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hp Finite Element Method pricing algorithms for lookback options in Lévy markets

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Chapter 1

Theory of Option Pricing

1.1 No-Arbitrage option pricing theory

In a frictionless market with a self financing strategy we cannot achieve a certain gain, an arbitrage, as discussed in [DeSch08]. Since the probability of making a negative or positive yield is always positive (or both are zero) we can find a measure under which the expected returns from our investment will be that of the risk free savings. We call this measure the risk-neutral measure, and it is shown in [DeSch94] that if there is no arbitrage in the market then such a measure will exist, and every uncertain gain can be priced by taking the payoff's expectation under the new measure and discounting it by the risk free interest rates.

Since [Ka49] we have that the value of such an expectation satisfies a backward Kolmogorov equation driven by the infinitesimal generator of the underlying price process.

In this thesis we will focus on solving these equations numerically arising from exponential Lévy models using Finite Element techniques.

1.2 The Black–Scholes framework

As described in [Sh04] the Black–Scholes framework models the stock price as a geometric Brownian motion under the risk-neutral measure

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Since the stock price follows a Markovian-process, due to the Feynman-Kac theorem the value of a European contract can be described with a function f defined as

$$f(t,x) = e^{-r(T-t)}\mathbb{E}\left[g(S_T) \middle| S_t = x \right],$$

and using Itô's formula, the process of $\left(f(\cdot, S_\cdot)_t\right)_{t\geq 0}$ will satisfy the SODE

$$d_t f(t,S_t) = \left(\frac{\partial}{\partial t} f(t,S_t) + rS_t \frac{\partial}{\partial S_t} f(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2} f(t,S_t) \right) dt + \frac{\partial}{\partial x} f(t,S_t)\, \sigma S_t dW_t.$$

Relying on the no-arbitrage argument we can state that under the risk-neutral measure the discounted gains from investing in this contract should follow a martingale. In order to achieve that the following condition for the drift should hold

$$\frac{\partial}{\partial t} f(t,S_t) + rS_t \frac{\partial}{\partial S_t} f(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2} f(t,S_t) = rf(t,S_t),$$

which is the pricing equation or backward Kolmogorov equation PDE, with the final condition

$$f(T,x) = g(x).$$

Now let $\mathcal{A}^{BS}$ denote the infinitesimal generator of a geometric Brownian motion,

$$\mathcal{A}^{BS} f = \frac{\partial}{\partial t} f + rS \frac{\partial}{\partial S_t} f + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S_t^2} f.$$

So the pricing equation for a European type payoff can be rephrased, after changing to time to maturity $s = T - t$ notation,

$$-\partial_s f + \mathcal{A}^{BS} f - rf = 0, \tag{1.1}$$

with initial condition

$$f(0,S) = g(S).$$

1.3 Lévy processes

The class of Lévy processes is a natural generalization of the Wiener process, in which discontinuous paths are incorporated. Certain Lévy models can replicate the observed path properties of stock returns, and in the meantime they give a good fit to option prices in the market.

Definition 1 *A positive, but not necessary bounded measure $\nu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}_{>0}$ is called a Lévy measure if it satisfies*

1. $\nu(\{0\}) = 0$
2. $\int_{\mathbb{R}} (1 \wedge x^2)\, \nu(dx) < \infty$

For an arbitrary measurable set $A \in \mathcal{B}(\mathbb{R})$ the Lévy measure $\nu(A)$ is the expected value of the *jump measure* associated with the Lévy process, which counts the jumps of sizes in A happening up till time 1 as

$$\begin{aligned} \nu(A) &= \mathbb{E}\left[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}\right] \\ &= \mathbb{E}\left[J([0,1], A)\right], \end{aligned}$$

where $J([0,t], B)$ is the jump measure associated with the Lévy process, meaning

$$J([t_1,t_2], A) = \#\{t \in (t_1,t_2] : \Delta X_t \neq 0, \Delta X_t \in A\}.$$

We call $\int_{\mathbb{R}} \nu(dx)$ the *intensity* of the Lévy measure, and if $\lambda := \int_{\mathbb{R}} \nu(dx) < \infty$, then we speak of a Lévy process with finite activity, which is a compound Poisson process if there is no drift or diffusion term.

