Chapter 1

Maxwell's Equations

Equations (6) summarize the knowledge of electromagnetism as it was understood by the mid 19th century. In 1873, however, James Clerk Maxwell introduced a critical modification that kick-started an era of wireless communication.

1.1 The Displacement Current

Eq. (7) is a statement of current conservation, that is, currents cannot be generated or destroyed, the net flux through a closed surface is zero. However, this law is flawed. For example, let's take a bunch of identical charges and hold them together (see Fig. 1.1). Once released, the charges will speed out because of Coulomb repulsion and there will be a net outward current. Evidently, the outward current is balanced by the decrease of charge inside the enclosing surface ∂V, and hence, Eq. (7) has to be corrected as follows

$$\int_{\partial V} \mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,da \;=\; -\frac{\partial}{\partial t}\int_V \rho(\mathbf{r},t)\,dV \tag{1.1}$$

This equation describes the conservation of charge. It's general form is found in many different contexts in physics and we will encounter it again when we discuss the conservation of energy (Poynting theorem).

Because Eq. (7) has been derived from Ampère's law, we need to modify the latter in order to end up with the correct conservation law of Eq. (1.1). This is where Maxwell comes in. He added an additional term to Ampère's law and arrived at

$$\int_{\partial A} \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{s} \;=\; \mu_0 \int_A \mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,da \,+\, \frac{1}{c^2}\frac{\partial}{\partial t}\int_A \mathbf{E}(\mathbf{r},t)\cdot\mathbf{n}\,da\;, \tag{1.2}$$

where $1/c^2 = \varepsilon_0\mu_0$. The last term has the form of a time-varying current. Therefore, $\varepsilon_0\,\partial\mathbf{E}/\partial t$ is referred to as the *displacement current*.

We again apply this equation to the end faces of a small cube (c.f. Fig. 6) and, as before, the left hand side vanishes. Thus,

$$\mu_0 \int_{\partial V} \mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,da \,+\, \frac{1}{c^2}\frac{\partial}{\partial t}\int_{\partial V} \mathbf{E}(\mathbf{r},t)\cdot\mathbf{n}\,da \;=\; 0\;. \tag{1.3}$$

Substituting Gauss' law from Eq. (6) for the second expression yields the desired charge continuity equation (1.1).

In summary, replacing Ampère's law in (6) by Eq. (1.2) yields a set of four equations for the fields $\mathbf{E}$ and $\mathbf{B}$ that are consistent with the charge continuity equation. These four equations define what is called Maxwell's integral equations.

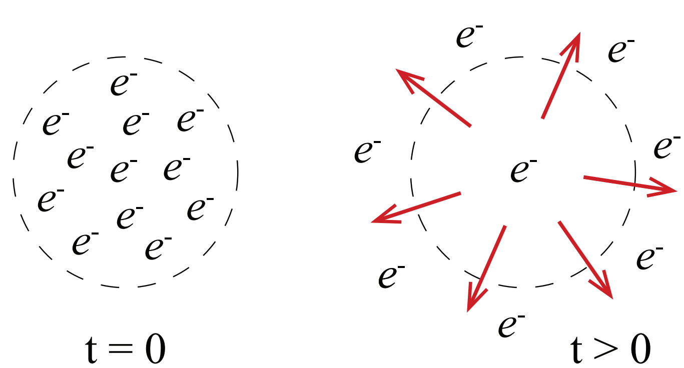

Figure 1.1: Illustration of charge conservation. A bunch of identical charges is held together at $t = 0$. Once released, the charges will spread out due to Coulomb repulsion, which gives rise to a net outward current flow.

1.2 Interaction of Fields with Matter

So far we have discussed the properties of the fields $\mathbf{E}$ and $\mathbf{B}$ in free space. The sources of these fields are charges ρ and currents $\mathbf{j}$, so-called *primary* sources. However, $\mathbf{E}$ and $\mathbf{B}$ can also interact with materials and generate *induced* charges and currents. These are then called *secondary* sources.

To account for these secondary sources we write

$$\rho_{\rm tot} \;=\; \rho + \rho_{\rm pol} \; , \tag{1.4}$$

where ρ is the charge density associated with primary sources. It is assumed that these sources are *not* affected by the fields $\mathbf{E}$ and $\mathbf{B}$. On the other hand, $\rho_{\rm pol}$ is the charge density induced in matter through the interaction with the electric field. It is referred to as the polarization charge density.[1] On a microscopic scale, the electric field slightly distorts the atomic orbitals in the material (see Fig. 1.2). On a macroscopic scale, this results in an accumulation of charges at the surface of the material (see Fig. 1.3). The net charge density inside the material remains zero.

