Big)$$

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$$\Delta(n) = X(n) - \mathsf{E}[X(n)]$$

Optimal Variance

Normalization for analysis:

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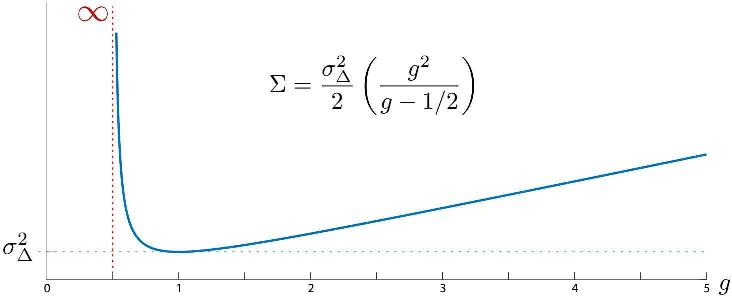

Asymptotic variance as a function of g

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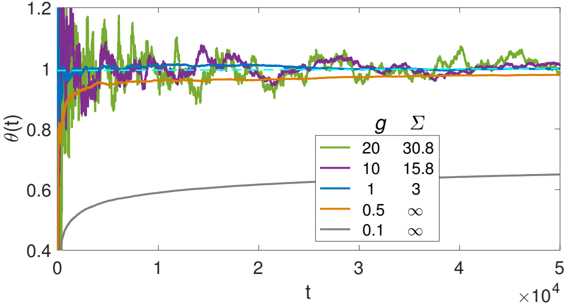

SA estimates of $\mathsf{E}[W^2]$, $W \sim N(0,1)$

Optimal Asymptotic Covariance and Zap-SNR

Zap-SNR (designed to emulate deterministic Newton-Raphson)

$$\text{Requires} \quad \widehat{A}_n \approx A(\theta_n) := \frac{d}{d\theta} \bar{f}\,(\theta_n)$$

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Always: $\alpha_n = 1/n$. Numerics that follow: $\gamma_n = (1/n)^\rho$, $\rho \in (0.5, 1)$

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ODE for Zap-SNR

$$\frac{d}{dt} x_t = -\big[A(x_t)\big]^{-1} \bar{f}(x_t), \qquad A(x) = \frac{d}{dx} \bar{f}(x)$$

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$$\frac{d}{dt} x_t = -\big[A(x_t)\big]^{-1} \bar{f}(x_t), \qquad A(x) = \frac{d}{dx} \bar{f}(x)$$

- Not necessarily stable (just like in deterministic Newton-Raphson)
- *General* conditions for convergence open

Detailed Comments:	1. In general, it is not clear how techniques from stochastic approximation can add to the literature of reinforcement learning. SA is concerned with stability and asymptotics, while RL is concerned with the efficiency of learning (sample complexity and regret). In general,I cannot convince myself why ppl care about the stability/asymptotic of SA in the context of RL. I think more justification is needed to bring together the theory of SA and RL.

Reinforcement Learning and Stochastic Approximation

condition Q1 in Theorem 1: (X,U) is an irreducible Markov chain. This assumption excludes the possibility of policy exploration and policy adaptation, which is key to RL. Under this theoretical limitation, the proposed method and analysis does not apply to general RL, making the stability/asymptotic results less interesting.

3. One contribution claimed in the paper is variance reduction. It seems that no theoretical justification is provided about how much is the variance reduced? Is it related to any condition

Reinforcement Learning and Stochastic Approximation

SA and RL Design

Functional equations in Stochastic Control

Always of the form

$$0 = \mathsf{E}[F(h^*, \Phi(n+1)) \mid \Phi_0 \ldots \Phi(n)]\,, \qquad h^* = ?$$

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$$\Phi(n) = (\text{state, action})$$

SA and RL Design

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Galerkin relaxation:

$0 = \mathsf{E}[F(h^{\theta^*}, \Phi(n+1))\zeta_n]\,, \qquad \theta^* = ?$

SA and RL Design

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Necessary Ingredients:

- Parameterized family $\{h^\theta : \theta \in \mathbb{R}^d\}$
- Adapted, d-dimensional stochastic process $\{\zeta_n\}$

Examples are TD- and Q-Learning

SA and RL Design

Functional equations in Stochastic Control

Always of the form

$$0 = \mathsf{E}[F(h^*, \Phi(n+1)) \mid \Phi_0 \dots \Phi(n)] , \qquad h^* = ?$$

Galerkin relaxation:

$$0 = \mathsf{E}[F(h^{\theta^*}, \Phi(n+1))\zeta_n] , \qquad \theta^* = ?$$

Necessary Ingredients:

- Parameterized family $\{h^\theta : \theta \in \mathbb{R}^d\}$
- Adapted, d-dimensional stochastic process $\{\zeta_n\}$

Examples are TD- and Q-Learning

These algorithms are thus special cases of stochastic approximation

(as we all know)

Stochastic Optimal Control

MDP Model

$\boldsymbol{X}$ is a stationary controlled Markov chain, with input $\boldsymbol{U}$

- For all states x and sets A,

  $\mathsf{P}\{X(n+1) \in A \mid X(n) = x,\ U(n) = u, \text{and prior history}\} = P_u(x, A)$
- $c\colon \mathsf{X} \times \mathsf{U} \to \mathbb{R}$ is a cost function
- $\beta < 1$ a discount factor

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Value function:

$$h^*(x) = \min_{\boldsymbol{U}} \sum_{n=0}^{\infty} \beta^n \mathsf{E}[c(X(n), U(n)) \mid X(0) = x]$$

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Bellman equation:

$$h^*(x) = \min_u \{c(x,u) + \beta \mathsf{E}[h^*(X(n+1)) \mid X(n) = x,\ U(n) = u]\}$$

Q-function

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Q-function: trick to swap expectation and minimum

Q-Learning and Galerkin Relaxation

Dynamic programming

Find function Q^* that solves

$$\mathsf{E}\Big[c(X(n),U(n)) + \beta \underline{Q}^*(X(n+1)) - Q^*(X(n),U(n)) \mid \mathcal{F}_n\Big] = 0$$

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That is,

$$0 = \mathsf{E}[F(Q^*, \Phi(n+1)) \mid \Phi_0 \ldots \Phi(n)]\,,$$

$$\text{with } \Phi(n+1) = (X(n+1), X(n), U(n)).$$

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Q-Learning

Find θ^* that solves

$$\mathsf{E}\Big[\Big(c(X(n),U(n)) + \beta \underline{Q}^{\theta^*}((X(n+1)) - Q^{\theta^*}((X(n),U(n))\Big)\zeta_n\Big] = 0$$

The family $\{Q^\theta\}$ and *eligibility vectors* $\{\zeta_n\}$ are part of algorithm design.

Watkins' Q-learning

Find θ^* that solves

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Watkin's algorithm is Stochastic Approximation

The family $\{Q^\theta\}$ and *eligibility vectors* $\{\zeta_n\}$ in this design:

- Linearly parameterized family of functions: $Q^\theta(x,u) = \theta^{\intercal}\psi(x,u)$
- $\zeta_n \equiv \psi(X_n, U_n)$
- $\psi_i(x,u) = 1\{x = x^i, u = u^i\}$ (complete basis)

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Asymptotic covariance is infinite for $\beta \geq 1/2$ [NIPS 2017]

Watkins' Q-learning

Big Question: *Can we Zap Q-Learning?*

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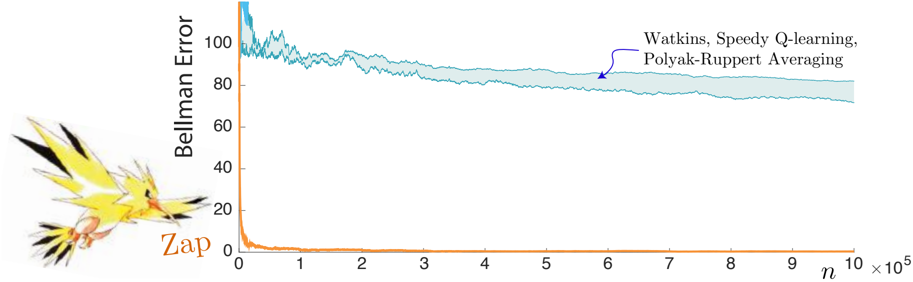
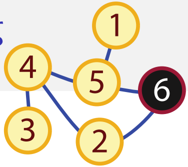


