ETH zürich

Antenna Laboratory 2

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D-ITET

THEORY

Initially in this theory section we will have review of some of the important figures of merits, radiation integrals, approximations in the far field and detailed theory behind the dipole and horn antenna.

Figures of Merit

This section is a short repetition on some of the important antenna parameters later needed in the experiment part of the laboratory.

Directivity

The ***directivity*** of an antenna is defined as the ratio of the radiation intensity in a given direction of an antenna to the radiation intensity averaged over all directions. The average radiation intensity is equal to the total power radiated by the antenna divided by 4π. If the direction is not specified, the direction of maximum radiation intensity is implied. Mathematical formulated:

$$D_{max} = \frac{4\pi U_{max}}{P_{rad}} \qquad \text{Eq. 1}$$

With U_{max} the maximum radiation intensity and P_{rad} the total radiated power. For an isotropic source the directivity is thus equal to unity.

Half Power Beam Width

Additional to directivity, antennas are also characterized by their beam widths and side lobe levels. This concept is best explained with an example. Assume the radiations pattern:

$$R(\theta) = \sin\theta \frac{\sin\big(4(\theta - \pi/2)\big)}{4(\theta - \pi/2)} \qquad \text{Eq. 2}$$

This pattern is plotted in Figure 1.The main beam is the region around the direction of maximum radiation. The main beam in the figure is centered at 90 degrees. The side lobes are the smaller beams that are away from the main beam. The side lobe radiation is generally undesired but cannot always be eliminated. These side lobes are roughly at 45 and 135 degrees.

The ***Half Power Beam Width (HPBW)*** is the angular separation in which the magnitude of the radiation pattern decreases by -3dB. In Figure 1 this is at 77.7 and 102.3 degrees. Hence the HPBW is $102.3° - 77.7° = 24.6°$.

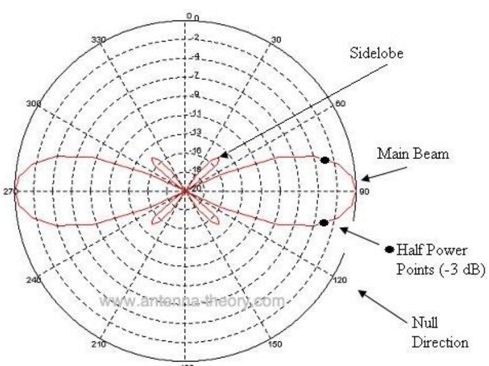

Figure 1: Radiation pattern for the Half Power Beam Width (HPBMW) example. The main lobe is at 90°. The two smaller generally undesired side lobes are also depicted. The two half power points are at 77.7° and 102.3°resulting in a HPBMW of 24.6°

Radiation

The usual procedure in the analysis of radiation problems consist in initially specifying the source and then calculate the fields radiated by these sources. In this section we will go through this procedure and develop some helpful methods in analyzing these electromagnetic fields.

Its common practice to introduce auxiliary functions, known as ***vector potentials*** which aid in the solving of the problems. Common vector potential function are the **A (*magnetic vector potential*)** and **F (*electric vector potential*).**

Magnetic vector potential

First we will introduce the magnetic vector potential **A** which will proof quite useful in solving radiation problems. The vector potential **A** is the result of a harmonic electric source current **J**.

Let's define **A** as:

$$\boldsymbol{H_A} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \qquad \text{Eq. 3}$$

We plug this into Maxwell's curl equation:

$$\boldsymbol{\nabla} \times \boldsymbol{E_A} = -j\omega\mu\boldsymbol{H_A} \qquad \text{Eq. 4}$$

$$\boldsymbol{\nabla} \times [\boldsymbol{E_A} + j\omega\boldsymbol{A}] = 0 \qquad \text{Eq. 5}$$

Using $\nabla \times (-\nabla\phi_e) = 0$:

$$\boldsymbol{E_A} = -\boldsymbol{\nabla}\phi_e - j\omega\boldsymbol{A} \qquad \text{Eq. 6}$$

With ϕ_e the electric scalar potential which is a function of position.

Taking the curl of Eq. 3 and using $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot A) - \boldsymbol{\nabla}^2\boldsymbol{A}$:

$$\boldsymbol{\nabla} \times \mu\boldsymbol{H_A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A} \qquad \text{Eq. 7}$$

Plugging in Maxwell's equation into (ref):

$$\boldsymbol{\nabla} \times \boldsymbol{H_A} = \boldsymbol{J} + j\omega\epsilon\boldsymbol{E_A} \qquad \text{Eq. 8}$$

$$\mu\boldsymbol{J} + j\omega\mu\epsilon\boldsymbol{E_A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot A) - \boldsymbol{\nabla}^2\boldsymbol{A} \qquad \text{Eq. 9}$$

Substituting $\boldsymbol{E_A}$ with Eq. 6

$$\boldsymbol{\nabla}^2\boldsymbol{A} + k^2\boldsymbol{A} = -\mu\boldsymbol{J} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A} + j\omega\mu\epsilon\phi_e) \qquad \text{Eq. 10}$$

With $k^2 = \omega^2\mu\epsilon$. We define the divergence of **A** as (*Lorentz condition*):

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = -j\omega\epsilon\mu\phi_e \qquad \text{Eq. 11}$$

Which finally leads to:

$$\nabla^2\boldsymbol{A} + k^2\boldsymbol{A} = -\mu\boldsymbol{J} \qquad \text{Eq. 12}$$

Once **A** is known, $\boldsymbol{H_A}$ and $\boldsymbol{E_A}$ can be calculated:

$$\boldsymbol{H_A} = \frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{A} \tag{Eq. 13}$$

$$\boldsymbol{E_A} = -\boldsymbol{\nabla}\phi_e - j\omega\boldsymbol{A} = -j\omega\boldsymbol{A} - j\frac{1}{\omega\mu\epsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) \tag{Eq. 14}$$

Electric vector potential

Although a magnetic current **M** appears to be physically unrealizable, equivalent magnetic currents arise when we use the volume or the surface equivalence theorems. Let's define the *electric vector potential* **F** by:

$$\boldsymbol{E_F} = -\frac{1}{\epsilon}\nabla \times \boldsymbol{F} \tag{Eq. 15}$$

By using Maxwell's equations we can again derive:

$$\boldsymbol{\nabla}^2\boldsymbol{F} + k^2\boldsymbol{F} = -\epsilon\boldsymbol{M} \tag{Eq. 16}$$

Once **F** is known we can similar to before calculate the electric and magnetic field due to the magnetic current:

$$\boldsymbol{H_F} = -j\omega\boldsymbol{F} - \frac{j}{\omega\mu\epsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) \tag{Eq. 17}$$

$$\boldsymbol{E_F} = -\frac{1}{\epsilon}\boldsymbol{\nabla} \times \boldsymbol{F} \tag{Eq. 18}$$

To summarize to procedure to find a field with the help of the vector potentials:

1. Specify **J** and **M** (electric and magnetic current density sources)
2. Find **A**(due to **J**): $\boldsymbol{A} = \frac{\mu}{4\pi}\iiint_V \boldsymbol{J}\frac{e^{-jkR}}{R}dv'$, which is a solution of the inhomogeneous vector wave equation and R the distance operation point to source.
3. Find **F**(due to **M**): $\boldsymbol{F} = \frac{\epsilon}{4\pi}\iiint_V \boldsymbol{M}\frac{e^{-jkR}}{R}dv'$
4. Find $\boldsymbol{H_A}, \boldsymbol{E_A}, \boldsymbol{E_F}$ and $\boldsymbol{H_F}$
5. The total fields are $\boldsymbol{E} = \boldsymbol{E_A} + \boldsymbol{E_F}$ and $\boldsymbol{H} = \boldsymbol{H_A} + \boldsymbol{H_F}$

Dipole Antennas

Infinitesimal dipole

Initially we will derivate the fields for a very long ($\lambda \gg l$) and very thin linear wire. The wire is oriented along the z-axis and the current through the wire is:

$$\boldsymbol{I}(z') = \boldsymbol{\hat{a}_z} I_0 \quad \text{Eq. 19}$$

We use the vector potential method to calculate the fields. The potential function **F** is zero and we only need to find **A**:

$$\boldsymbol{A}(x,y,z) = \frac{\mu}{4\pi} \int_C \boldsymbol{I}_e(x',y',z') \frac{e^{-jkR}}{R} dl' \quad \text{Eq. 20}$$

Where C is along the length of the source. We can write:

$$\boldsymbol{A}(x,y,z) = \boldsymbol{\hat{a}_z} I_0 \frac{\mu}{4\pi r} e^{-jkr} \int_{-l/2}^{l/2} dz' = \boldsymbol{\hat{a}_z} I_0 \frac{\mu l}{4\pi r} e^{-jkr} \quad \text{Eq. 21}$$

The transformation between rectangular and spherical components is given, in matrix form, by:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\, cos\phi & \sin\theta\, sin\phi & \cos\theta \\ \cos\theta\, cos\phi & \cos\theta\, sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{Eq. 22}$$

