Chapter 5

Energy and Momentum

The equations established so far describe the behavior of electric and magnetic fields. They are a direct consequence of Maxwell's equations and the properties of matter. Although the electric and magnetic fields were initially postulated to explain the forces in Coulomb's and Ampere's laws, Maxwell's equations do not provide any information about the energy content of an electromagnetic field. As we shall see, Poynting's theorem provides a plausible relationship between the electromagnetic field and its energy content.

5.1 Poynting's Theorem

If the scalar product of the field $\mathbf{E}$ and Eq. (1.34) is subtracted from the scalar product of the field $\mathbf{H}$ and Eq. (1.33) the following equation is obtained

$$\mathbf{H}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H}) \;=\; -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \mathbf{j}\cdot\mathbf{E}\,. \tag{5.1}$$

Noting that the expression on the left is identical to $\nabla\cdot(\mathbf{E}\times\mathbf{H})$, integrating both sides over space and applying Gauss' theorem the equation above becomes

$$\int_{\partial V}(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n}\,da \;=\; -\int_V\left[\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{j}\cdot\mathbf{E}\right]dV \tag{5.2}$$

Although this equation already forms the basis of Poynting's theorem, more insight is provided when $\mathbf{B}$ and $\mathbf{D}$ are substituted by the generally valid equations (1.20).

Eq. (5.2) then reads as

$$\int_{\partial V} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} \, da \; + \; \frac{\partial}{\partial t} \int_V \frac{1}{2} \Big[\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H} \Big] \, dV \;\; = \; - \int_V \mathbf{j} \cdot \mathbf{E} \, dV \qquad (5.3)$$

$$- \frac{1}{2} \int_V \left[\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{P} \cdot \frac{\partial \mathbf{E}}{\partial t} \right] dV \; - \; \frac{\mu_0}{2} \int_V \left[\mathbf{H} \cdot \frac{\partial \mathbf{M}}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{H}}{\partial t} \right] dV.$$

This equation is a direct conclusion of Maxwell's equations and has therefore the same validity. Poynting's theorem is more or less an interpretation of the equation above. The left hand side has the general appearance of a *conservation law*, similar to the conservation of charge encountered previously in Eq. (1.1).

If we set $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu_0 \mu \mathbf{H}$ then the second integrand becomes $(1/2) \, [\varepsilon_0 \varepsilon |\mathbf{E}|^2 + \mu_0 \mu |\mathbf{H}|^2]$, which is recognized as the sum of electric and magnetic energy density. Thus, the second term in Eq. (5.3) corresponds to the time rate of change of electromagnetic energy in the volume V and, accordingly, the first term is the flux of energy in or out of V. The remaining terms on the right side are equal to the rate of energy dissipation inside V. According to this interpretation

$$W \; = \; \frac{1}{2} \Big[\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H} \Big] \qquad (5.4)$$

represents the density of electromagnetic energy, and

$$\mathbf{S} \; = \; (\mathbf{E} \times \mathbf{H}) \qquad (5.5)$$

is the energy flux density. $\mathbf{S}$ is referred to as the *Poynting vector*, discovered in 1883 by John Poynting and independently by Oliver Heaviside. In principle, the curl of any vector field can be added to $\mathbf{S}$ without changing the conservation law (5.3), but it is convenient to make the choice as stated in (5.5).

If the medium within V is linear and non-dispersive, the two last terms in Eq. (5.3) equal zero and the only term accounting for energy dissipation is $\mathbf{j} \cdot \mathbf{E}$. To understand this term, we consider the work done per unit time on a single charge q. In terms of the velocity $\mathbf{v}$ of the charge and the force $\mathbf{F}$ acting on it, the work per unit time is $dW/dt = \mathbf{F} \cdot \mathbf{v}$. Using the Lorentz force in Eq. (1) gives $dW/dt = q \, \mathbf{E} \cdot \mathbf{v} + q \, [\mathbf{v} \times \mathbf{B}] \cdot \mathbf{v}$. Because $[\mathbf{v} \times \mathbf{B}] \cdot \mathbf{v} = [\mathbf{v} \times \mathbf{v}] \cdot \mathbf{B} = 0$ we obtain

$\mathbf{F} = q\,\mathbf{v}\cdot\mathbf{E}$, which corresponds to the $\mathbf{j}\cdot\mathbf{E}$ term in Eq. (5.3).Thus, we find that the magnetic field does no work and that it is only the electric field that gives rise to dissipation of electromagnetic energy. The energy removed from the electromagnetic field is transferred to the charges in matter and ultimately to other forms of energy, such as heat.