Definition 2 (Lévy process) *On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions from [DeSch08, p. 72], the adapted càdlàg stochastic process $(L_t)_{t\geq 0}$ is called a Lévy-process iff*

1. *L has independent increments: $L_t - L_s$ is independent of $\mathcal{F}_s$, for $0 \leq s < t$.*
2. *L has stationary increments: $L_t - L_s \sim L_{t-s}$, for $0 \leq s < t$.*
3. *L is stochastically continuous: $\forall \varepsilon > 0$, $\lim_{t\to s} P\left[|L_t - L_s| > \varepsilon\right] = 0$.*

4. L starts at zero: $L_0 = 0$.

The distribution of a Lévy process $(L_t)_{t\geq 0}$ at time $t>0$ can be represented with a telescopic sum as $L_t = \sum_{i=1}^{n}\left(L_{\frac{ti}{n}} - L_{\frac{t(i-1)}{n}}\right)$ $\forall n\in\mathbb{N}$. In this representation every increment is independent from the other (due to the first property of *Definition 2*) and they have identical distributions (due to the second property), so a Lévy process at a given time follows an infinitely divisible distribution.

Following the results by [Lu70], the characteristic function of an infinitely divisible distribution at a given time can be expressed as

$$\varphi_{L_t}(u) = \mathbb{E}\left[e^{iuL_t}\right] = \left(\mathbb{E}\left[e^{iuL_1}\right]\right)^t.$$

Theorem 3 *The Lévy–Khinchin representation of the characteristic function of an infinitely divisible distribution, as in [Sa99, Theorem 8.1], states that for* $\psi(u) = \ln\left(\mathbb{E}\left[e^{iuL_1}\right]\right)$ *we have*

$$\psi(u) = i\gamma u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1}\right)\nu(dx), \tag{1.2}$$

where ν *is a Lévy measure,* $\gamma\in\mathbb{R}$, $\sigma\in\mathbb{R}_{\geq 0}$ *and the triplet* (γ,σ,ν) *uniquely characterizes the Lévy process.*

The characteristic triplet (γ,σ,ν) *is also called Lévy triplet or generating triplet of the Lévy process* $(L)_{t\geq 0}$.

Truncating the integrand in (1.2) with a function $f(x) = \mathcal{O}(|x|)$ as $|x|\to 0$ (in our case with $x\mathbb{1}_{|x|\leq 1}$) is required to make $e^{iux}-1$ integrable in the neighborhood of 0, because

$$e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1} = \mathcal{O}\left(|x|^2\right) \quad \text{as } |x|\to 0.$$

No such truncation is necessary if the Lévy measure ν satisfies

$$\int_{|x|\leq 1}|x|\,\nu(dx) < \infty. \tag{1.3}$$

For simplicity from here on we will assume the Lévy processes we will be looking at satisfy (1.3).

From the Lévy–Ito decomposition [Sa99, Theorems 19.2 and 19.3], every Lévy process can be decomposed uniquely to the sum of three independent processes, $L = L^{(1)} + L^{(2)} + L^{(3)}$, such that $L^{(1)}$ is a linear transformation of a Brownian-motion, $L^{(2)}$ is a compound Poisson process with jump sizes bigger than 1 $\mathbf{P}$-a.s. and $L^{(3)}$ is a pure jump martingale with finite quadratic variation.

Remark 4 *Examples of Lévy processes are arithmetic Brownian motion (Gaussian Lévy process) with the Lévy triplet of* $\left(\mu - \frac{\sigma^2}{2},\sigma,0\right)$*; variance gamma process [MaSe90]; CGMY (tempered* α*-stable process, [CGMY02]) with the triplet* $(\gamma_0,0,\nu)$ *where* $\nu(dx) = k(x)\,dx$ *and* $k(x)$ *is*

$$k(x) = \frac{\exp(-G|x|)}{|x|^{1+Y}}\mathbb{1}_{x<0} + \frac{\exp(-M|x|)}{|x|^{1+Y}}\mathbb{1}_{x>0}.$$

Theorem 5 *The price of a European option given as the discounted expectation*

$$f(t,x) = e^{-r(T-t)}\mathbb{E}_Q\left[g\left(S_0e^{L_T}\right)\middle|\, L_t = x\right]$$

with terminal payoff function $g(S_T)$ in an exponential Lévy market model solves the partial integro-differential equation (PIDE)

$$\partial_t f + \mathcal{A}^L f - rf = 0,$$

where $\mathcal{A}^L$ is the infinitesimal generator of the underlying Lévy process.