To account for polarization charges we introduce the polarization $\mathbf{P}$ which, in analogy to Gauss' law in Eq. (6), is defined as

$$\int_{\partial V} \mathbf{P}(\mathbf{r},t)\cdot\mathbf{n}\, da \;=\; -\int_V \rho_{\rm pol}(\mathbf{r},t)\, dV \; . \tag{1.5}$$

$\mathbf{P}$ has units of Cb/m^2, which corresponds to dipole moment (Cb / m) per unit vol-

[1]The $\mathbf{B}$-field interacts only with currents and not with charges.

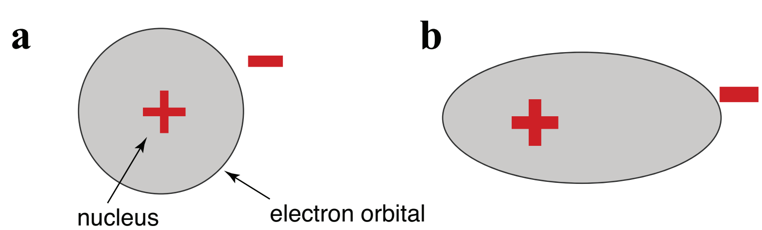

Figure 1.2: Microscopic polarization. An external electric field $\mathbf{E}$ distorts the orbital of an atom. (a) Situation with no external field. (b) Situation with external field.

ume (m^3). Inserting Eqs. (1.4) and (1.5) into Gauss' law yields

$$\int_{\partial V} [\varepsilon_0 \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)] \cdot \mathbf{n}\, da = \int_V \rho(\mathbf{r},t)\, dV \; . \tag{1.6}$$

The expression in brackets is called the *electric displacement*

$$\mathbf{D} \;=\; \varepsilon_0 \mathbf{E} + \mathbf{P} \; . \tag{1.7}$$

Time-varying polarization charges give rise to polarization currents. To see this, we take the time-derivative of Eq. (1.5) and obtain

$$\int_{\partial V} \frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t) \cdot \mathbf{n}\, da \;=\; \frac{\partial}{\partial t} \int_V \rho_{\rm pol}(\mathbf{r},t)\, dV \; , \tag{1.8}$$

which has the same appearance as the charge conservation law (1.1). Thus, we identify $\partial\mathbf{P}/\partial t$ as the *polarization current density*

$$\mathbf{j}_{\rm pol}(\mathbf{r},t) \;=\; \frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t) \; . \tag{1.9}$$

To summarize, the interaction of the $\mathbf{E}$-field with matter gives rise to polarization charges and polarization currents. The magnitude and the dynamics of these secondary sources depends on the material properties [$\mathbf{P} = f(\mathbf{E})$], which is the subject of solid-state physics.

An electric field interacting with matter not only gives rise to polarization currents but also to conduction currents. We will denote the conduction current density as $\mathbf{j}_{\rm cond}$. Furthermore, according to Ampère's law, the interaction of matter with

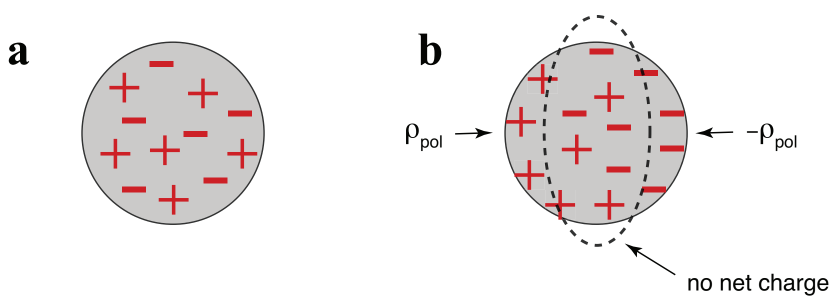

Figure 1.3: Macroscopic polarization. An external electric field $\mathbf{E}$ accumulates charges at the surface of an object. (a) Situation with no external field. (b) Situation with external field.

magnetic fields can induce magnetization currents. We will denote the magnetization current density as $\mathbf{j}_{\rm mag}$. Taken all together, the total current density can be written as

$$\mathbf{j}_{\rm tot}(\mathbf{r},t) \;=\; \mathbf{j}_0(\mathbf{r},t) + \mathbf{j}_{\rm cond}(\mathbf{r},t) + \mathbf{j}_{\rm pol}(\mathbf{r},t) + \mathbf{j}_{\rm mag}(\mathbf{r},t)\;, \tag{1.10}$$

where $\mathbf{j}_0$ is the source current density. In the following we will not distinguish between source current and conduction current and combine the two as

$$\mathbf{j}(\mathbf{r},t) \;=\; \mathbf{j}_0(\mathbf{r},t) + \mathbf{j}_{\rm cond}(\mathbf{r},t)\;. \tag{1.11}$$

$\mathbf{j}$ is simply the current density due to *free* charges, no matter whether primary or secondary. On the other hand, $\mathbf{j}_{\rm pol}$ is the current density due to *bound* charges, that is, charges that experience a restoring force to a point of origin. Finally, $\mathbf{j}_{\rm mag}$ is the current density due to *circulating* charges, associated with magnetic moments.