In most practical applications we are interested in the mean value of **S**, that is, the value of $\mathbf{S}$ averaged over several oscillation periods. This quantity describes the net power flux density and is needed, for example, for the evaluation of radiation patterns. Assuming that the fields are harmonic in time, linear and non-dispersive, then the two last terms in Eq. (5.3) disappear. Furthermore, we assume that the energy density (5.4) only accounts for polarization and magnetization currents that a re loss-free, that is, all losses are associated with the $\mathbf{j}\cdot\mathbf{E}$ term. The time average of Eq. (5.3) then becomes

$$\int_{\partial V} \langle \mathbf{S}(\mathbf{r})\rangle \cdot \mathbf{n}\; da \;=\; -\frac{1}{2}\int_V \mathrm{Re}\left\{\mathbf{j}^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\right\}\, dV \tag{5.6}$$

where we have used complex notation. The term on the right defines the mean energy dissipation within the volume V. $\langle\mathbf{S}\rangle$ represents the time average of the Poynting vector

$$\langle \mathbf{S}(\mathbf{r})\rangle \;=\; \frac{1}{2}\,\mathrm{Re}\left\{\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\right\} \tag{5.7}$$

The magnitude of $\langle\mathbf{S}\rangle$ is called the intensity $I(\mathbf{r}) = |\langle\mathbf{S}(\mathbf{r})\rangle|$.

In the far-field, that is, far from sources and material boundaries, the electromagnetic field can be locally approximated by a plane wave (see Section 2.1.1). The electric and magnetic fields are in phase, perpendicular to each other, and the ratio of their amplitudes is constant. $\langle\mathbf{S}\rangle$ can then be expressed by the electric field alone as

$$\langle \mathbf{S}(\mathbf{r})\rangle \;=\; \frac{1}{2}\,\frac{1}{Z_i}\,|\mathbf{E}(\mathbf{r})|^2\,\mathbf{n}_r \;=\; \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,n_i\,|\mathbf{E}(\mathbf{r})|^2 \tag{5.8}$$

where $\mathbf{n}_r$ represents the unit vector in radial direction, $n_i = \sqrt{\varepsilon_i\,\mu_i}$ is the index of refraction, and Z_i is the wave impedance (4.33).

The surface integral of $\langle \mathbf{S} \rangle$ correspond to the total power generated or dissipated inside the enclosed surface, that is,

$$\bar{P} \;=\; \int_{\partial V} \langle \mathbf{S}(\mathbf{r}) \rangle \cdot \mathbf{n}\; da \;=\; \int_{\partial V} I(\mathbf{r})\; da \tag{5.9}$$

5.1.1 Example: Energy Transport by Evanescent Waves

Let us consider a single dielectric interface that is irradiated by a plane wave under conditions of *total internal reflection* (TIR) (c.f. Section 4.4). For non-absorbing media and for supercritical incidence, all the power of the incident wave is reflected. One can anticipate that because no losses occur upon reflection at the interface there is no net energy transport into the medium of transmittance. In order to prove this fact we have to investigate the time-averaged energy flux across a plane parallel to the interface. This can be done by considering the z-component of the Poynting vector (cf. Eq. (5.7))

$$\langle S \rangle_z = \frac{1}{2} \mathrm{Re}\big(E_x H_y^* - E_y H_x^*\big) \;, \tag{5.10}$$

where all fields are evaluated in the upper medium, i.e. the medium of transmittance. Applying Maxwell's equation (2.32) to the special case of a plane or evanescent wave, allows us to express the magnetic field in terms of the electric field as

$$\mathbf{H} = \sqrt{\frac{\varepsilon_0 \varepsilon}{\mu_0 \mu}} \left[\left(\frac{\mathbf{k}}{k} \right) \times \mathbf{E} \right] \;. \tag{5.11}$$

Introducing the expressions for the transmitted field components of **E** and **H** into Eq. (5.10), it is straightforward to prove that $\langle \mathbf{S} \rangle_z$ vanishes and that there is no net energy transport in the direction *normal* to the interface.