With Eq. 22, we change into spherical components:

$$A_r = \frac{\mu I_0 l e^{-jkr}}{4\pi r} \cos\theta \quad \text{Eq. 23}$$

$$A_\theta = -\frac{\mu I_0 l e^{-jkr}}{4\pi r} \sin\theta \quad \text{Eq. 24}$$

$$A_\phi = 0 \quad \text{Eq. 25}$$

Using Eq. 13 we can calculate the magnetic field:

$$\boldsymbol{H} = \boldsymbol{\widehat{a_\phi}} \frac{1}{\mu_r} \left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta} \right] \quad \text{Eq. 26}$$

Thus:

$$H_r = H_\theta = 0 \quad \text{Eq. 27}$$

$$H_\phi = j \frac{k I_0 l \sin\theta}{4\pi r} \left[1 + \frac{1}{jkr} \right] e^{-jkr} \quad \text{Eq. 28}$$

We can also calculate the electric field with Eq. 14:

$$E_r = \eta \frac{I_0 l \cos\theta}{2\pi r^2} \left[1 + \frac{1}{jkr} \right] e^{-jkr} \quad \text{Eq. 29}$$

$$E_\theta = j\eta \frac{kI_0 l \cos\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right] e^{-jkr} \quad \text{Eq. 30}$$

$$E_\phi = 0 \quad \text{Eq. 31}$$

Far Field

Using far field approximations ($kr \gg 1$) we can simplify:

$$E_\theta \cong j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \sin\theta \quad \text{Eq. 32}$$

$$E_r \cong E_\phi = H_r = H_\theta = 0 \quad \text{Eq. 33}$$

$$H_\phi \cong j \frac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta \quad \text{Eq. 34}$$

We also define the *wave impedance* as the ration of E_θ and H_ϕ:

$$Z_w = \frac{E_\theta}{H_\phi} \cong \eta \quad \text{Eq. 35}$$

Where η is the *intrinsic impedance* (377 ohms for free space).

Finite length dipole

Now we will derive the fields for a finite length dipole. We write the current as:

$$I_e(x' = 0, y' = 0, z') = \begin{cases} \hat{\boldsymbol{a}}_{\boldsymbol{z}} I_0 \sin\left[k\left(\frac{l}{2} - z'\right)\right], & 0 \le z' \le l/2 \\ \hat{\boldsymbol{a}}_{\boldsymbol{z}} I_0 \sin\left[k\left(\frac{l}{2} + z'\right)\right], & -l/2 \le z' \le 0 \end{cases} \quad \text{Eq. 36}$$

We subdivide the finite dipole antenna into infinitesimal dipoles of length $\Delta z'$ and using equations Eq. 32Eq. 32 Eq. 33, Eq. 34:

$$dE_\theta \cong j\eta \frac{kI_e(x', y', z') e^{-jkr}}{4\pi R} \sin\theta \, dz' \quad \text{Eq. 37}$$

$$dE_r \cong dE_\phi = dH_r = dH_\theta = 0 \quad \text{Eq. 38}$$

$$dH_\phi \cong j \frac{kI_e(x', y', z') l e^{-jkr}}{4\pi R} \sin\theta dz' \quad \text{Eq. 39}$$

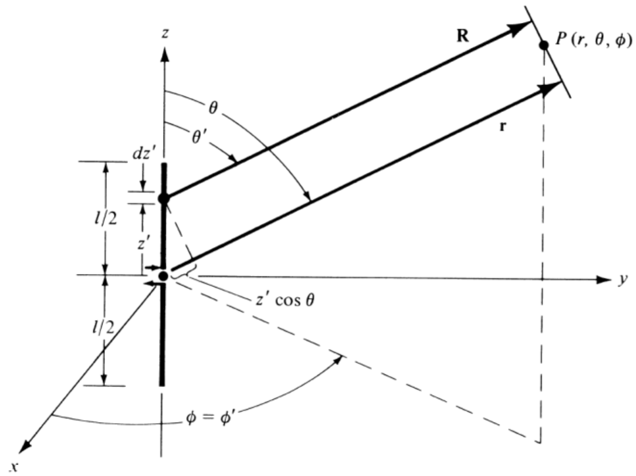

Figure 2 Finite length dipole geometry and far field approximation

With $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} = \sqrt{x^2 + y^2 + (z-z')^2}$

Using the far field approximations we get the fields:

$$E_\theta \cong j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] \quad \text{Eq. 40}$$

$$H_\phi \cong \frac{E_\theta}{\eta} \cong j\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] \quad \text{Eq. 41}$$

Half wavelength dipole

One of the most common used antennas is the half-wavelength ($l = \lambda/2$) dipole. To calculate the fields we plug in the length of the antenna into Eq. 40 and Eq. 41.

$$E_\theta \cong j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right] \quad \text{Eq. 42}$$

$$H_\phi \cong j\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right] \quad \text{Eq. 43}$$

Three and two dimensional radiation patterns are shown below for a half-wavelength dipole.

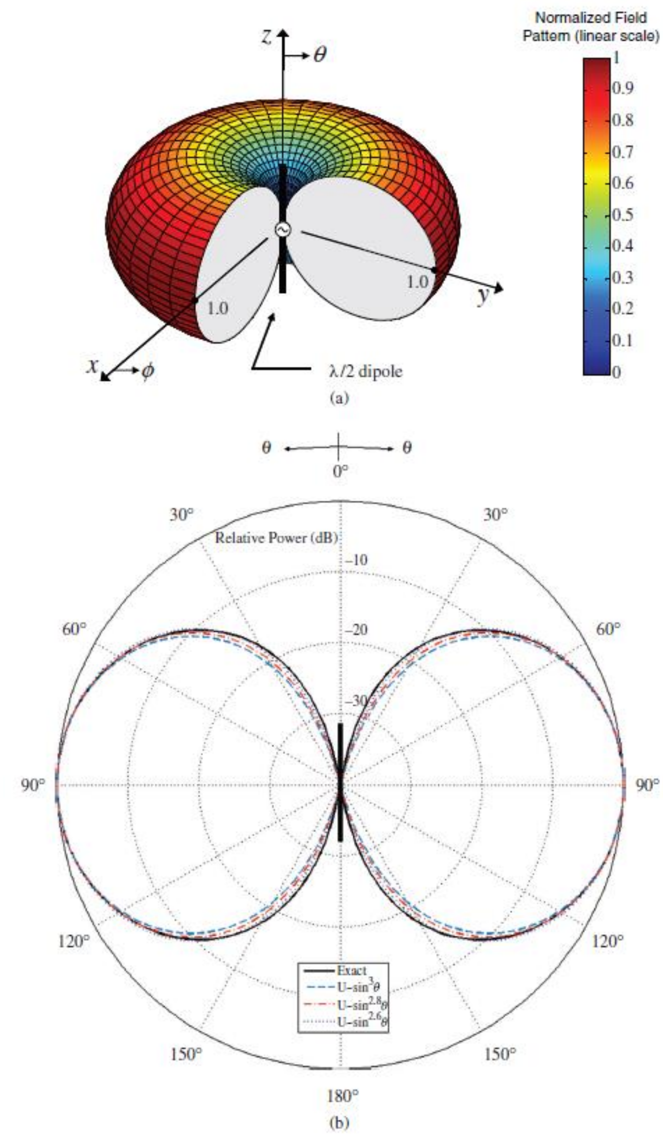

Figure 3 Radiation profile of a dipole

Horn Antennas

The Horn antenna is widely used microwave antennas. Its development dates back to the late 1800s and popularity increased in the 1930s due to a high interest in microwaves and waveguide transmission lines during World War II. This style of antennas are often used as a feed element in large radio astronomy, satellite tracking and communication dishes. The wide spread of this design is thanks to its simplicity in construction, ease of excitation, versatility, large gain and preferred overall performance. The Horn antenna can take various forms (Figure 4)Figure 1 but in basic principle it's a hollow pipe of different cross sections which has been tapered or flared to a larger opening. In the following section the fundamental theory of a horn antenna will be investigated.

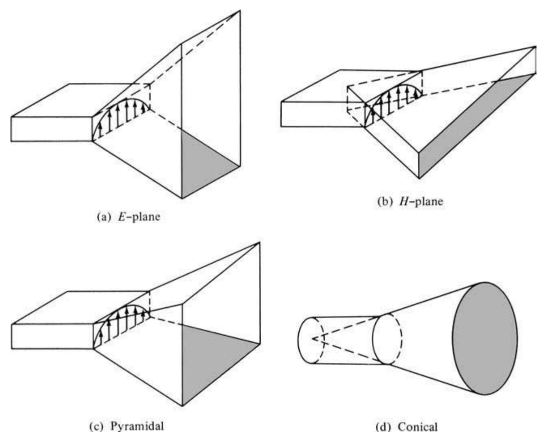

Figure 4 Different geometrical configurations for horn antennas

As previously described the horn antenna can be separated into two parts: The waveguide section and the flare. We will discussion these sections in order.

Waveguide Section

Waveguides are low-loss transmission lines with high-power handling capability. Transmission lines can be characterized impedance, attenuation and propagation constant parameters. There are different types of waveguides such as parallel plate waveguides, rectangular waveguides and circular waveguides.