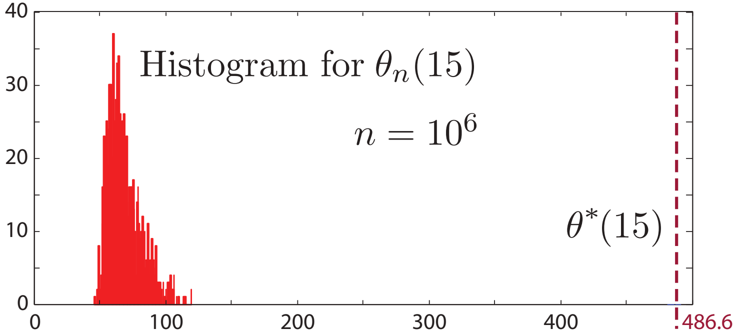
Zap Q-Learning

Asymptotic Covariance of Watkins' Q-Learning

Improvements are needed!



Histogram of parameter estimates after 10^6 iterations.



Example from Devraj & M 2017

Zap Q-learning

Zap Q-Learning $\equiv$ Zap-SNR for Q-Learning

$$0 = \bar{f}(\theta) = \mathsf{E}\big[f(\theta, W(n))\big]$$
$$:= \mathsf{E}\Big[\zeta_n\Big(c(X(n), U(n)) + \beta \underline{Q}^\theta(X(n+1)) - Q^\theta(X(n), U(n))\Big)\Big]$$

$A(\theta) = \frac{d}{d\theta}\bar{f}(\theta)$; At points of differentiability:

$$A(\theta) = \mathsf{E}\Big[\zeta_n\big[\beta\psi(X(n+1), \phi^\theta(X(n+1))) - \psi(X(n), U(n))\big]^T\Big]$$
$$\phi^\theta(X(n+1)) := \arg\min_u Q^\theta(X(n+1), u)$$

Zap Q-learning

Zap Q-Learning $\equiv$ Zap-SNR for Q-Learning

Algorithm:

$$\theta(n+1) = \theta(n) + \alpha_n (-\widehat{A}_n)^{-1} f(\theta(n), \Phi(n)) \,, \qquad \widehat{A}_n = \widehat{A}_{n-1} + \gamma_n (A_n - \widehat{A}_{n-1})$$

$$\begin{aligned} A_{n+1} &:= \frac{d}{d\theta} f\left(\theta_n, \Phi(n)\right) \\ &= \zeta_n \big[\beta\psi(X(n+1), \phi^{\theta_n}(X(n+1))) - \psi(X(n), U(n))\big]^{T} \end{aligned}$$

Zap Q-learning

Zap Q-Learning $\equiv$ Zap-SNR for Q-Learning

ODE Analysis: change of variables $q = \mathcal{Q}^*(\varsigma)$

Functional $\mathcal{Q}^*$ maps cost functions to Q-functions:

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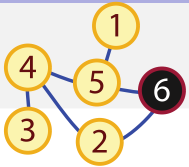
ODE for Zap-Q

$$q_t = \mathcal{Q}^*(\varsigma_t), \qquad \frac{d}{dt}\varsigma_t = -\varsigma_t + c$$

$\Rightarrow$ convergence, optimal covariance, ...