Proof. Assume that under a risk neutral measure the stock price process $(S_t)_{t\geq 0}$ follows an exponential Lévy process

$$S_t = S_0 e^{rt+L_t}, \tag{1.4}$$

where r is the risk-free interest rate and L_t is a Lévy process that has finite exponential moments. The process can be characterized with the characteristic triplet (γ_0, σ, ν), where γ_0 is chosen such that

$$\gamma_0 = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1)\,\nu(dx),$$

to ensure that $\exp(L_t)$ is a martingale. Rephrasing (1.4), $(S_t)_{t\geq 0}$ solves the SODE

$$dS_t = (r + \gamma_0) S_{t-} dt + \sigma S_{t-} dW_t + \int_{\mathbb{R}} (e^x - 1) S_{t-} J(dt, dx),$$

with initial condition $(S_t)_{t=0} = S_0$.

Since S is a (function of a) Markovian process, one can use the (extended) Feynmann–Kac theorem [Sg95, Theorem 4.1.] stating

$$f(t,x) = e^{-r(T-t)} \mathbb{E}_Q \left[g\left(S_0 e^{L_T}\right) \big| L_t = x \right],$$

then by Itô's formula for scalar Lévy processes [CoTa04, prop. 8.15] we get

$$df(t, L_t) = \partial_t f(t, L_{t-})\,dt + \frac{\sigma^2}{2} \partial_{xx} f(t, L_t)\,dt + \partial_x f(t, L_{t-})\,dL_t + \Delta f(t, L_t) - \Delta L_t \partial_x f(t, L_{t-}),$$

which is, by the martingale decomposition for dL_t as in [CoTa04, Prop. 8.16], is equivalent to

$$\begin{aligned} df(t, L_t) \quad = \quad & \left(\partial_t f(t, L_{t-}) + (r + \gamma_0)\,\partial_x f(t, L_{t-}) + \frac{\sigma^2}{2} \partial_{xx} f(t, L_t) \right) dt \\ & + \left(\int_{\mathbb{R}\setminus\{0\}} \left(f(t, L_{t-} + y) - f(t, L_{t-}) - \partial_x f(t, L_{t-})\,y \right) \nu(dy) \right) dt \\ & + \underbrace{\sigma \partial_x f(t, L_{t-})\,dW_t}_{(*)} + \underbrace{\int_{\mathbb{R}\setminus\{0\}} \left(f(t, L_{t-} + y) - f(t, L_{t-}) \right) \left(J(dt, dy) - \nu(dy)\,dt \right)}_{(**)} \end{aligned}$$

where the last two terms, $(*)$ and $(**)$, are martingales, because $J(dt, dy) - \nu(dy)\,dt$ (the $-\nu(dy)$ comes from the $-\Delta L_t \partial_x f(t, L_{t-})$ term) is a compensated jump measure.

Finally by using the no-arbitrage arguments, we get that the drift should equal $rf(t, L_{t-})$. This yields a partial integro-differential equation of

$$\partial_t f(t, L_{t-}) + \frac{\sigma^2}{2} \partial_{xx} f(t, L_t) + \int_{\mathbb{R}\setminus\{0\}} \left(f(t, x + y) - f(t, x) - \partial_x f(t, x)\,y \right) \nu(dy) = rf(t, x). \tag{1.5}$$

Let $\mathcal{A}^L$ denote the infinitesimal generator of a Lévy process be defined as

$$A^L f(x) = \lim_{t \searrow 0} \frac{\mathbb{E}\left[f(x + L_t) - f(x)\right]}{t},$$

where the convergence is in the sense of the supremum norm on C^0 and f is such that the right-hand side exists. Then from [CoTa04, Proposition 3.16] the generator has the form

$$\mathcal{A}^L f(t,x) = (r + \gamma_0)\, \partial_x f(t,x) + \frac{\sigma^2}{2} \partial_{xx} f(t,x) + \int_{\mathbb{R}\setminus\{0\}} \left(f(t, x+y) - f(t,x) - \partial_x f(t,x)\, y\right) \nu(dy), \tag{1.6}$$

so the pricing equation for a European type payoff can be rephrased as $\partial_t f + \mathcal{A}^L f - rf = 0$. ■

Chapter 2

hp Finite Element Method

The Finite Element Method (FEM) is a general projection method based on a variational formulation of a boundary value problem. FEM is widely used to solve partial differential equations arising from physics and finance by discretizing the variational formulation on a finite dimensional function space of piecewise polynomial functions. In such spaces linear operators can be represented in matrix form.