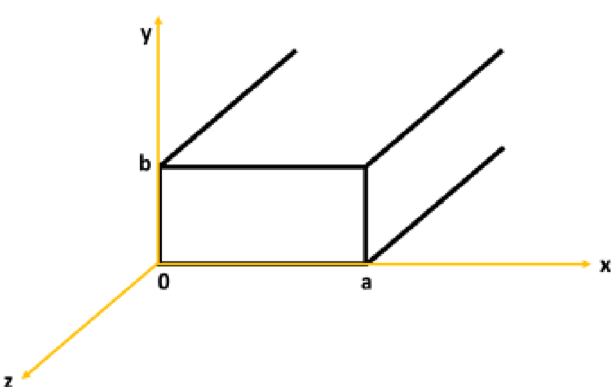

Figure 5 Rectangular hollow core waveguide in z-direction with dimensions a, b.

Let's consider a rectangular waveguide consisting of a hollow core and a metallic shell as seen in Figure 5. The waveguide has a cross-section with the length a in x-direction and height b in y-direction. The waves is guided along the z-axis this waveguide supports two kind of waves:

- Transverse electric waves (TE-waves). $\boldsymbol{E} = (E_x, E_y, 0)$ and $\boldsymbol{H} = (H_x, H_y, H_z)$
- Transverse magnetic waves (TM-waves). $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{H} = (H_x, H_y, 0)$

These waves have to satisfy the Maxwell equations and the boundary conditions are given by the material (The tangential components of the electric field and the derivative of the tangential component of the magnetic field have to be zero at the boundaries).

TE-Waves

To derive the solution for TE-waves we directly start with the Helmholtz equations which can be derived by using the Maxwell equations:

$$\boldsymbol{\nabla}^2 \boldsymbol{F} + k^2 \boldsymbol{F} = -\epsilon \boldsymbol{M} \qquad \text{Eq. 44}$$

We use Eq. 18 and Eq. 17to derivate the relations:

$E_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y}$	$E_y = \frac{1}{\epsilon}\frac{\partial F_z}{\partial x}$	$E_Z = 0$
$H_x = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial x \partial z}$	$H_y = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial y \partial z}$	$H_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)F_z$

By finding F_z we can calculate the other E- and H- fields. In this case the region is source free and from Eq. 44 we get:

$$\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2} + k^2 F_z = 0 \qquad \text{Eq. 45}$$

We use $F_z(x, y, z) = X(x)Y(y)Z(z)$ to solve this equation:

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0 \qquad \text{Eq. 46}$$

Dividing by XYZ:

$$\frac{X''}{X} + \frac{Y''}{Y} + k^2 = -\frac{Z''}{Z} \qquad \text{Eq. 47}$$

The left side of the equation is independent of z while the right side is independent of x and y. So both sides are constant which we will call k_z^2. Thus:

$$Z'' + k_z^2 Z = 0 \qquad \text{Eq. 48}$$

This equation has the solution:

$$Z(z) = c_1 e^{-jk_z z} + c_2 e^{jk_z z} \qquad \text{Eq. 49}$$

This solutions corresponds to waves traveling in positive (negative) z direction.

For the other components:

$$\frac{X''}{X} + k^2 - k_z^2 = -\frac{Y''}{Y} \qquad \text{Eq. 50}$$

Again both sides have to be constant (k_y^2) and we get:

$$Y'' + k_y^2 Y = 0 \quad \text{Eq. 51}$$

With the boundary conditions: $Y'(0) = Y'(b) = 0$.

This equation has the solution:

$$Y(y) = c_3 \cos\left(k_y y\right) + c_4 \sin\left(k_y y\right) \quad \text{Eq. 52}$$

With $k_y = \left(\frac{n\pi}{b}\right)^2$ and $c_4 = 0$.

The last component:

$$X'' + \left(k^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2\right) X = 0 \quad \text{Eq. 53}$$

We introduce $k_x^2 = k^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2$ and with the given boundary conditions ($X'(0) = X'(a) = 0$) we get:

$$X(x) = c_5 \cos(k_x x) + c_6 \sin(k_x x) \quad \text{Eq. 54}$$

With $k_x = \left(\frac{m\pi}{a}\right)^2$ and $c_6 = 0$

We are interested in forward propagating waves ($c_2 = 0$) thus we can write F_z as:

$$F_z = Acos\ (k_x x) \cos\left(k_y y\right) e^{-jk_z z} = A \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z} \quad \text{Eq. 55}$$

With this we can calculate the electric and magnetic fields:

$E_x = A_{mn} \frac{n\pi}{b\epsilon} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$	$H_x = A_{mn} \frac{m\pi kz}{a\omega\mu\epsilon} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$	Eq. 56
$E_y = -A_{mn} \frac{m\pi}{a\epsilon} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$	$H_y = A_{mn} \frac{m\pi kz}{b\omega\mu\epsilon} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$	Eq. 57
$E_z = 0$	$H_z = -jA_{mn} \frac{k^2 - k_z^2}{\omega\mu\epsilon} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e_z^{-jk} z$	Eq. 58

The m, n values are naturals numbers (0,1,2,…) but $(m, n) \neq (0,0)$. The different field configurations of m and n are known as *modes***.** In this case the transverse electric wave modes or TE_{mn}, where m indicates the number of half-cycle variations within the waveguide in x-direction and n the number of half-cycle variations in y-direction.

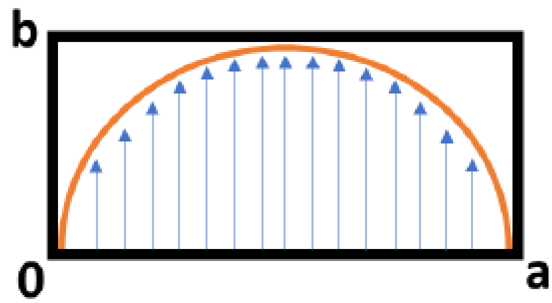

Figure 6 TE_{10} mode of a rectangular waveguide

Cut off frequency

k_z is the z-component of the wave vector. An interesting question at this point is which frequencies are supported by the waveguide. To answer this question we will have a closer look at the wave vector:

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \left(\frac{\omega}{c}\right)^2 = \left(\frac{2\pi f}{c}\right)^2 \quad \text{Eq. 59}$$

Solving for k_z.

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \text{Eq. 60}$$

We can distinguish two cases where once the mode is propagating if k_z is real and once the field is exponentially decaying when k_z is imaginary. With this we have an equation for our lowest propagating frequency.

$$k_z = 0 \rightarrow \left(\frac{2\pi f}{c}\right)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \text{Eq. 61}$$

To get the cutoff frequency we look at the lowest possible mode TE_{10} and solve for f:

$$f_c = \frac{c}{2a} \quad \text{Eq. 62}$$

Any frequency below f_c will exponentially decay and the waveguide does not transport the energy. Every mode within a waveguides has its own cutoff frequency. That is, for a given mode to propagate the operating frequency need to be above the cut-off of the TE_{mn} mode given by:

$$f_c^{mn} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \text{Eq. 63}$$

For example given a waveguide with dimensions $a = 20mm$ and $b = 15mm$ results in the TE modes listed in Table 1. An operating frequency of 6 GHz does not work due to being below the cut-off frequency $f_{c_{10}}$. At 9 GHz only the fundamental mode TE_{10} is propagating and at frequencies >10 GHz multiple modes are present in the waveguide.

Mode	**m**	**n**	$\boldsymbol{f_{c_{m,n}} [GHz]}$
TE	1	0	7,5
TE	0	1	10

TE, TM	1	1	12,5
TE	2	0	15
TE, TM	2	1	18,02

Table 1 Modes and the cut off frequency for a rectangular hollow core waveguide with dimensions $a = 20mm$ and $b = 15mm$.

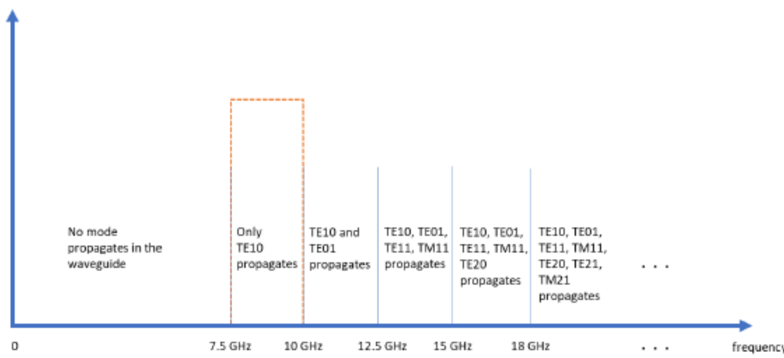

Figure 7 Plotting of Table 1 the different modes on the frequency axis and their cut-off frequencies

TM-waves

The derivation of the TM modes is similar to the TE modes.

$$\boldsymbol{\nabla}^2 \boldsymbol{A} + k^2 \boldsymbol{A} = 0 \qquad \text{Eq. 64}$$

and the relation between $\boldsymbol{A}$ and the fields are:

$E_x = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial x \partial z}$	$E_y = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial y \partial z}$	$E_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z$
$H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y}$	$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x}$	$H_z = 0$

Solving Eq. 64:

$$A_z = A_{mn} sin\,(k_x x)\sin(k_y y)e^{-jk_z z} = A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z z} \qquad \text{Eq. 65}$$

$E_x = B_{mn}\frac{k_x k_y}{\omega\mu\epsilon}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$	$H_x = B_{mn}\frac{k_y}{\mu}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$	Eq. 66
$E_y = -B_{mn}\frac{k_x k_y}{\omega\mu\epsilon}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$	$H_y = B_{mn}\frac{k_x}{\mu}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$	Eq. 67
$E_z = -jB_{mn}\frac{k^2 - k_z^2}{\omega\mu\epsilon}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e_z^{-jk}z$	$H_z = 0$	Eq. 68

Again some modes are not possible$(m, n) = (0,1)$ and $(m, n) = (1,0)$ thus the lowest order mode for TM is TM_{11}. For cutoff frequency of the TM_{mn} modes is the same as for TE_{mn} modes:

$$f_c^{mn} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \text{Eq. 69}$$

Going back to the previous example Table 1 illustrates the different modes of the waveguide depending on the operation frequency.