On the other hand, when considering the energy transport along the interface ($\langle \mathbf{S} \rangle_x$), a non-zero result is found:

$$\langle \mathbf{S} \rangle_{\mathrm{x}} = \frac{1}{2} \sqrt{\frac{\varepsilon_2 \mu_2}{\varepsilon_1 \mu_1}} \sin\theta_1 \left(|t^{\mathrm{s}}|^2 \left| \mathbf{E}_1^{(\mathrm{s})} \right|^2 + |t^{\mathrm{p}}|^2 \left| \mathbf{E}_1^{(\mathrm{p})} \right|^2 \right) \mathrm{e}^{-2\gamma z} \,. \tag{5.12}$$

Thus, an evanescent wave transports energy along the surface, in the direction of the transverse wavevector.

5.1.2 Energy density in dispersive and lossy media

The two last terms in Eq. (5.3) strictly vanish only in a linear medium with no dispersion and no losses. The only medium fulfilling these conditions is vacuum. For all other media, the last two terms only vanish approximately. In this section we consider a linear medium with a frequency-dependent and complex ε and μ.

Let us return to the Poynting theorem stated in Eq. (5.2). While the left hand side denotes the power flowing in or out of the volume V, the right hand side denotes the power dissipated or generated in the volume V. The three terms on the right hand side are of similar form and so we start by considering first the electric energy term $\mathbf{E}\cdot(\partial\mathbf{D}/\partial t)$. The electric energy density w_E at the time t is

$$w_E(\mathbf{r},t) \;=\; \int_{-\infty}^{t} \mathbf{E}(\mathbf{r},t') \cdot \frac{\partial \mathbf{D}(\mathbf{r},t')}{\partial t'}\, dt' \;. \tag{5.13}$$

We now express the fields $\mathbf{E}$ and $\mathbf{D}$ in terms of their Fourier transforms as $\mathbf{E}(t')=\int \hat{\mathbf{E}}(\omega)\exp[-\mathrm{i}\omega t']\,\mathrm{d}\omega$ and $\mathbf{D}(t')=\int \hat{\mathbf{D}}(\omega)\exp[-\mathrm{i}\omega t']\,\mathrm{d}\omega$, respectively. In the last expression we substitute $\omega=-\omega'$ and obtain $\mathbf{D}(t')=\int \hat{\mathbf{D}}^*(\omega')\exp[\mathrm{i}\omega' t']\,\mathrm{d}\omega'$, where we used $\hat{\mathbf{D}}(-\omega')=\hat{\mathbf{D}}^*(\omega')$ since $\mathbf{D}(t)$ is real (c.f. Eq. (2.22)). Using the linear relation $\hat{\mathbf{D}}=\varepsilon_0\,\varepsilon\hat{\mathbf{E}}$ and inserting the Fourier transforms in Eq. (5.13) yields

$$w_E(\mathbf{r},t) \;=\; \varepsilon_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\omega'\,\varepsilon^*(\omega')}{\omega'-\omega}\, \hat{\mathbf{E}}(\omega)\cdot\hat{\mathbf{E}}^*(\omega')\; \mathrm{e}^{i(\omega'-\omega)t}\; d\omega'\, d\omega \;, \tag{5.14}$$