In one dimensional space FEM works on a given *domain* $G = [a, b]$ with $a, b \in \mathbb{R}$, on which a *mesh* τ_h that is defined as $\tau_h = \{x_k : k = 0, \ldots, N+1\}$, $N > 0$, with a sequence of *mesh points*

$$a = x_0 < x_1 < \ldots < x_{N+1} = b,$$

and the *elements* are the disjoint subintervals of G, $K_i = [x_i, x_{i+1}] \quad \forall i \in \{0, \ldots, N\}$.

The distance between elements is the *mesh witdth*

$$h_k = x_{k+1} - x_k \quad \text{for } k = 0, ..., N+1.$$

Definition 6 *A uniform mesh* $\mathcal{U}^N(0,1)$ *is a series of intervals* $K_k = [x_k, x_{k+1}]$, *called* elements *defined by the grid points*

$$x_k = \frac{k}{N+1}, \quad \textit{for } k = 0, ..., N+1.$$

A uniform mesh can also be denoted with τ_h with $h_k \equiv h$. A plot of an example of a uniform mesh is displayed in *Figure 2.1*.

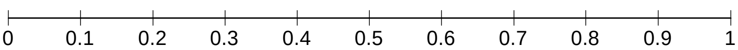


Figure 2.1: A uniform mesh with $N = 9$ refinements

We denote the space of piecewise polynomial functions on G with a partitioning mesh τ_h as

$$S^p(G, \tau_h) = \{f : G \to \mathbb{R} : f|_{K_i} \text{ is a polynomial}\}.$$

2.1 h- and p- FEM for analytic functions

Functions f that are analytic in the whole domain are locally given by convergent power series and satisfy

$$\sup_{x\in G}\left|\frac{d^k f}{dx^k}(x)\right| \leq C^{k+1}k!,$$

for some constant $C > 0$ and for every k, which means all of their derivatives are square integrable and so are in $H^\infty(G) := \cap_{k\in\mathbb{N}} H^k(G)$. Using finite element methods for these type of functions p-refinement is enough to achieve exponential convergence rates, as we will see in this section. But for functions that have singularities on known points in space or time, we will need hp-refinement, which is discussed in the next section.

Increasing the uniform polynomial order on all elements in FEM, or p-FEM is understood as in [Sw98, Remark 3.18.] the refinement from $S^1(G, \tau_h)$ increasing p in $S^p(G, \tau_h)$. Increasing the uniform mesh width h on all elements is called the h-FEM and is understood as decreasing h uniformly in $S^p(G, \tau_h)$. hp-FEM will be explained in *Section 2.2.*

2.1.1 Basis functions on the reference element

To approximate functions by piecewise polynomial functions we chose Legendre polynomials (Figure 2.2) that are defined on $(-1, 1)$ by applying the Gram-Schmidt process in $L^2(-1, 1)$ on the monomials $\{\xi^p\}_{p=0}^\infty$. The Legendre polynomials satisfy the Legendre differential equation

$$\begin{aligned}\left(\left(1-\xi^2\right)L_p'(\xi)\right)' + p(p+1)L_p(\xi) &= 0\\ L_p(1) &= 1,\end{aligned}$$

and more importantly the orthogonality property

$$\int_{-1}^{1} L_n(\xi)L_m(\xi)\,d\xi = \frac{2}{2n+1}\delta_{n=m} \qquad n, m \geq 2, \tag{2.1}$$

which leads to the stiffness matrix derived from the diffusion term being diagonal in higher polynomial orders, as described in *Section 2.1.6.*

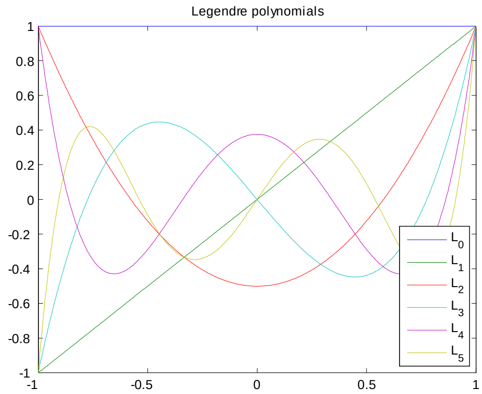


Figure 2.2: Legendre polynomials

The reference element shape functions $\{N_i\}_{i=0}^{p}$ can be built from Legendre polynomials. They read as follows, on the reference element $\hat{K} = (-1, 1)$

$$
\begin{aligned}
\hat{N}_0(\xi) &= \frac{1+\xi}{2}, \quad \hat{N}_1(\xi) = \frac{1-\xi}{2}, \\
\hat{N}_p(\xi) &= \frac{1}{\sqrt{2(2p-1)}} \left(L_p - L_{p-2}\right)(\xi) \qquad p \geq 2,
\end{aligned}
$$

and are depicted in *Figure 2.3*.

