Numerical methods for fractional SPDEs with applications to spatial statistics

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Abstract: The numerical approximation of the solution u to a stochastic partial differential equation (SPDE) with additive spatial white noise on a bounded domain in $\mathbb{R}^d$ is considered. The differential operator is given by the fractional power L^β, $\beta \in (0,1)$, of an integer order elliptic differential operator L and is therefore non-local. Its inverse $L^{-\beta}$ is represented by a Bochner integral from the Dunford–Taylor functional calculus. For this integral representation, a quadrature has been proposed in [1]. In this way, the inverse fractional power operator $L^{-\beta}$ is approximated by a weighted sum of non-fractional resolvents $(I + t_j^2 L)^{-1}$ at certain quadrature nodes $t_j > 0$. The resolvents are then discretized in space by a standard finite element method.

In this work, the approach of [1] is combined with an approximation of the white noise, which is based only on the mass matrix of the finite element discretization. Thus, an efficient numerical algorithm for computing samples of the approximate solution is obtained. For the resulting approximation $u_{h,k}^Q$ of u, a concise analysis of the strong mean-square error [2] and the weak error [3] is performed and explicit rates of convergence are derived. Specifically, for a twice continuously Fréchet differentiable real-valued function φ with second derivative of polynomial growth, the weak error $|\mathbb{E}[\varphi(u)] - \mathbb{E}[\varphi(u_{h,k}^Q)]|$ is analyzed, and it is shown that the component of the convergence rate stemming from the stochasticity is doubled compared to the corresponding strong rate.

A key property of the presented scheme is that it does not require the knowledge of the eigenfunctions of the differential operator, which is necessary, e.g., for approximations based on truncated spectral Karhunen-Loève expansions of the noise term. For this reason, the method is particularly interesting for real-world applications in spatial statistics, such as to employ solutions to fractional order SPDEs as approximations of Gaussian Matérn fields, where $L = \kappa^2 - \Delta$, $\kappa > 0$, augmented with appropriate boundary conditions. This application is taken up in numerical experiments on the unit cube $(0,1)^d$ in $d = 1, 2, 3$ spatial dimensions for varying $\beta \in (0,1)$ to illustrate and attest the theoretical results. Furthermore, the method is used in a statistical application for estimating the posterior mean of the random field u conditioned on data [4].

References

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