where we have carried out the differentiation and integration over time and assumed that the fields were zero at $t\rightarrow-\infty$. For later purposes it is advantageous to represent the above result in different form. Using the substitutions $u'=-\omega$ and $u=-\omega'$ and making use of $\hat{\mathbf{E}}(-u)=\hat{\mathbf{E}}^*(u)$ and $\varepsilon(-u)=\varepsilon^*(u)$ gives an expression similar to Eq. (5.14) but in terms of u and u'. Finally, we add this expression to Eq. (5.14) and take one half of the resulting sum, which yields

$$w_E(\mathbf{r},t) \;=\; \frac{\varepsilon_0}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\omega'\varepsilon^*(\omega')-\omega\varepsilon(\omega)}{\omega'-\omega}\right] \hat{\mathbf{E}}(\omega)\cdot\hat{\mathbf{E}}^*(\omega')\; \mathrm{e}^{i(\omega'-\omega)t}\; d\omega'\, d\omega \;. \tag{5.15}$$

Similar expressions are obtained for the magnetic term $\mathbf{H}\cdot(\partial\mathbf{B}/\partial t)$ and the dissipative term $\mathbf{j}\cdot\mathbf{E}$ in Eq. (5.2).

If $\varepsilon(\omega)$ is a complex function then w_E not only accounts for the energy density built up in the medium but also for the energy transferred to the medium, such as heat dissipation. This contribution becomes indistinguishable from the term $\mathbf{j}\cdot\mathbf{E}$ in Eq. (5.2). Thus, the imaginary part of ε can be included in the conductivity σ (c.f. Eq. (3.12)) and accounted for in the term $\mathbf{j}\cdot\mathbf{E}$ through the linear relationship $\hat{\mathbf{j}}=\sigma\hat{\mathbf{E}}$. Therefore, to discuss the energy density it suffices to consider only the real part of ε, which we're going to denote as ε'.

Let us now consider a monochromatic field represented by $\hat{\mathbf{E}}(\mathbf{r},\omega) = \mathbf{E}_0(\mathbf{r})[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]/2$. Inserting in Eq. (5.15) yields four terms: two that are constant in time and two that oscillate in time. Upon averaging over an oscillation period $2\pi/\omega_0$ the oscillatory terms vanish and only the constant terms survive. For these terms we must view the expression in brackets in Eq. (5.15) as a limit, that is,

$$\lim_{\omega'\to\omega}\left[\frac{\omega'\varepsilon'(\omega') - \omega\varepsilon'(\omega)}{\omega'-\omega}\right] \;=\; \left.\frac{d\,[\omega\,\varepsilon'(\omega)]}{d\omega}\right|_{\omega=\omega_0} . \tag{5.16}$$

Thus, the cycle average of Eq. (5.15) yields

$$\bar{w_E}(\mathbf{r}) \;=\; \frac{\varepsilon_0}{4}\left.\frac{d\,[\omega\,\varepsilon'(\omega)]}{d\omega}\right|_{\omega=\omega_0} |\mathbf{E}_0(\mathbf{r})|^2 . \tag{5.17}$$

A similar result can be derived for the magnetic term $\mathbf{H}\cdot(\partial\mathbf{B}/\partial t)$.

It can be shown that Eq. (5.17) also holds for quasi-monochromatic fields which have frequency components ω only in a narrow range about a center frequency ω_0. Such fields can be represented as

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{\tilde{\mathbf{E}}(\mathbf{r},t)\} = \mathrm{Re}\{\mathbf{E}_0(\mathbf{r},t)\,\mathrm{e}^{-i\omega_0 t}\} , \tag{5.18}$$

which is known as the *slowly varying amplitude approximation*. Here, $\mathbf{E}_0(\mathbf{r},t)$ is the slowly varying (complex) amplitude and ω_0 is the 'carrier' frequency. The envelope $\mathbf{E}_0$ spans over many oscillations of frequency ω_0.