Zap Q-Learning

Example: Stochastic Shortest Path



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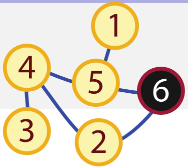
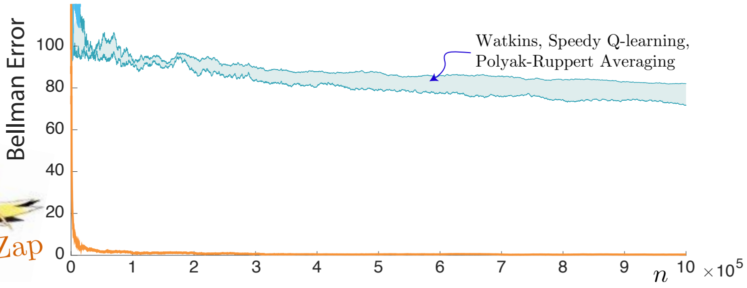


Convergence with Zap gain $\gamma_n = n^{-0.85}$

Watkins' algorithm has infinite asymptotic covariance with $\alpha_n = 1/n$



Convergence of Zap-Q Learning

Discount factor: $\beta = 0.99$

Zap Q-Learning

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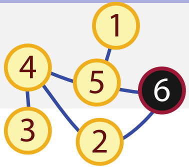

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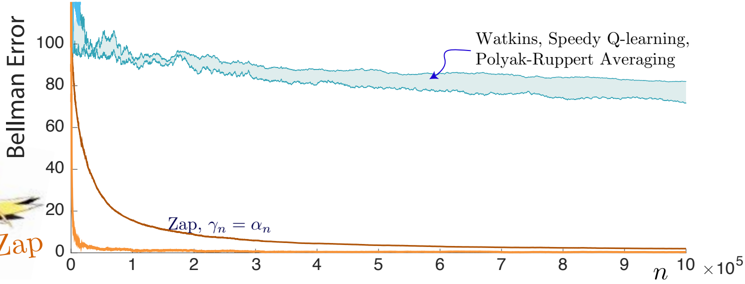

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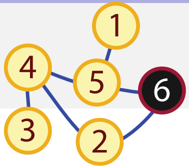
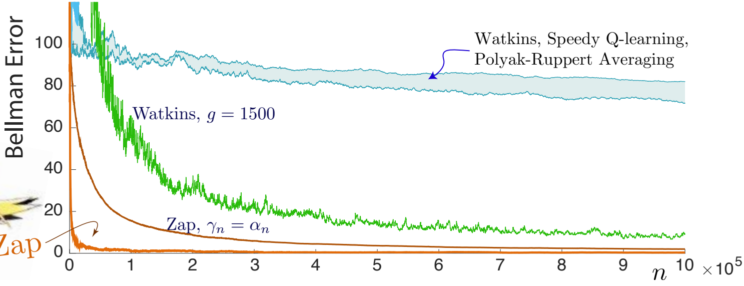


Convergence with Zap gain $\gamma_n = n^{-0.85}$

Watkins' algorithm has infinite asymptotic covariance with $\alpha_n = 1/n$

Optimal scalar gain is approximately $\alpha_n = 1500/n$



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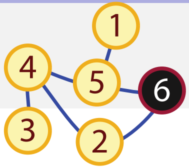
Zap Q-Learning

Optimize Walk to Cafe



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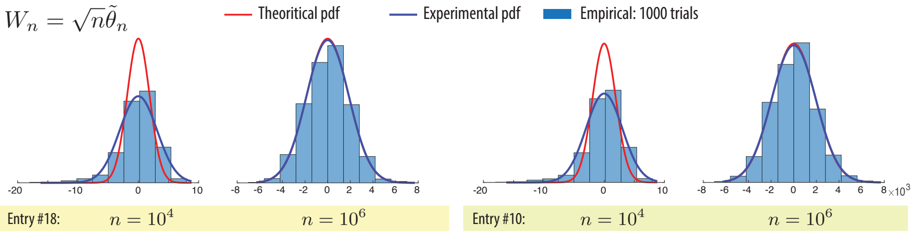