The flare section

The flare section directly follows the waveguide part of the horn antenna. Simplified the horn antenna can be seen as an opened out waveguide where the flare is the transition from waveguide to air. A simple open ended waveguide would also radiate but to better match the impedance of free space the horn is used. As seen in Figure 8 the flare can be separated in different subgroups. For this discussion we will focus on the E-Plane, H-Plane and Pyramidal configurations.

E-Plane Horn

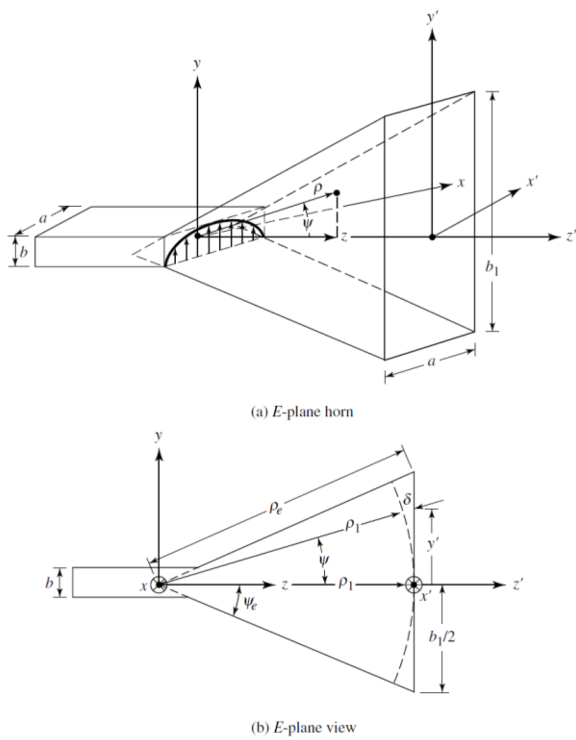

Figure 8 E-Plane Horn antenna geometric configuration. The flare of the antenna is along the E-Plane and the H-Plane sections remains unchanged.

To get the radiation pattern of the E-Plane horn antenna we must first know the electric field along the aperture of the horn. These fields can be derivate by treating the horn as a radial waveguide. We would not only find the field at the aperture but also inside of the horn. This process is straightforward but tedious and is thus omitted but can be found in the book by Constantine Balanis (Balanis, Constantine A. *Antenna theory: analysis and design*. John wiley & sons, 2015.).The mode at the aperture of the horn are given by:

$$E'_z = E'_x = H'_y = 0 \quad \text{Eq. 70}$$

$$E_y{}'(x',y') \approx E_1 \cos\left(\frac{\pi}{a}x'\right) e^{-j[ky'^2/(2\rho_1)]} \qquad \text{Eq. 71}$$

$$H_z'(x',y') \approx jE_1 \left(\frac{\pi}{ka\eta}\right) \sin\left(\frac{\pi}{a}x'\right) e^{-j[ky'^2/(2\rho_1)]} \qquad \text{Eq. 72}$$

$$H_x'(x',y') \approx -\frac{E_1}{\eta} \cos\left(\frac{\pi}{a}x'\right) e^{-j[ky'^2/(2\rho_1)]} \qquad \text{Eq. 73}$$

$$\rho_1 = \rho_3 \cos(\psi_e) \qquad \text{Eq. 74}$$

Where E_1 is a constant and $\eta = \sqrt{\mu/\epsilon}$ the intrinsic impedance of the medium. The primes indicated the fields at the aperture. Comparing with the solution from the rectangular waveguide Eq. 56 the expression is similar to the TE_{10} mode. The only difference is the complex exponential term which is used to present the quadratic phase variations of the fields over the aperture of the horn. This term can be illustrated geometrically. In Figure 8 assume an imaginary apex of the horn (shown dashed) there exists a line source radiating cylindrical waves. As the wave travel outward in radial direction, the wave fronts are cylindrical. At any point y' at the aperture of the horn the phase is not the same as at the origin point $y' = 0$.

Radiation

To solve the electric field of the horn antenna we will focus on the solution in the far field. Using the previous determined procedure with vector potentials and far field approximations we end up with some quite complex field functions. This derivation is outside of the scope of this experiment thus we will only have a look at the results:

$$E_r = 0 \qquad \text{Eq. 75}$$

$$E_\theta = -j\frac{a\sqrt{\pi k \rho_1} E_1 e^{-jkr}}{8r} \times \left\{ e^{j\left(\frac{k_y^2 \rho_1}{2k}\right)} \sin\phi\,(1+\cos\theta) \left[\frac{\cos\frac{k_x a}{2}}{\left(\frac{k_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2}\right] F(t_1,t_2) \right\} \qquad \text{Eq. 76}$$

$$E_\phi = -j\frac{a\sqrt{\pi k \rho_1} E_1 e^{-jkr}}{8r} \times \left\{ e^{j\left(\frac{k_y^2 \rho_1}{2k}\right)} \cos\phi\,(1+\cos\theta) \left[\frac{\cos\frac{k_x a}{2}}{\left(\frac{k_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2}\right] F(t_1,t_2) \right\} \qquad \text{Eq. 77}$$

$$F(t_1,t_2) = [C(t_2) - C(t_1)] - j[S(t_2) - S(t_1)] \qquad \text{Eq. 78}$$

With $C(x)$ and $S(x)$ the cosine and sine Fresnel integrals. To get a better understanding of these fields we will look at radiation pattern (Figure 9). As expected the E-plane pattern is much narrower than the H-plane due to the flaring.

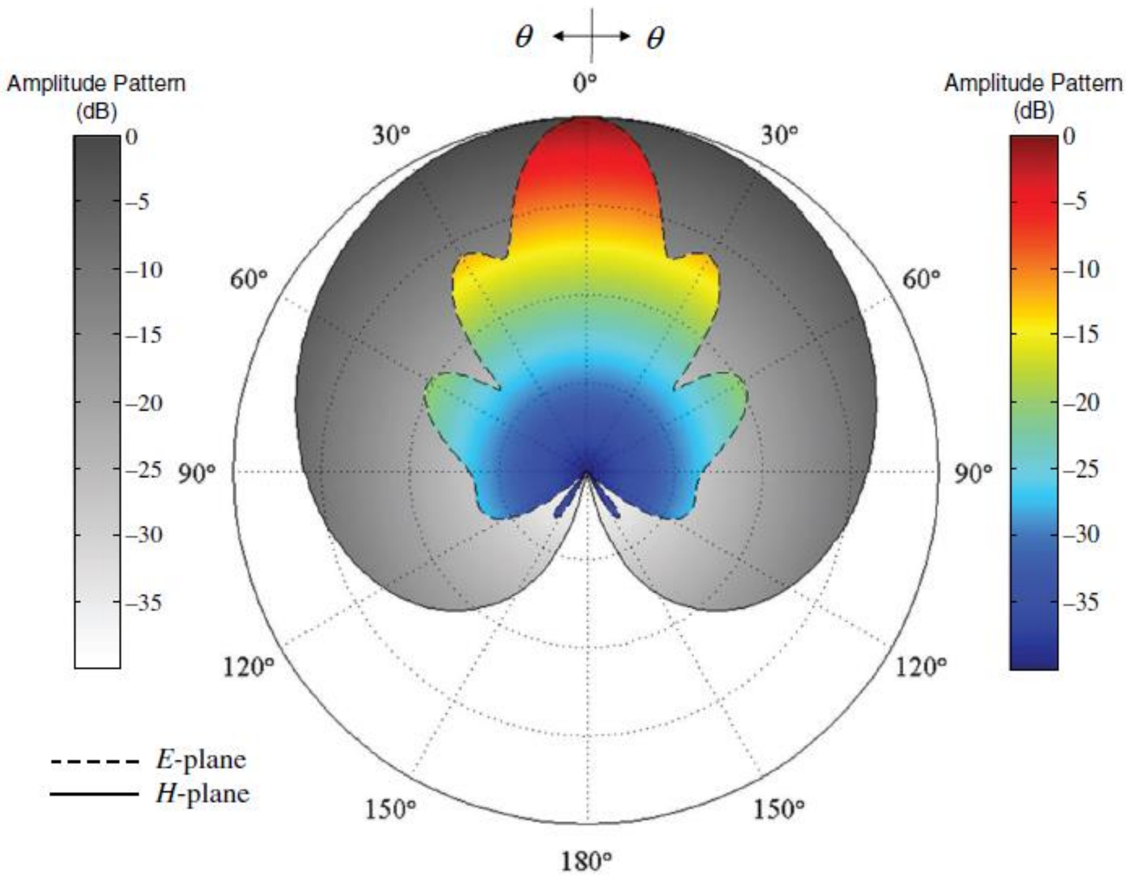

Figure 9 Radiation pattern of the E-plane horn antenna. The E-Plane is much narrower compared to the H-Plane.