Expressing the field amplitudes in terms of time-averages, that is $|\mathbf{E}_0|^2 = 2\,\langle\mathbf{E}(t)\cdot\mathbf{E}(t)\rangle$, we can express the total cycle-averaged energy density $\bar{W}$ as

$$\bar{W} \;=\; \left[\varepsilon_0\frac{d\,[\omega\,\varepsilon'(\omega)]}{d\omega}\Big\langle\mathbf{E}\cdot\mathbf{E}\Big\rangle \;+\; \mu_0\frac{d\,[\omega\,\mu'(\omega)]}{d\omega}\Big\langle\mathbf{H}\cdot\mathbf{H}\Big\rangle\right] \tag{5.19}$$

where $\mathbf{E}=\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}=\mathbf{H}(\mathbf{r},t)$ are the time-dependent fields. Notice, that ω is the center frequency of the spectra of $\mathbf{E}$ and $\mathbf{H}$. For a medium with negligible dispersion this expression reduces to the familiar $\bar{W} = (1/2)\,[\varepsilon_o\varepsilon'|\mathbf{E}_0|^2 + \mu_o\mu'\,|\mathbf{H}_0|^2]$, which follows from Eq. (5.4) using the dispersion-free constitutive relations. Because of $d(\omega\varepsilon')/d\omega > 0$ and $d(\omega\mu')/d\omega > 0$ the energy density is always positive, even for metals with $\varepsilon' < 0$.

5.2 The Maxwell Stress Tensor

In this section we use Maxwell's equations to derive the conservation law for linear momentum in an electromagnetic field. The net force exerted on an arbitrary object is entirely determined by Maxwell's stress tensor. In the limiting case of an infinitely extended object, the formalism renders the known formulas for radiation pressure.

The general law for forces in electromagnetic fields is based on the conservation law for linear momentum. To derive this conservation law we will consider Maxwell's equations in vacuum. In this case we have $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$. Later we will relax this constraint. The conservation law for linear momentum is entirely a consequence of Maxwell's equations (1.16) - (1.19) and of the Lorentz force law (5), which connects the electromagnetic world with the mechanical one.

If we operate on Maxwell's first equation by $\times\varepsilon_0\mathbf{E}$, on the second equation by $\times\mu_0\mathbf{H}$, and then add the two resulting equations we obtain

$$\varepsilon_0(\nabla\times\mathbf{E})\times\mathbf{E} + \mu_0(\nabla\times\mathbf{H})\times\mathbf{H} = \mathbf{j}\times\mathbf{B} - \frac{1}{c^2}\left[\frac{\partial\mathbf{H}}{\partial t}\times\mathbf{E}\right] + \frac{1}{c^2}\left[\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{H}\right] \tag{5.20}$$

We have omitted the arguments $(\mathbf{r},t)$ for the different fields and we used $\varepsilon_0\mu_0 = 1/c^2$. The last two expressions in Eq. (5.20) can be combined to $(1/c^2)\,\mathrm{d}/\mathrm{d}t\,[\mathbf{E}\times\mathbf{H}]$. For the first expression in Eq. (5.20) we can write

$$\begin{aligned}\varepsilon_0(\nabla\times\mathbf{E})\times\mathbf{E} \;&= \qquad (5.21)\\ &\varepsilon_0\begin{bmatrix}\partial/\partial x(E_x^2-E^2/2) & +\partial/\partial y(E_xE_y) & +\partial/\partial z(E_xE_z)\\ \partial/\partial x(E_xE_y) & +\partial/\partial y(E_y^2-E^2/2) & +\partial/\partial z(E_yE_z)\\ \partial/\partial x(E_xE_z) & +\partial/\partial y(E_yE_z) & +\partial/\partial z(E_z^2-E^2/2)\end{bmatrix} - \varepsilon_0\mathbf{E}\,\nabla\cdot\mathbf{E}\\ &= \nabla\cdot[\varepsilon_0\mathbf{E}\mathbf{E} - (\varepsilon_0/2)E^2\,\overleftrightarrow{\mathbf{I}}] - \rho\mathbf{E}\;.\end{aligned}$$