CLT gives good prediction of finite-n performance

Zap Q-Learning

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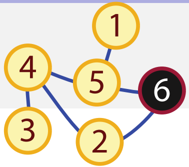
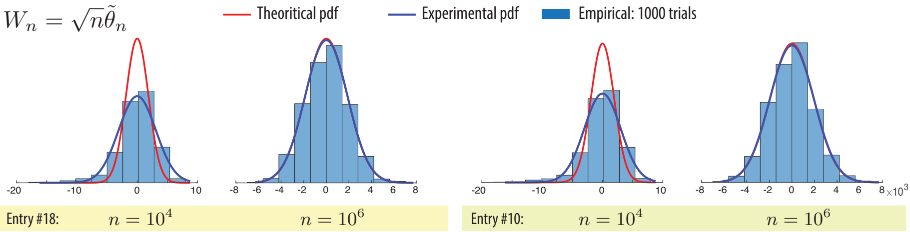


Convergence with Zap gain $\gamma_n = n^{-0.85}$

$W_n = \sqrt{n}\tilde{\theta}_n$



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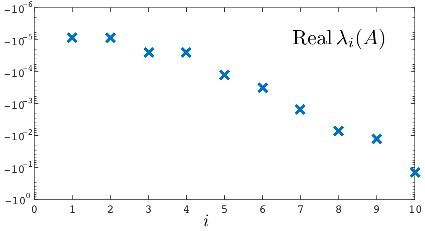
Zap Q-Learning

Model of Tsitsiklis and Van Roy: **Optimal Stopping Time in Finance**

State space: $\mathbb{R}^{100}$

Parameterized Q-function: Q^θ with $\theta \in \mathbb{R}^{10}$



$\mathrm{Real}\, \lambda > -\frac{1}{2}$ for every eigenvalue λ

Asymptotic covariance is infinite

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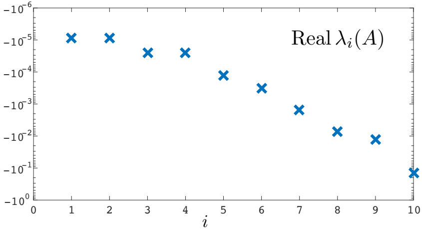

Real $\lambda > -\frac{1}{2}$ for every eigenvalue λ

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Authors observed slow convergence
Proposed a matrix gain sequence

$\{G_n\}$ (see refs for details)

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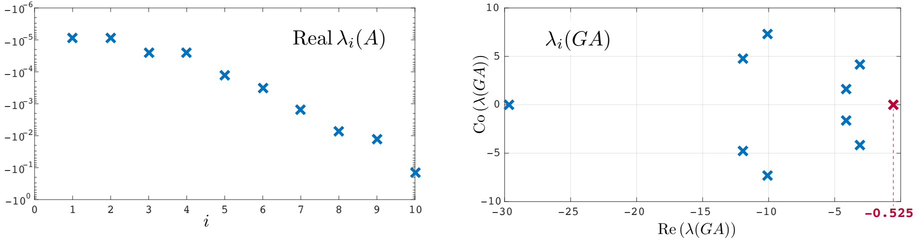

Eigenvalues of A and GA for the finance example

Favorite choice of gain in [25] barely meets the criterion $\mathrm{Re}(\lambda(GA)) < -\frac{1}{2}$

Zap Q-Learning

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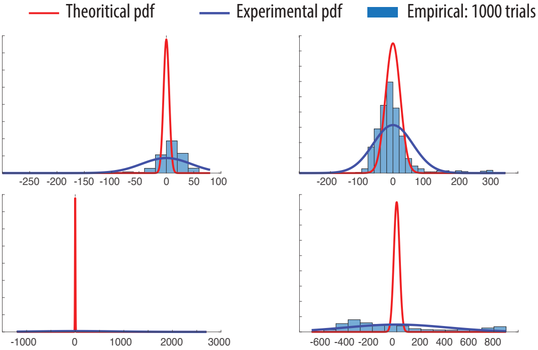
Zap-Q