H- Plane Horn

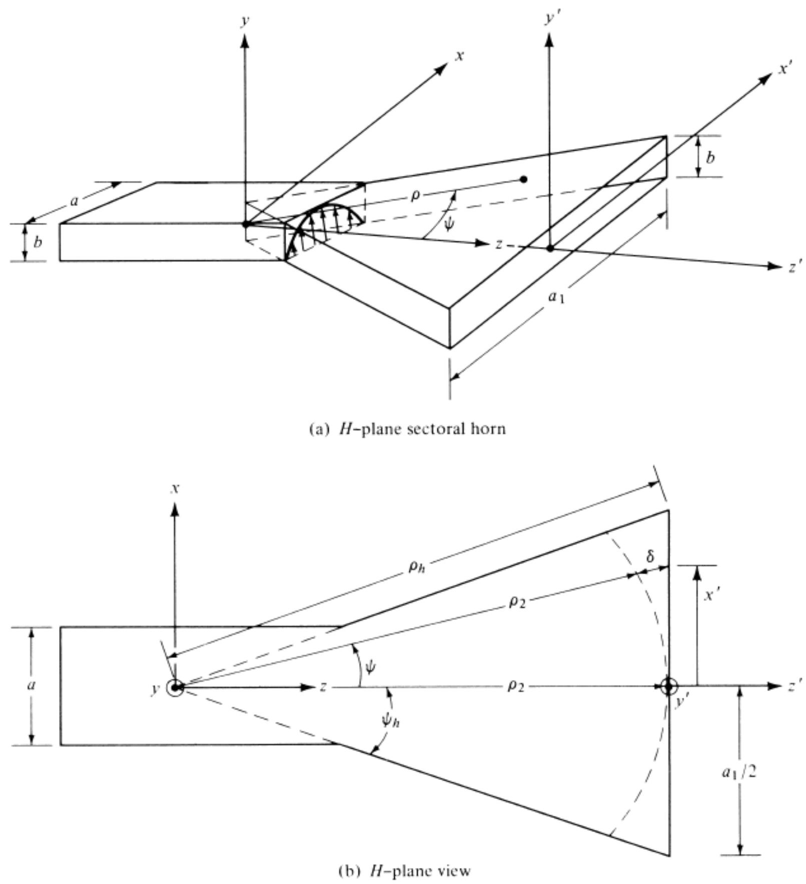

Figure 10 H-Plane Horn antenna geometric configuration. The flare is along the H-plane while the E-plane remains unchanged.

The procedure for the H-Plane sectoral horn is similar to the E-plane horn. The summary is as follows:

Aperture Fields

$$E'_x = H'_y = 0 \qquad \text{Eq. 79}$$

$$E_y{}'(x') \approx E_2 \cos\left(\frac{\pi}{a_1}x'\right) e^{-j[kx'^2/(2\rho_2)]} \qquad \text{Eq. 80}$$

$$H'_x(x') \approx \frac{E_2}{\eta} \cos\left(\frac{\pi}{a_1}x'\right) e^{-j[kx'^2/(2\rho_2)]} \qquad \text{Eq. 81}$$

$$\rho_2 = \rho_h \cos(\psi_h) \qquad \text{Eq. 82}$$

Radiated Fields

$$E_r = 0 \qquad \text{Eq. 83}$$

$$E_\theta = -jE_2 \frac{b}{8}\sqrt{\frac{k\rho_2}{\pi}} \frac{e^{-jkr}}{r} \times \left\{\sin\phi\,(1+\cos\theta)\frac{\sin Y}{Y}\left[e^{jf_1}F(t'_1,t'_2) + e^{jf_2}F(t''_1,t''_2)\right]\right\} \qquad \text{Eq. 84}$$

$$E_\phi = -jE_2 \frac{b}{8}\sqrt{\frac{k\rho_2}{\pi}}\frac{e^{-jkr}}{r} \times \left\{\cos\phi\,(1+\cos\theta)\frac{\sin Y}{Y}\left[e^{jf_1}F(t_1',t_2') + e^{jf_2}F(t_1'',t_2'')\right]\right\}$$ Eq. 85

$$F(t_1,t_2) = [C(t_2) - C(t_1)] - j[S(t_2) - S(t_1)]$$ Eq. 86

$Y = \frac{kb}{2}\sin\theta\sin\phi;\ f_1 = \frac{k_x'^2\rho_2}{2k};\ ;\ f_1 = \frac{k_x''^2\rho_2}{2k}$

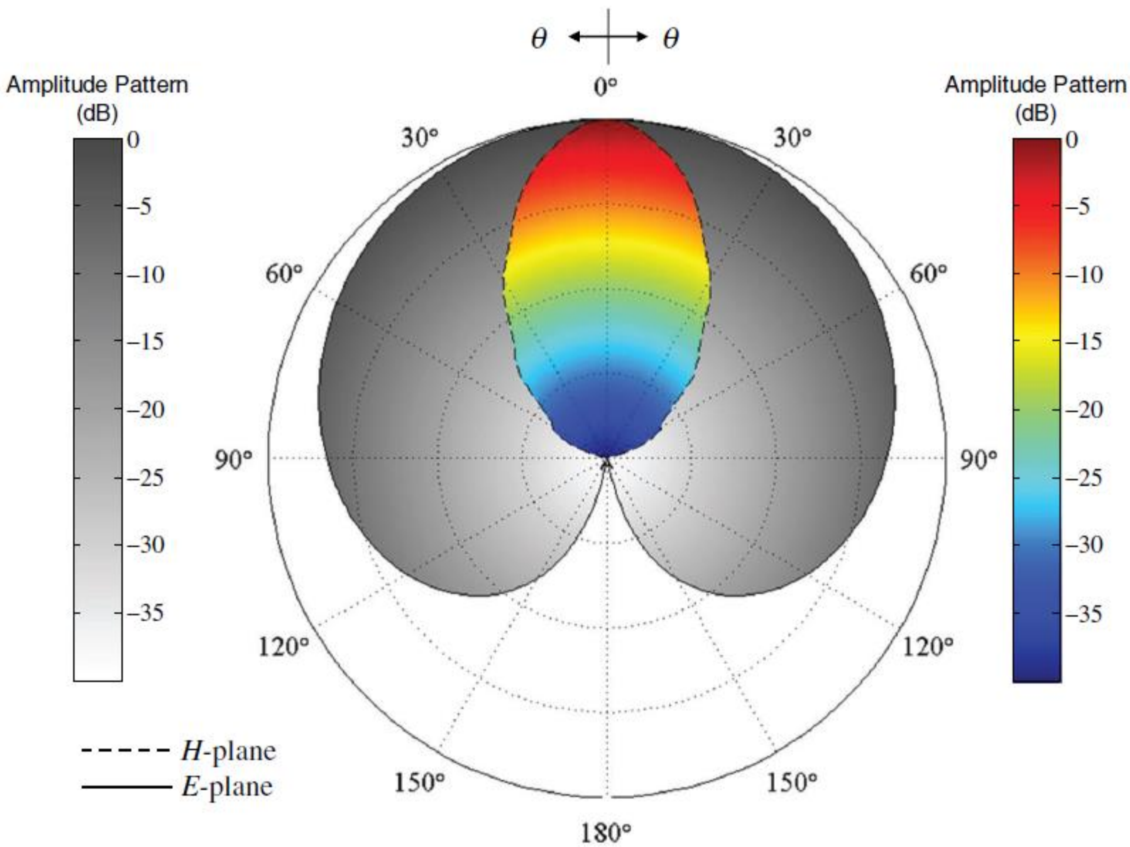

Figure 11 Radiation pattern of the H-Plane horn antenna. The H-plane is much narrower than the E-plane.

Contrary to the E-Plane Horn the directivity of the fields is switched instead of the E-Plane now the H-Plane is much narrower.

Pyramidal Horn

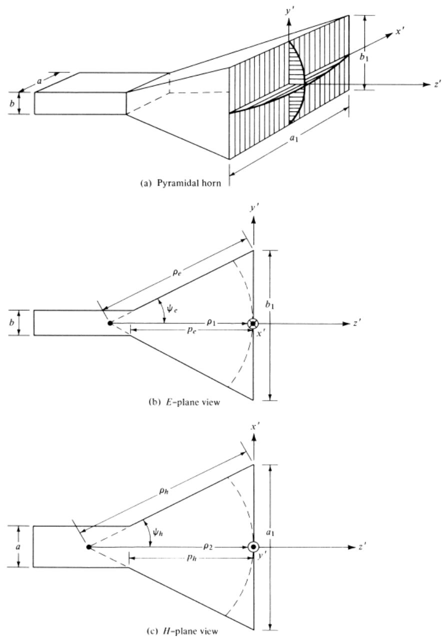

Figure 12 Pyramidal horn geometry. The horn is flared in both the E-Plane and the H-Plane

Finally we will have a look a pyramidal horn antenna where both planes are flared. We approximate again the fields at the aperture:

$$E_y{}'(x',y') \approx E_0 \cos\left(\frac{\pi}{a}x'\right) e^{-j\left[k\left(\frac{x'^2}{\rho_2}+\frac{y'^2}{\rho_1}\right)/2\right]} \qquad \text{Eq. 87}$$

$$H_x'(x',y') \approx -\frac{E_0}{\eta}\cos\left(\frac{\pi_1}{a_1}x'\right) e^{-j\left[k\left(\frac{x'^2}{\rho_2}+\frac{y'^2}{\rho_1}\right)/2\right]} \qquad \text{Eq. 88}$$