We now introduce (1.10) into Ampère's modified law (1.2) and obtain

$$\int_{\partial A} \mathbf{B}\cdot d\mathbf{s} \;=\; \mu_0 \int_A \left[\mathbf{j} \,+\, \left(\mathbf{j}_{\rm pol} + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}\right) \,+\, \mathbf{j}_{\rm mag}\right]\cdot \mathbf{n}\, da\;, \tag{1.12}$$

where we have skipped the arguments $(\mathbf{r},t)$ for simplicity. According to Eqs. (1.9) and (1.7), the term inside the round brackets is equal to $\partial\mathbf{D}/\partial t$. To relate the induced magnetization current to the $\mathbf{B}$-field we define in analogy to Eq. (1.5)

$$\int_{\partial A} \mathbf{M}(\mathbf{r},t)\cdot d\mathbf{s} \;=\; \int_A \mathbf{j}_{\rm mag}(\mathbf{r},t)\cdot\mathbf{n}\, da\;, \tag{1.13}$$

with $\mathbf{M}$ being the magnetization. Inserting into Eq. (1.12) and rearranging terms leads to

$$\int_{\partial A} \left[\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right]\cdot d\mathbf{s} \;=\; \int_A \left[\mathbf{j} \,+\, \frac{\partial}{\partial t}\mathbf{D}\right]\cdot\mathbf{n}\, da\,, \tag{1.14}$$

The expression in brackets on the left hand side is called the *magnetic field*

$$\mathbf{H} \;=\; \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\;. \tag{1.15}$$

It has units of A/m. The magnitude and the dynamics of magnetization currents depends on the specific material properties [$\mathbf{M} = f(\mathbf{B})$].

1.3 Maxwell's Equations in Integral Form

Let us now summarize our knowledge electromagnetism. Accounting for Maxwell's displacement current and for secondary sources (conduction, polarization and magnetization) turns our previous set of four equations (6) into

$$\int_{\partial V} \mathbf{D}(\mathbf{r},t)\cdot\mathbf{n}\,da \;=\; \int_V \rho(\mathbf{r},t)\,dV \tag{1.16}$$

$$\int_{\partial A} \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{s} \;=\; -\frac{\partial}{\partial t}\int_A \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,da \tag{1.17}$$

$$\int_{\partial A} \mathbf{H}(\mathbf{r},t)\cdot d\mathbf{s} \;=\; \int_A \left[\mathbf{j}(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right]\cdot\mathbf{n}\,da \tag{1.18}$$

$$\int_{\partial V} \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,da \;=\; 0 \tag{1.19}$$

The displacement $\mathbf{D}$ and the induction $\mathbf{B}$ account for secondary sources through

$$\mathbf{D}(\mathbf{r},t) \;=\; \varepsilon_0\mathbf{E}(\mathbf{r},t)+\mathbf{P}(\mathbf{r},t)\,, \quad \mathbf{B}(\mathbf{r},t) \;=\; \mu_0\left[\mathbf{H}(\mathbf{r},t)+\mathbf{M}(\mathbf{r},t)\right] \tag{1.20}$$

These equations are always valid since they don't specify any material properties. To solve Maxwell's equations (1.16)-(1.19) we need to invoke specific material properties, i.e. $\mathbf{P}=f(\mathbf{E})$ and $\mathbf{M}=f(\mathbf{B})$, which are denoted *constitutive relations*.

1.4 Maxwell's Equations in Differential Form

For most of this course it will be more convenient to express Maxwell's equations in differential form. Using Stokes' and Gauss' theorems we can easily transform Eq. (1.16)-(1.19). However, before doing so we shall first establish the notation that we will be using.

Differential Operators

The gradient operator (grad) will be represented by the nabla symbol (∇) and is defined as a Cartesian vector

$$\nabla \equiv \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} . \tag{1.21}$$

It can be transformed to other coordinate systems in a straightforward way. Using ∇ we can express the divergence operator (div) as $\nabla \cdot$. To illustrate this, let's operate with $\nabla \cdot$ on a vector $\mathbf{F}$

$$\nabla \cdot \mathbf{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z\,. \tag{1.22}$$

In other words, the divergence of $\mathbf{F}$ is the scalar product of ∇ and $\mathbf{F}$.