where Eq. (1.32) has been used in the last step. The notation $\mathbf{EE}$ denotes the outer product, $E^2 = E_x^2 + E_y^2 + E_z^2$ is the electric field strength, and $\overleftrightarrow{\mathbf{I}}$ denotes the unit tensor. A similar expression can be derived for $\mu_0(\nabla \times \mathbf{H}) \times \mathbf{H}$. Using these two expressions in Eq. (5.20) we obtain

$$\nabla \cdot \left[\varepsilon_0 \mathbf{EE} - \mu_0 \mathbf{HH} - \frac{1}{2}\left(\varepsilon_0 E^2 + \mu_0 H^2\right) \overleftrightarrow{\mathbf{I}}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c^2}[\mathbf{E} \times \mathbf{H}] + \rho\,\mathbf{E} + \mathbf{j} \times \mathbf{B}\,. \tag{5.22}$$

The expression in brackets on the left hand side is called Maxwell's stress tensor in vacuum, usually denoted as $\overleftrightarrow{\mathbf{T}}$. In Cartesian components it reads as

$$\overleftrightarrow{\mathbf{T}} = \left[\varepsilon_0 \mathbf{EE} - \mu_0 \mathbf{HH} - \frac{1}{2}\left(\varepsilon_0 E^2 + \mu_0 H^2\right) \overleftrightarrow{\mathbf{I}}\right] = \tag{5.23}$$

$$\begin{bmatrix} \varepsilon_0(E_x^2 - E^2/2) + \mu_0(H_x^2 - H^2/2) & \varepsilon_0 E_x E_y + \mu_0 H_x H_y & \varepsilon_0 E_x E_z + \mu_0 H_x H_z \\ \varepsilon_0 E_x E_y + \mu_0 H_x H_y & \varepsilon_0(E_y^2 - E^2/2) + \mu_0(H_y^2 - H^2/2) & \varepsilon_0 E_y E_z + \mu_0 H_y H_z \\ \varepsilon_0 E_x E_z + \mu_0 H_x H_z & \varepsilon_0 E_y E_z + \mu_0 H_y H_z & \varepsilon_0(E_z^2 - E^2/2) + \mu_0(H_z^2 - H^2/2) \end{bmatrix}$$

After integration of Eq. (5.22) over an arbitrary volume V which contains all sources ρ and $\mathbf{j}$ we obtain

$$\int_V \nabla \cdot \overleftrightarrow{\mathbf{T}}\ \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c^2}\int_V [\mathbf{E} \times \mathbf{H}]\,\mathrm{d}V + \int_V [\rho\,\mathbf{E} + \mathbf{j} \times \mathbf{B}]\,\mathrm{d}V\,. \tag{5.24}$$

The last term is recognized as the mechanical force (cf. Eq. (5)). The volume integral on the left can be transformed to a surface integral using Gauss's integration law

$$\int_V \nabla \cdot \overleftrightarrow{\mathbf{T}}\ \mathrm{d}V = \int_{\partial V} \overleftrightarrow{\mathbf{T}} \cdot \mathbf{n}\,\mathrm{d}a\,. \tag{5.25}$$

∂V denotes the surface of V, $\mathbf{n}$ the unit vector perpendicular to the surface, and $\mathrm{d}a$ an infinitesimal surface element. We then finally arrive at the *conservation law for linear momentum*

$$\int_{\partial V} \overleftrightarrow{\mathbf{T}}(\mathbf{r},t) \cdot \mathbf{n}(\mathbf{r})\,\mathrm{d}a = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{G}_{\text{field}} + \mathbf{G}_{\text{mech}}\right] \tag{5.26}$$

Here, $\mathbf{G}_{\text{mech}}$ and $\mathbf{G}_{\text{field}}$ denote the mechanical momentum and the field momentum, respectively. In Eq. (5.26) we have used Newton's expression of the mechanical force $\mathbf{F} = \mathrm{d}/\mathrm{d}t\,\mathbf{G}_{\mathrm{mech}}$ and the definition of the *field momentum* (Abraham density)

$$\mathbf{G}_{\mathrm{field}} \;=\; \frac{1}{c^2}\int_V [\mathbf{E}\times\mathbf{H}]\,\mathrm{d}V \tag{5.27}$$