The above expressions contain cosinusoidal amplitude distribution in the x' direction and quadratic phase variations in both the x' and y' directions, similar to those of the sectoral E- and H-plane horns. Using these source terms we get the solution to the E- and H-field components.

$$E_r = 0 \qquad \text{Eq. 89}$$

$$E_\theta = j\frac{ke^{jkr}}{4\pi r}[\sin\phi(1+\cos\theta\, I_1 I_2)] \qquad \text{Eq. 90}$$

$$E_\phi = j\frac{kE_0e^{-jkr}}{4\pi r}[\cos\phi(\cos\theta+1)I_1I_2] \qquad \text{Eq. 91}$$

$$I_1 = \frac{1}{2}\sqrt{\frac{\pi\rho_2}{k}}\Big(e^{j(k_x'^2\rho_2/2k)}\{[C(t_2')-C(t_1{}')]-j[S(t_2')-S(t_1{}')]\} + e^{j(k_x''^2\rho_2/2k)}\{[C(t_2'')-C(t_1{}'')]-j[S(t''_2)-S(t_1{}'')]\}\Big) \qquad \text{Eq. 92}$$

$$I_1 = \frac{1}{2}\sqrt{\frac{\pi\rho_2}{k}}\Big(e^{j(k_x'^2\rho_2/2k)}\{[C(t_2')-C(t_1{}')]-j[S(t_2')-S(t_1{}')]\} + e^{j(k_x''^2\rho_2/2k)}\{[C(t_2'')-C(t_1{}'')]-j[S(t''_2)-S(t_1{}'')]\}\Big) \qquad \text{Eq. 93}$$

$$I_2 = \sqrt{\frac{\pi\rho_1}{k}}e^{j(k_y^2\rho_1/2k)}\{[C(t_2)-C(t_1)]-j[S(t_2)-S(t_1)]\}$$

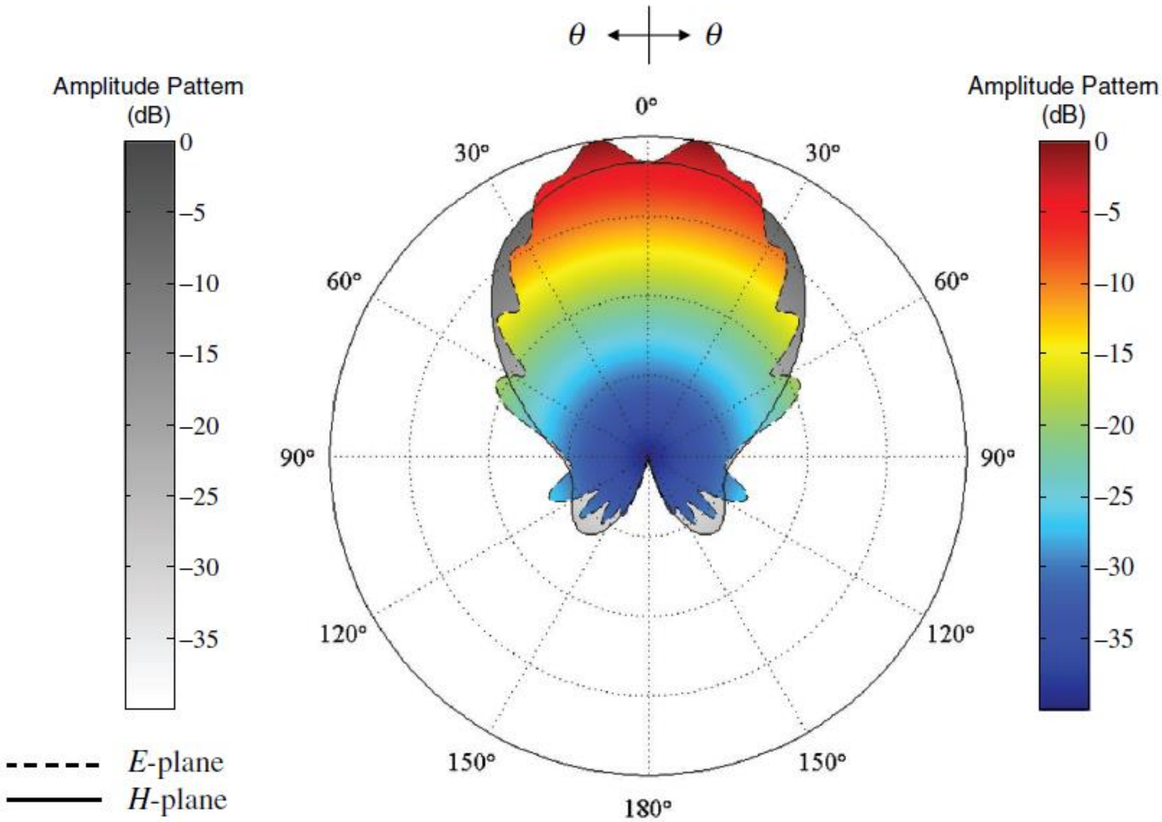

Figure 13 Radiation Pattern of the pyramidal antenna. Both E-Plane and H-Plane are directed.

Directivity

To calculate the directivity of a Horn antenna it is necessary to find the maximum radiation:

$$\mathrm{U}_{max} = \frac{r^2}{2\eta}|\boldsymbol{E}|^2_{max} \tag{Eq. 94}$$

Horn antennas generally have the maximum radiation nearly along the z-axis. $\theta = 0$. Using our previous derivate fields for the pyramidal horn antenna we can find the directivity:

$$D_E = \frac{64a\rho_1}{\pi\lambda b_1}\left[C^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right) + S^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right)\right] \tag{Eq. 95}$$

$$D_H = \frac{64b\rho_2}{a_1\lambda} \times \{[C(u) - C(v)]^2 + [S(u) - S(v)]^2\} \tag{Eq. 96}$$

$$u = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a_1} + \frac{a_1}{\sqrt{\lambda\rho_2}}\right) \tag{Eq. 97}$$

$$v = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a_1} - \frac{a_1}{\sqrt{\lambda\rho_2}}\right) \tag{Eq. 98}$$

$$D_p = \frac{4\pi U_{max}}{P_{rad}} = \frac{\pi\lambda^2}{32a_1b_1}D_E D_H = \frac{4\pi}{\lambda^2}b_1a_1e_A \tag{Eq. 99}$$

With D_E (D_H) the directivity of the E-plane (H-plane) horn antenna and e_A the aperture efficiency. To summarize to procedure to find the directivity of a horn antenna:

1. Calculate:

$$A = \frac{a_1}{\lambda}\sqrt{\frac{50}{\rho_h/\lambda}} \quad \text{Eq. 100}$$

$$B = \frac{b_1}{\lambda}\sqrt{\frac{50}{\rho_e/\lambda}} \quad \text{Eq. 101}$$

2. Using A and B, find G_H and G_E respectively using Figure 13 and Figure 14. if either A or B or both are smaller than 2, calculate G_H and G_E with:

$$G_E = \frac{32}{\pi}B \quad \text{Eq. 102}$$

$$G_H = \frac{32}{\pi}A \quad \text{Eq. 103}$$

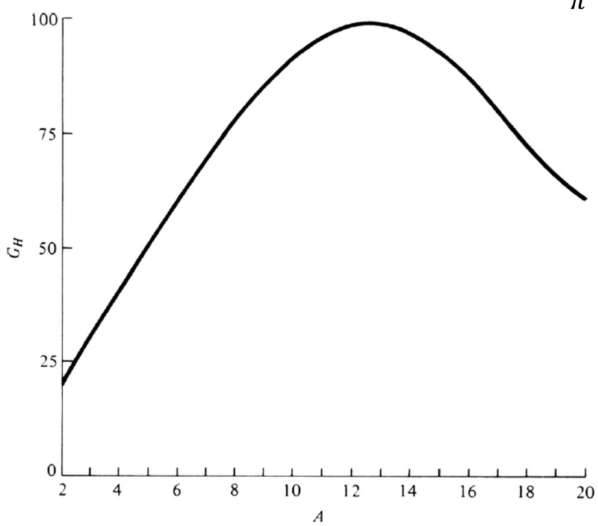

Figure 14 G_H as a function of A

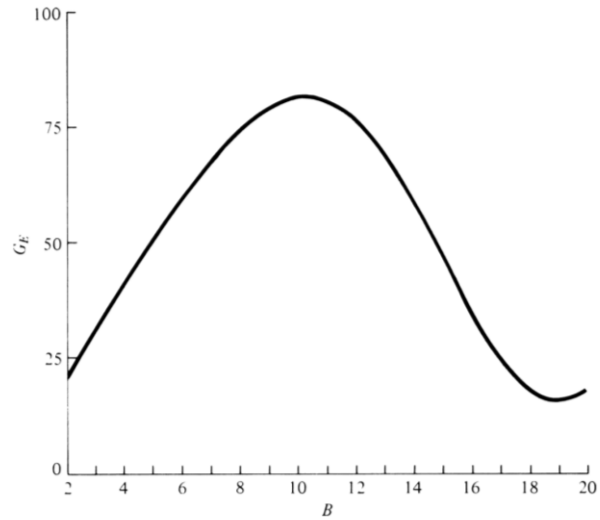

Figure 15 G_E as a function of B

3. Calculate D by using values of G_E and G_H with .

$$D_p = \frac{G_E G_H}{\frac{32}{\pi}\sqrt{\frac{50}{\rho_e/\lambda}}\sqrt{\frac{50}{\rho_h/\lambda}}} \quad \text{Eq. 104}$$