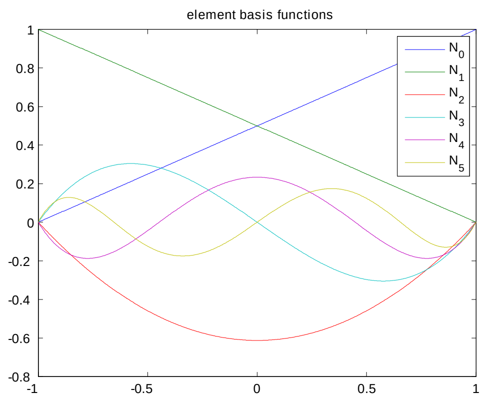


Figure 2.3: Polynomial basis functions on the reference element $\hat{K}$

The shape functions of the elements K_i are defined via the mapping $\varphi_i : \hat{K} \rightarrow K_i$

$$
N_j^i(x) = \begin{cases} \hat{N}_j\left(\varphi_i^{-1}(x)\right) & \forall x \in K_i \\ 0 & \text{otherwise} \end{cases}, \tag{2.2}
$$

where φ_i is a diffeomoprhishm, meaning it is a bijective mapping with $\varphi_i \in C^1\left(\hat{K}, K_i\right)$ and $\varphi_i^{-1} \in C^1\left(K_i, \hat{K}\right)$.

On each element the external shape functions N_0 and N_1 form hat functions on two neighbouring elements: N_0^i on K_i and N_1^{i+1} on K_{i+1} as depicted in *Figure 2.4*. The higher order polynomials, N_k^i with $k \geq 2$ are internal shape functions, with 0 values on the boundaries of the element they are defined on.

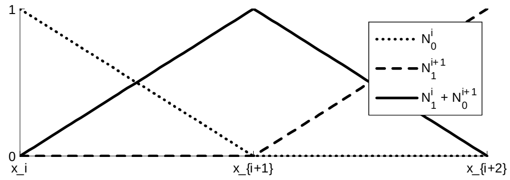


Figure 2.4: Linear shape functions forming hat functions on two consecutive mehs elements

Space of piecewise polynomial functions on G, with a partitioning mesh τ_h can then be characterized as

$$S^p(G, \tau_h) = \operatorname{span}\left(\left\{N_j^i\right\}_{i\in\{0,\ldots,N\}, j\in\{1,\ldots,p_i\}}\right) \text{ with } N_j^i \text{ as in (2.2).}$$

2.1.2 Basis function numbering and assembly

On the elements K_i with $i \in \{0, \ldots, N\}$ we introduce a numbering scheme for the basis functions in a way that we first index the external shape functions $\{b_i\}_{i\in\mathbf{N}}$ from 1 to $N+2$, so the neighbouring N_0^i and N_1^{i+1} linear shape functions form hat functions, and their contributions will add up during assembly. Then we number the internal shape functions from $N+3$ to M, where M is the number of all basis functions in a continuous Galerkin (cG) setting, and

$$M = N + 2 + \sum_{i=0}^{N}(p_i - 1) = 2 + \sum_{i=0}^{N} p_i,$$

where p_i is the polynomial degree on element K_i, and we have altogether $N+2$ linear hat functions.