This is the momentum carried by the electromagnetic field within the volume V. It is created by the dynamic terms in Maxwell's curl equations. The time-derivative of the field momentum is zero when it is averaged over one oscillation period and hence the average mechanical force becomes

$$\langle\mathbf{F}\rangle \;=\; \int_{\partial V} \langle \overleftrightarrow{\mathbf{T}}\,(\mathbf{r},t)\rangle\cdot\mathbf{n}(\mathbf{r})\;\mathrm{d}a \tag{5.28}$$

with $\langle ...\rangle$ denoting the time average. Equation (5.28) is of general validity. It allows the mechanical force acting on an arbitrary body within the closed surface ∂V to be calculated (see Figure 5.1). The force is entirely determined by the electric and magnetic fields on the surface ∂V. It is interesting to note that no material properties enter the expression for the force; the entire information is contained in the electromagnetic field. The only material constraint is that the body is rigid. If the body deforms when it is subject to an electromagnetic field we have to include electrostrictive and magnetostrictive forces. Since the enclosing surface is

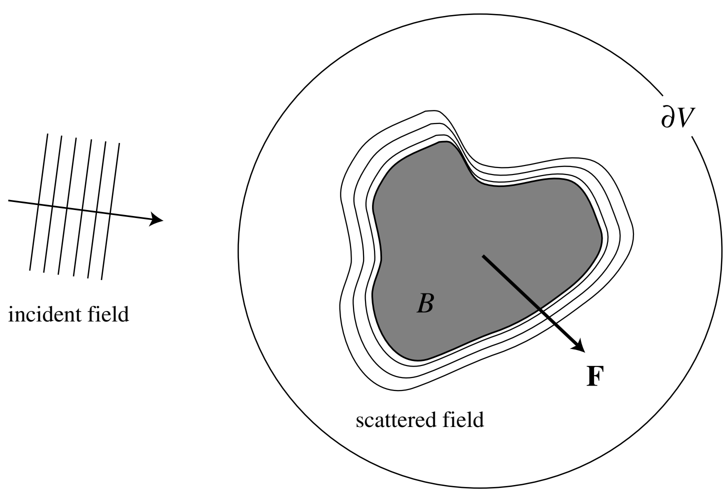

Figure 5.1: The mechanical force $\mathbf{F}$ acting on the object B is entirely determined by the electric and magnetic fields at an arbitrary surface ∂V enclosing B.

arbitrary the same results are obtained whether the fields are evaluated at the surface of the body or in the far-field. It is important to note that the fields used to calculate the force are the self-consistent fields of the problem, which means that they are a superposition of the incident and the scattered fields. Therefore, prior to calculating the force, one has to solve for the electromagnetic fields. If the object B is surrounded by a medium that can be represented accurately enough by the dielectric constant ε and magnetic susceptibility μ, the mechanical force can be calculated in the same way if we replace Maxwell's stress tensor Eq. (5.23) by

$$\overleftrightarrow{\mathbf{T}} = \left[\varepsilon_0\varepsilon\mathbf{E}\mathbf{E} - \mu_0\mu\mathbf{H}\mathbf{H} - \frac{1}{2}\left(\varepsilon_0\varepsilon E^2 + \mu_0\mu H^2\right)\overleftrightarrow{\mathbf{I}}\right] \tag{5.29}$$

5.3 Radiation pressure

Here, we consider the radiation pressure on a medium with an infinitely extended planar interface as shown in Fig. 5.2. The medium is irradiated by a monochromatic plane wave at normal incidence to the interface. Depending on the material properties of the medium, part of the incident field is reflected at the interface. Introducing the complex reflection coefficient r, the electric field outside the medium can be written as the superposition of two counter-propagating plane waves

$$\mathbf{E}(\mathbf{r},t) = E_0\,\mathrm{Re}\left\{\left[\mathrm{e}^{\mathrm{i}kz} + r\,\mathrm{e}^{-\mathrm{i}kz}\right]\mathrm{e}^{-\mathrm{i}\omega t}\right\}\mathbf{n}_x\,. \tag{5.30}$$