Half power Beam Width

The H-plane HPBW is mainly related to a_1 and E-plane HPBW is mainly related to b_1 dimensions. The exact dimensions for a target HPBW must be found via iterations. But there are some empirical formulas taken from Volakis, John L. *Antenna engineering handbook*. McGraw-Hill Education, 2007.:

$$HPBW\ in\ H_{plane}(degrees) = \theta_{3dB_{H_{plane}}} = (40\lambda/a_1) + 15° \quad \text{Eq. 105}$$

$$HPBW\ in\ E_{plane}(degrees) = \theta_{3dB_{E_{plane}}} = 44\lambda/b_1$$

EXPERIMENTS

Design of a Horn Antenna

The goal of this experiment is to design a horn antenna such as the one shown in Figure 16. In order to start the design, first the waveguide must be designed for the target frequency range. The dimensions a and b must be calculated for the lowest and highest operating frequencies. We want the waveguide to work at TE10 mode only. Therefore, the waveguide frequency range must be between the cut-off frequencies of two modes. The lowest mode is always the TE10 mode.

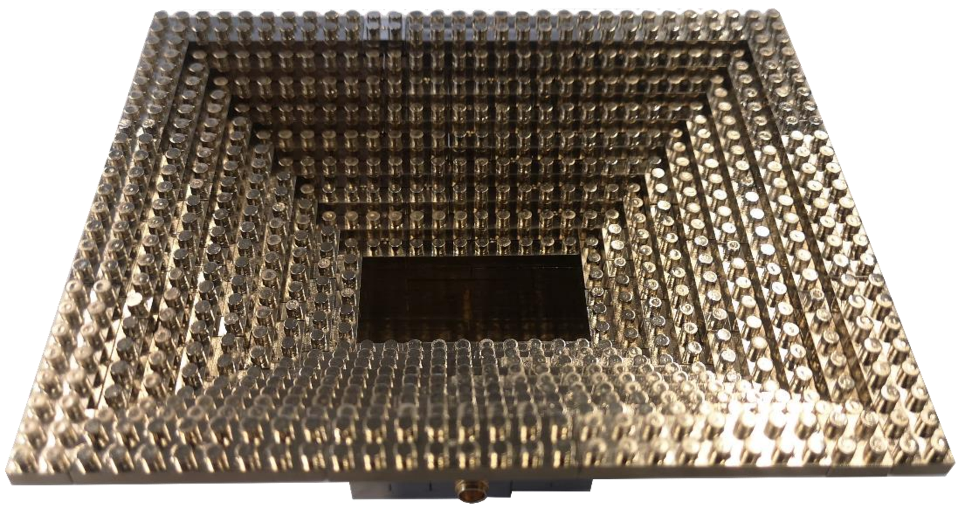

Figure 16: Example of a finished horn antenna built with the experiment kit.

We start with the waveguide and locate the RF connector. The RF connector must be mounted on the “a” side of the waveguide and it must be in the middle of a.

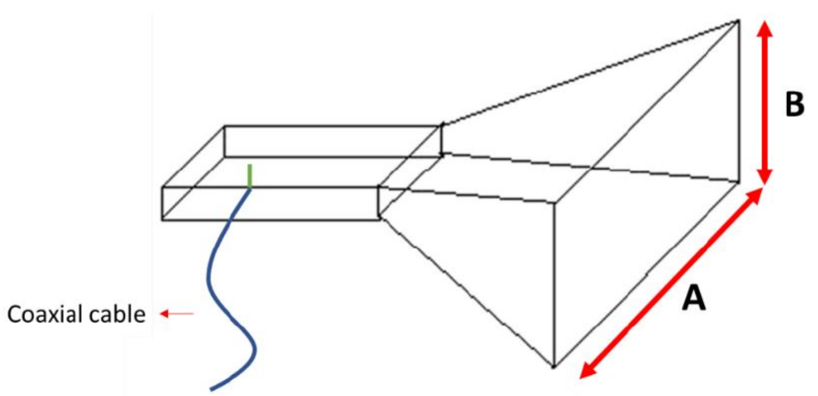

Figure 17: Position of the RF connector and design parameters A and B of the horn antenna.

The distance of the feed point from the back plane of the waveguide is a design parameter. This distance changes the reflection coefficient. The designer needs to find the right position to get the lowest reflection coefficient. Another design parameter to get the best reflection coefficient is the feed length. Figure 21 shows the feed block for the horn antenna experiment. There is threat over the inner conductor of the connector and there is a ring over the inner conductor of the feed. The designer needs to find the best position of the ring in order to get the lowest reflection coefficient. You can change the position of the ring with your hands.

After building the waveguide, the horn needs to be built. The horn aperture is directly related to the directivity and HPBW both in H-plane and E-plane. Another design parameter is the flare angle. The flare angle changes the reflection coefficient of the horn antenna. The designer needs to calculate the horn aperture dimensions for the given HPBW and directivity. You can take the aperture efficiency as 1 in this experiment. In the real antenna, this parameter is expected to be close to between 0.5 to 1.

The calculations must be applied to the inner dimensions of the waveguide and horn. Because we deal with the inner dimensions for this antenna. Therefore, the calculation of the horn aperture gives us the inner dimensions of the horn and the calculation of the waveguide gives us the inner dimensions of the waveguide.

Blocks

The size of the blocks used to build the antenna is shown in Figure 18.

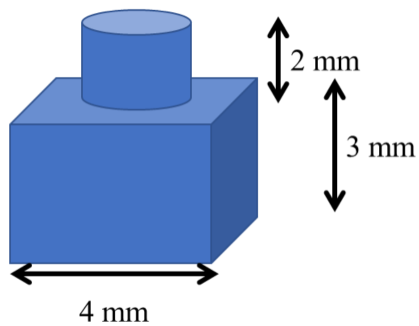

Figure 18: Dimensions of the blocks used for this experiment.

Be careful when removing blocks to avoid damaging them. In this experiment, use your hands to remove the cells. If you try to remove the cells with your hands but you can’t remove them, then use the removing plier. For 1X3 and larger cells, hold the cylinder at the center. If you can’t remove the cell easily, change to another cylinder. Figure 19 shows an example usage of the removing plier.

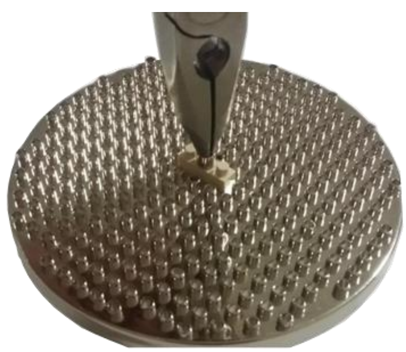

Figure 19: Usage of removing plier.

Experiment Steps

1. Set up the measurement as shown in Figure 20 (without LAN cable). Use an SMA cable to connect the RF OUT connector of the VNA to the RF IN connector of the TX module.

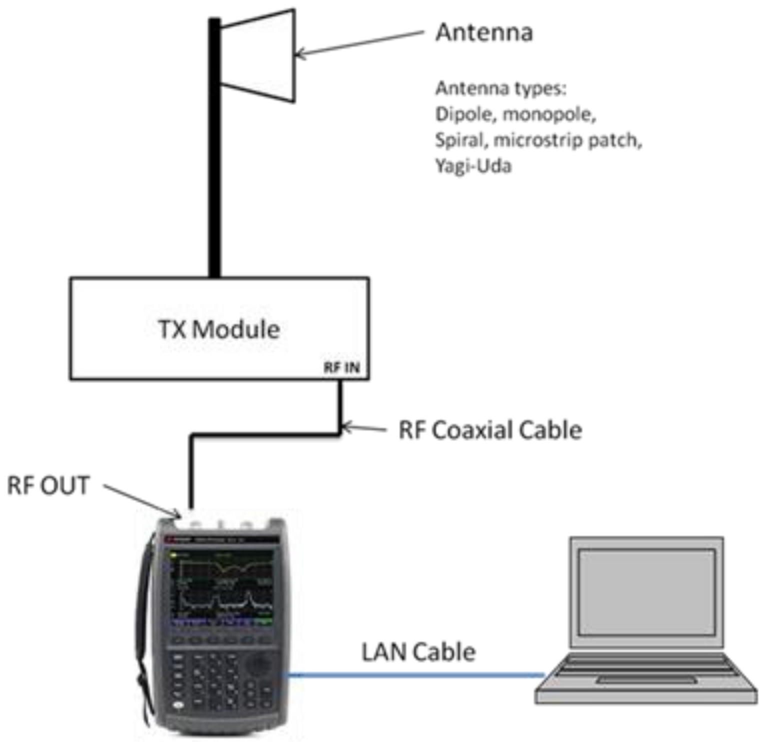

Figure 20 Measurement Setup for S11 parameter

2. Under Mode, select the NA option and the S11 parameter.
3. Press the *Freq/Dist* button on the VNA and set the start frequency to 2 GHz and the stop frequency to 6 GHz. The target frequency is 6 GHz. This corresponds to the frequency range of the designed Horn antenna
4. Press the *Scale/Amptd* button on the VNA and select *Autoscale*.
5. Calculate the waveguide dimensions for a waveguide operating between 3.75 and 7.5 GHz. These are the cut-off frequencies of the lowest mode and the second lowest mode. The second lowest mode for this experiment is TE01. Write down the waveguide dimensions and the cutoff frequencies of the modes and check if a and b are the correct dimensions. Check your results with the supervisor.

a:			
b:			
Mode	**m**	**n**	$f_{c_{m,n}}[GHz]$
TE	1	0	
TE	0	1	
TE, TM	1	1	

6. Design the waveguide section:
   a. a and b are the **inner** dimensions of the waveguide.
   b. Start with building the ground plane by using the 2X8 cells. The ground plane can be slightly larger than the waveguide.

c. Place the horn feed block (Figure 21) one brick above the ground plane (Figure 22). The feed block should be in the middle of side a.
d. Build the waveguide with the 1X1, 1X2, 1X3, 1X4 and 1X6 cells
e. The height of the waveguide is 0.72λ at 6 GHz.