G-Q



Entry #1: $n = 2 \times 10^6$

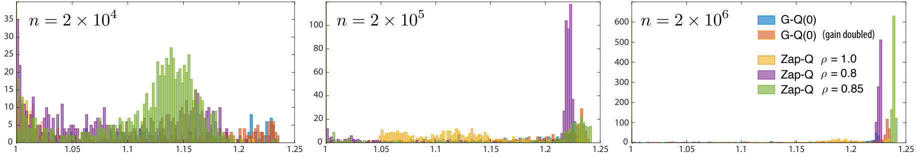
Entry #7: $n = 2 \times 10^6$

Zap Q-Learning

Model of Tsitsiklis and Van Roy: **Optimal Stopping Time in Finance**

State space: $\mathbb{R}^{100}$.
Parameterized Q-function: Q^θ with $\theta \in \mathbb{R}^{10}$

Histograms of the average reward obtained using the different algorithms:



Zap-Q $\gg$ G-Q

Conclusions & Future Work

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  We need better design tools to improve performance

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  Example: $g^* = 1500$ was chosen based on **asymptotic** covariance

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    - *Optimal stopping time problems*
- Adaptive optimization of algorithm parameters

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- Zapped Momentum Methods [2]

Conclusions & Future Work

Opportunities for the controls community

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We Are Q!

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$$\frac{d}{dt}x_t = f(x_t, u_t)\,, \quad J^*(x) = \min_{\boldsymbol{u}} \int_0^\infty c(x_t, u_t)\,dt\,,\ x_0 = x$$

Conclusions & Future Work

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Forget about Markov chains and randomized policies!

Follow your heart

Conclusions & Future Work

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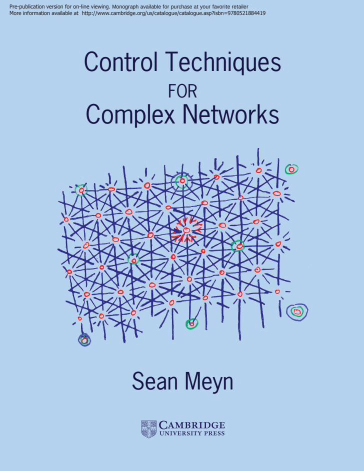
Follow your heart

Åström, Cassandras, Jovanovic, Jain, Kristic, friends@NREL ... *we are Q!*

Thank you!


Pre-publication version for on-line viewing. Monograph available for purchase at your favorite retailer
More information available at http://www.cambridge.org/us/catalogue/catalogue.asp?isbn=9780521884419
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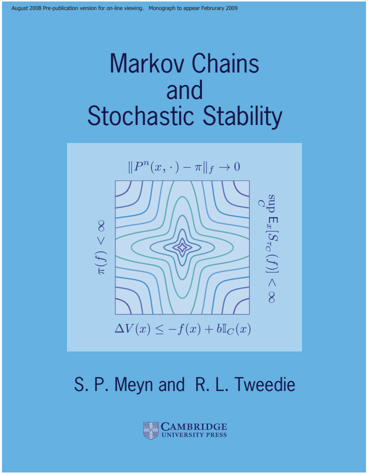
August 2008 Pre-publication version for on-line viewing. Monograph to appear Februrary 2009
Markov Chains
and
Stochastic Stability
$\|P^n(x,\cdot)-\pi\|_f \to 0$
$\pi(f) < \infty$
$\sup_C \mathsf{E}_x[S_{\tau_C}(f)] < \infty$
$\Delta V(x) \le -f(x) + b\mathbb{I}_C(x)$
S. P. Meyn and R. L. Tweedie
CAMBRIDGE
UNIVERSITY PRESS

References

This lecture

- A. M. Devraj and S. P. Meyn, *Zap Q-learning.* *Advances in Neural Information Processing Systems (NIPS).* Dec. 2017.
- A. M. Devraj and S. P. Meyn, *Fastest convergence for Q-learning.* Available on *ArXiv*. Jul. 2017.

Berkeley short course, March 2018

- Part I (Basics, with focus on variance of algorithms)

  https://www.youtube.com/watch?v=dhEF5pfYmvc
- Part II (Zap Q-learning)

  https://www.youtube.com/watch?v=Y3w8f1xIb6s

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