The resulting numbering of the basis functions will be then on $b_i(x)$ for $i \in \{1, ..., M\}$, first the linear hat functions from 1 to $N+2$

$$\begin{aligned}
b_n(x) &= N_0^i(x) + N_1^{i+1}(x) \quad x \in K_i \cup K_{i+1}, \quad 2 \leq n \leq N+1, \quad i = n-2 \\
b_1(x) &= N_0^0(x) \quad x \in K_0 \\
b_{N+2}(x) &= N_N^1(x) \quad x \in K_N,
\end{aligned}$$

and then the higher order polynomial shape function are numbered elementwise as

$$\begin{aligned}
b_n(x) &= N_k^i(x) \quad x \in K_i, \quad n > N+1, \\
i &= \max\left\{m \in \mathbb{N} : 2 + N + \sum_{j=0}^{m}(p_j - 1) \leq n\right\}, \\
k &= n - \left(2 + N + \sum_{j=0}^{n}(p_j - 1)\right).
\end{aligned}$$

Using a cG scheme we sum up the contributions from neighbouring linear shape functions, thus forming hat functions which will only allow continuous functions within $S^p(G, \tau_h)$

Example 7 *The numbering of shape functions in $N+1=4$ elements where the polynomial order on each element is 3 reads as follows*

shape function:	N_0	N_1	N_2	N_3
element K_0	2	1	6	7
element K_1	3	2	8	9
element K_2	4	3	10	11
element K_3	5	4	12	13

2.1.3 Static condensation

Since the first $N+1$ basis functions represent the linear external shape functions, and the rest are internal, higher order polynomial shape functions, solving a system of linear equations $\mathbf{A}\underline{u} = \underline{f}$ can be done separately, where $\mathbf{A} \in \mathbb{R}^{M\times M}$ representing a linear operator in $S^p(G, \tau_h)$ and $\underline{u}, \underline{f} \in \mathbb{R}^M$ are representing two piecewise polynomial functions, $u, f \in S^p(G, \tau_h)$.

First let us partition the matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{10} \\ \mathbf{A}_{01} & \mathbf{A}_{11} \end{pmatrix}$$

where

$$\begin{aligned}
\mathbf{A}_{00} &= \{\mathbf{A}_{i,j}\}, i,j \in \{1,\ldots,N+2\}, \\
\mathbf{A}_{10} &= \{\mathbf{A}_{i,j}\}, i \in \{1,\ldots,N+2\}, j \in \{N+3,\ldots,M\}, \\
\mathbf{A}_{01} &= \{\mathbf{A}_{i,j}\}, i \in \{N+3,\ldots,M\}, j \in \{1,\ldots,N+2\}, \\
\mathbf{A}_{11} &= \{\mathbf{A}_{i,j}\}, i,j \in \{1,\ldots,N+2\},
\end{aligned}$$

and the vectors $\underline{u}$ and $\underline{f}$ as

$$\underline{u} = \begin{pmatrix} \underline{u}_0 \\ \underline{u}_1 \end{pmatrix} \quad \text{and} \quad \underline{f} = \begin{pmatrix} \underline{f}_0 \\ \underline{f}_1 \end{pmatrix}.$$

Then we first have to solve for only the degrees of freedom of external shape functions, i.e.

$$\left(\mathbf{A}_{00} - \mathbf{A}_{10}(\mathbf{A}_{11})^{-1}\mathbf{A}_{01}\right)\underline{u}_0 = \underline{f}_0 - \mathbf{A}_{10}(\mathbf{A}_{11})^{-1}\underline{f}_1,$$

then solve for the internal shape functions

$$\mathbf{A}_{11}\underline{u}_1 = \underline{f}_1 - \mathbf{A}_{01}\underline{u}_0.$$

The advantage of this procedure is that the condition numbers for the submatrices $\mathbf{A}_{11}$ and $\left(\mathbf{A}_{00} - \mathbf{A}_{10}(\mathbf{A}_{11})^{-1}\mathbf{A}_{01}\right)$ are several orders of magnitude lower than that of $\mathbf{A}$, and therefore we can achieve higher accuracy partitioning the system this way.

2.1.4 Solving the pricing PIDE

Discretizing the weak formulation of (1.5) in the FE space $S^p(G, \tau_h)$ for functions $f(t,x) \in S^p(G, \tau_h)$, in the form

$$f(t,x) = \sum_{i=1}^{M} \underline{u}(t)\, b_i(x) \quad x \in \overline{G},$$

with the basis functions from *Section 2.1.2* will lead to a semi-discrete form

$$\left\langle \sum_{j=1}^{M} \partial_t \underline{u}_j(t)\, b_j(x), \sum_{i=1}^{M} \underline{v}_i b_i(x) \right\rangle_{L^2} + \frac{\sigma^2}{2} \left\langle \sum_{j=1}^{M} \underline{u}_j(t)\, b_j'(x), \sum_{i=1}^{M} \underline{v}_i b_i'(x) \right\rangle_{L^2} \tag{2.3}$$
$$+a^L \left(\sum_{j=1}^{M} \underline{u}_j(t)\, b_j(x), \sum_{i=1}^{M} \underline{v}_i b_i(x) \right) = r \left\langle \sum_{j=1}^{M} \underline{u}_j(t)\, b_j(x), \sum_{i=1}^{M} \underline{v}_i b_i(x) \right\rangle_{L^2},$$

