Pólya-type criteria for conditionally strictly positive definite functions on (Hilbert) spheres

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Abstract

Isotropic positive definite functions are used in approximation theory and are for example applied in geostatistics and physiology. They are also of importance in statistics where they occur as correlation functions of homogeneous random fields on spheres. We study a class of function applicable for interpolation of arbitrary scattered data on $\mathbb{S}^{d-1}$ by linear combination of shifts of an isotropic basis function ϕ and low order spherical harmonics.

A class of functions for which the resulting interpolation problem is uniquely solvable for any distinct point set $\Xi \subset \mathbb{S}^{d-1}$ is the class of conditionally strictly positive definite functions $(CSPD_m(\mathbb{S}^{d-1}))$, where m indicates the order of spherical harmonics added. Using Schoenbergs famous representations of isotropic positive definite functions on $\mathbb{S}^{d-1}$ and $\mathbb{S}^{\infty}$ [2] as starting point we derive new sufficient conditions for conditionally strict positive definiteness on all spheres $\mathbb{S}^{d-1}$, with $d > 2$, which only require monotonicity properties of the basis functions.

The results extend a characterisation of Ma described in [1] for positive definite functions. To be precise, we prove that a function $\phi = \varphi(\cos(\cdot))$ for which

$$\varphi(-x) - \varphi(x) \text{ and } \varphi(x) + \varphi(-x), \quad x \in [0, 1),$$

are absolute monotone of order k and no polynomials is strictly conditionally positive definite of this order. Further we prove that this condition is necessary and sufficient for strictly positive definite functions and conditionally negative definite functions. As a result a theorem of Gneiting is generalised showing that all functions $\phi \in C([0, \pi])$ which are completely monotone on $(0, \pi)$ and no linear polynomials are strictly positive definite on all spheres. Similar results are presented for conditionally positive definite functions of order 1.

References

[1] C. Ma: *Isotropic Covariance Matrix Functions On All Spheres*, Mathematical Geosciences, 47:699-717, 2015.

[2] I.J. Schoenberg: *Positive definite functions on the sphere*, Duke Math. J., 9:96-108, 1942.