Figure 21: Horn feed block.

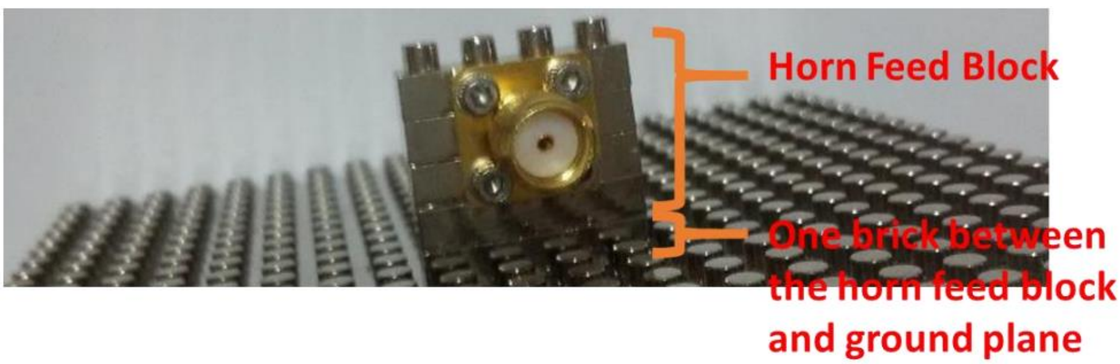

Figure 22: Location of the horn feed block.

Figure 23: Example build of the waveguide section.

7. Connect the open-ended waveguide to TX Module. Change the position of the ring over the inner conductor of the feed block to have the best reflection coefficient for 4.3 - 6 GHz frequency range.
8. Discuss the reflection measurement.

9. Design the flare section:
    a. Calculate the A and B dimensions for 50.7º HPBW in H-plane and 61º HPBW in E-plane.

A dimension Flare	
B dimension Flare:	

    b. Find the directivity of the antenna.

Directivity Flare	

    c. The flare angle for this experiment is 15.4º. The flare angle is for the inner dimensions of the horn. Calculate the height of the flare.

Height Flare	

    d. Build the horn with 2X1, 2X2, 2X4 and 2X8 metal cells similar to Figure 24.

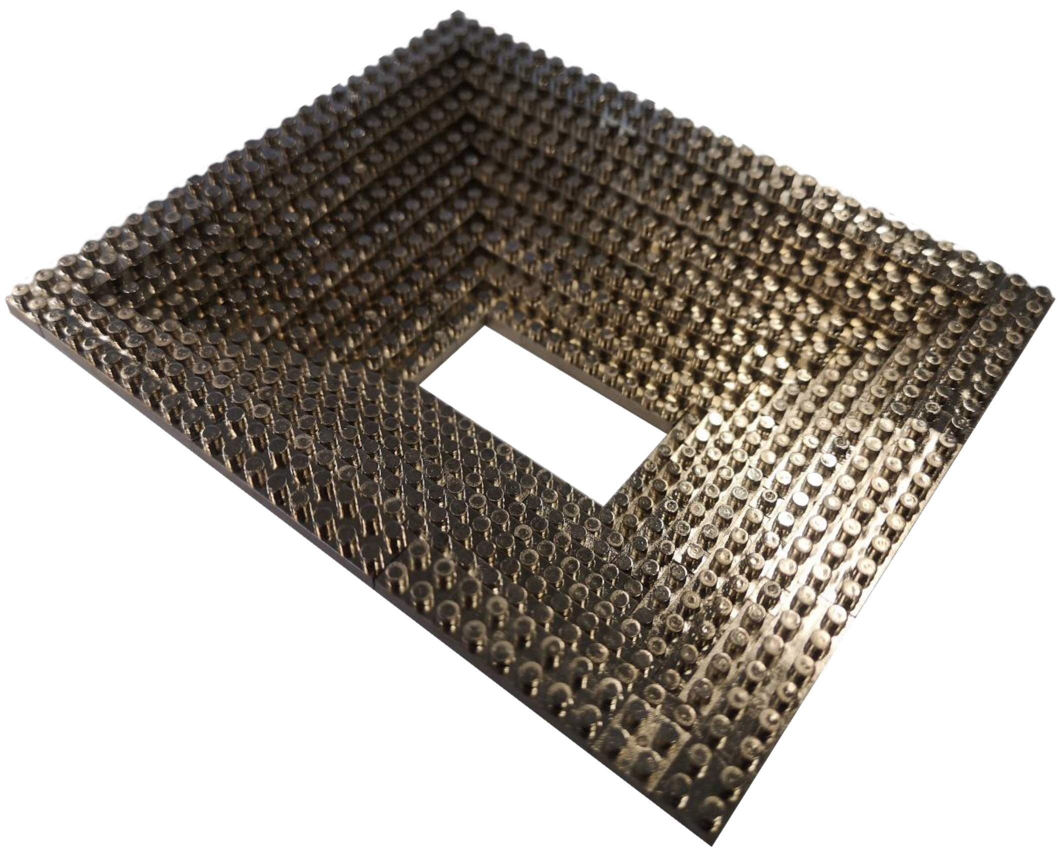

Figure 24: Example build of the flare section.

10. Measure the H-plane pattern of the wave guide and the horn antenna at 6 GHz with the setup shown in Figure 25.

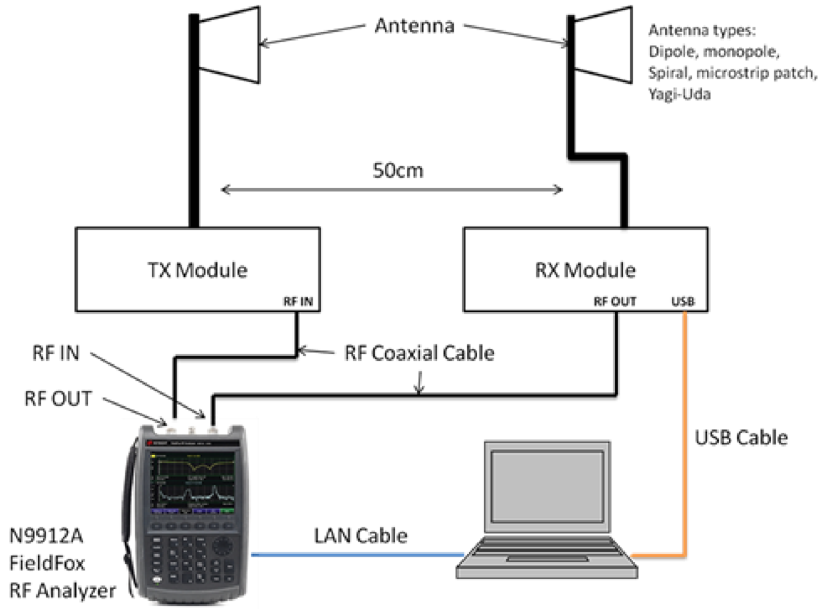

Figure 25 Setup for antenna measurements.

a. Use an SMA cable to connect the RF OUT connector of the RX module to the RF IN connector of the VNA.
b. Ask the supervisor for the transmitter antenna. Mount this antenna to the TX Module.
c. Place your waveguide on the RR Module.
d. Set the start frequency to your desired frequency range such that the center frequency is at 6GHz. Set the output power to 0 dBm. Then click on *Configure* and after completing the configuration click on *Save & Exit*.
e. Change the distance between the antennas to about 50 cm and make sure that no objects or people are near the antennas.
f. Click on *2D Radiation Pattern* to measure the S12 parameter (transmission).
g. Under *Rotator Control*, set the value of *Step Size* to 5 Deg/Step and click on *Play* to start the measurement.
h. Measure only the waveguide without a flare and then the waveguide with the two designed flares attached. Compare the measurements and discuss the differences.
i. Write down the HPBW of the H-Plane of the antennas.

HPBW Waveguide:	
HPBW Flare :	

Questions

After the experiment, answer the following questions:

1. What parameters affect the reflection coefficient of horn antennas?
2. Which parameters affect the HPBW of the horn antenna?