Using Maxwell's curl equation (1.33) we find for the magnetic field

$$\mathbf{H}(\mathbf{r},t) = \sqrt{\varepsilon_0/\mu_0}\,E_0\,\mathrm{Re}\left\{\left[\mathrm{e}^{\mathrm{i}kz} - r\,\mathrm{e}^{-\mathrm{i}kz}\right]\mathrm{e}^{-\mathrm{i}\omega t}\right\}\mathbf{n}_y\,. \tag{5.31}$$

To calculate the radiation pressure P we integrate Maxwell's stress tensor on an infinite planar surface A parallel to the interface as shown in Fig. 5.2. The radiation pressure can be calculated by using Eq. (5.28) as

$$P\,\mathbf{n}_z = \frac{1}{A}\int_A \langle\overleftrightarrow{\mathbf{T}}(\mathbf{r},t)\rangle\cdot\mathbf{n}_z\,\mathrm{d}a\,. \tag{5.32}$$

We do not need to consider a closed surface ∂V since we are interested in the pressure exerted on the interface of the medium and not in the mechanical force

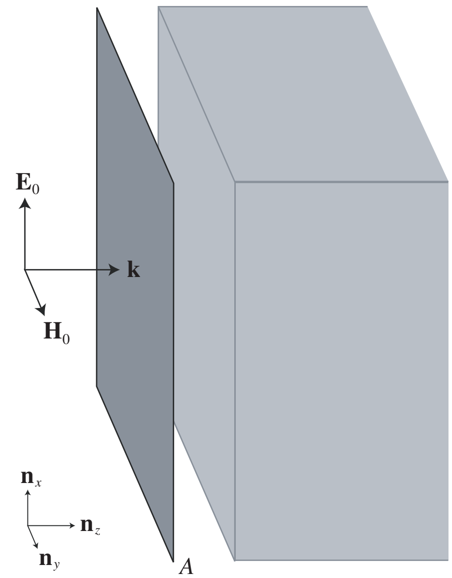

Figure 5.2: Configuration used to derive the radiation pressure.

acting on the medium. Using the fields of Eqs. (5.30) and (5.31) we find that the first two terms in Maxwell's stress tensor Eq. (5.23) give no contribution to the radiation pressure. The third term yields

$$\langle \overleftrightarrow{\mathbf{T}}(\mathbf{r},t)\rangle \cdot \mathbf{n}_z \;=\; -\frac{1}{2}\,\langle \varepsilon_0 E^2 + \mu_0 H^2\rangle\, \mathbf{n}_z \;=\; \frac{\varepsilon_0}{2}\, E_0^2\,[1+|r|^2]\,\mathbf{n}_z\;. \tag{5.33}$$

Using the definition of the intensity of a plane wave $I_0 \,=\, (\varepsilon_0/2)\,c\,E_0^2$, c being the vacuum speed of light, we can express the radiation pressure as

$$P \;=\; \frac{I_0}{c}\;[1\,+\,R]\;, \tag{5.34}$$

with $R \,=\, |r|^2$ being the reflectivity. For a perfectly absorbing medium we have $R = 0$, whereas for a perfectly reflecting medium $R = 1$. Therefore, the radiation pressure on a perfectly reflecting medium is twice as high as for a perfectly absorbing medium.

To conclude this chapter we should emphasize the importance of electromagnetic energy and momentum. Energy and momentum are fundamental concepts of physics that make transitions between different fields feasible. For example, electromagnetic energy can be transferred to heat, which is a concept of thermodynamics, and electromagnetic momentum can be transferred to mechanical forces, which is a concept of classical mechanics. Electromagnetic energy and momentum are also used in Lagrangian and Hamiltonian formalisms, which form the stepping stones to quantum mechanics and quantum electrodynamics. Thus, energy and momentum make it possible to transition between different fields of physics. Such transitions cannot be accomplished by more standard electrical concepts such as voltage and current.