Similarly, we express the rotation operator (rot) as $\nabla\times$ which, when applied to a vector $\mathbf{F}$ yields

$$\nabla \times \mathbf{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \partial F_z/\partial y - \partial F_y/\partial z \\ \partial F_x/\partial z - \partial F_z/\partial x \\ \partial F_y/\partial x - \partial F_x/\partial y \end{bmatrix} . \tag{1.23}$$

Thus, the rotation of $\mathbf{F}$ is the vector product of ∇ and $\mathbf{F}$.

Finally, the Laplacian operator (Δ) can be written as $\nabla \cdot \nabla = \nabla^2$. Applied to a scalar ψ we obtain

$$\nabla^2\psi = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{bmatrix} = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial z^2}\psi\,. \tag{1.24}$$

Very often we will encounter sequences of differential operators, such as $\nabla \times \nabla\times$. The following identities can be easily verified and are helpful to memorize

$$\nabla \times \nabla\psi = 0 \tag{1.25}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{1.26}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}\,. \tag{1.27}$$

The last term stands for the vector $\nabla^2\mathbf{F} = [\nabla^2 F_x,\, \nabla^2 F_y,\, \nabla^2 F_z]^T$.

The Theorems of Gauss and Stokes

The theorems of Gauss and Stokes have been derived in *Analysis II*. We won't reproduce the derivation and only state their final forms

$$\int_{\partial V} \mathbf{F}(\mathbf{r},t)\cdot\mathbf{n}\,da \;=\; \int_{V} \nabla\cdot\mathbf{F}(\mathbf{r},t)\,dV \qquad \text{Gauss' theorem} \tag{1.28}$$

$$\int_{\partial A} \mathbf{F}(\mathbf{r},t)\cdot d\mathbf{s} \;=\; \int_{A} [\nabla\times\mathbf{F}(\mathbf{r},t)]\cdot\mathbf{n}\,da \qquad \text{Stokes' theorem} \tag{1.29}$$

Using these theorems we can turn Maxwell's integral equations (1.16)-(1.19) into differential form.

Differential Form of Maxwell's Equations

Applying Gauss' theorem to the left hand side of Eq. (1.16) replaces the surface integral over ∂V by a volume integral over V. The same volume integration is performed on the right hand side, which allows us to write

$$\int_{\partial V} [\nabla\cdot\mathbf{D}(\mathbf{r},t) \,-\, \rho(\mathbf{r},t)]\; dV \;=\; 0\;. \tag{1.30}$$

This result has to hold for any volume V, which can only be guaranteed if the integrand is zero, that is,

$$\nabla\cdot\mathbf{D}(\mathbf{r},t) \;=\; \rho(\mathbf{r},t)\;. \tag{1.31}$$

This is Gauss' law in differential form. Similar steps and arguments can be applied to the other three Maxwell equations, and we end up with *Maxwell's equations in differential form*

$$\nabla\cdot\mathbf{D}(\mathbf{r},t) \;=\; \rho(\mathbf{r},t) \tag{1.32}$$

$$\nabla\times\mathbf{E}(\mathbf{r},t) \;=\; -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \tag{1.33}$$

$$\nabla\times\mathbf{H}(\mathbf{r},t) \;=\; \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) \,+\, \mathbf{j}(\mathbf{r},t) \tag{1.34}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},t) \;=\; 0 \tag{1.35}$$

It has to be noted that it was Oliver Heaviside who in 1884 has first written Maxwell's equations in this compact vectorial form. Maxwell had written most of his equations in Cartesian coordinates, which yielded long and complicated expressions.

Maxwell's equations form a set of four coupled differential equations for the fields $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$. The components of these vector fields constitute a set of 16 unknowns. Depending on the considered medium, the number of unknowns can be reduced considerably. For example, in linear, isotropic, homogeneous and source-free media the electromagnetic field is entirely defined by two scalar fields. Maxwell's equations combine and complete the laws formerly established by Faraday, Oersted, Ampère, Gauss, Poisson, and others. Since Maxwell's equations are differential equations they do not account for any fields that are constant in space and time. Any such field can therefore be added to the fields.

Let us remind ourselves that the concept of fields was introduced to explain the transmission of forces from a source to a receiver. The physical observables are therefore forces, whereas the fields are definitions introduced to explain the troublesome phenomenon of the "action at a distance".

The conservation of charge is implicitly contained in Maxwell's equations. Taking the divergence of Eq. (1.34), noting that $\nabla \cdot \nabla \times \mathbf{H}$ is identical zero, and substituting Eq. (1.32) for $\nabla \cdot \mathbf{D}$ one obtains the continuity equation

$$\nabla \cdot \mathbf{j}(\mathbf{r},t) + \frac{\partial}{\partial t}\rho(\mathbf{r},t) \;=\; 0 \qquad (1.36)$$

consistent with the integral form (1.1) derived earlier.