for all $\underline{v} \in \mathbb{R}^M$. This can be rephrased to

$$\underline{v}^\top \left(\partial_t \underline{u}(t) + \frac{\sigma^2}{2} \mathbf{A} \underline{u}(t) + \mathbf{S} \underline{u}(t) \right) = \underline{v}^\top \left(r \mathbf{M} \underline{u}(t) \right) \quad \forall \underline{v} \in \mathbb{R}^M,$$

and so the semi-discrete version of (1.5) reads as follows.

$$\begin{aligned} \partial_t \underline{u}(t) + \frac{\sigma^2}{2} \mathbf{A} \underline{u}(t) + \mathbf{S} \underline{u}(t) &= r \mathbf{M} \underline{u}(t), \\ \mathbf{M} \underline{u}(0) &= \underline{g} \end{aligned}$$

The matrices $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{S}$ are defined in (2.5), (2.9) and (2.12) respectively.

The task of discretization can thus be broken down to three smaller problems. First is the L^2 projection of an arbitrary function $f : G \to \mathbb{R}$ to $S^p(G, \tau_h)$ using the mass matrix, the second task is a Laplace-operator equation using the stiffness matrix derived from the diffusion term and the third is a Lévy operator equation, using the stiffness matrix derived from the jump term.

2.1.5 L^2 projection

Projecting an arbitrary function $g \in L^2$ into the finite element space $S^p(G, \tau_h) = \text{span}\{b_i\}$ corresponds to solving

$$u_s = \underset{u_s \in S^p(G,\tau_h)}{\arg\min} \; \|u - g\|_{L^2(G)}. \tag{2.4}$$

Using the weak formulation, we need to find $u \in S^p(G, \tau_h)$ such that $\int_G uv d\mu = \int_G gv d\mu$ $\forall v \in H^1(G)$, or using the bilinear form notation $\langle u, v \rangle_{L^2} = \langle g, v \rangle_{L^2}$, which can be discretized as in (2.3) to

$$\mathbf{M} \underline{u} = \underline{g} \tag{2.5}$$

where $\mathbf{M}_{i,j} = \langle b_j, b_i \rangle_{L^2}$, and $\underline{g}_i = \langle g, b_i \rangle_{L^2}$.

Since the basis functions are defined on the reference element, every entry of the mass matrix can be written as integrals on the reference element. Every block of the mass matrix $\widehat{\mathbf{M}}^p_{i,j}$ which corresponds to an element-pair $K_i \times K_j$ is a $p \times p$ matrix, p being the uniform polynomial order on all elements, and $\widehat{\mathbf{M}}^p_{i,j}$ has entries

$$\widehat{\mathbf{M}}^p_{k,l} = \int_{-1}^{1} \hat{N}_l(\xi)\, \hat{N}_k(\xi)\, d\xi \quad k, l \in \{0, ..., p\}.$$

Once the local matrices are computed, the contribution of each shape function is assembled to form the global mass matrix

$$\mathbf{M} = \frac{h_0}{2} \begin{pmatrix} \widehat{\mathbf{M}}^{p_0} & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + \frac{h_1}{2} \begin{pmatrix} 0 & & \\ & \widehat{\mathbf{M}}^{p_1} & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + ... + \frac{h_{N+1}}{2} \begin{pmatrix} 0 & & \\ & 0 & \\ & & \ddots & \\ & & & \widehat{\mathbf{M}}^{p_{N+1}} \end{pmatrix}.$$

Theorem 8 *Solving (2.4) with uniform meshes and uniform polynomial orders on elements will lead to the error of the approximation* u_s*, the solution of (2.5), being*

$$\|g - u_s\|_{L^2} \leq C \|g'\|_{H^k} \frac{h^{\min\{p,k\}+1}}{p^{k+1}}, \quad \forall g \in H^k(G), \tag{2.6}$$

where C *is independent of* h *and* p *but depends on* k*.*

Proof. See [Sw98, Theorem 3.17]. ∎

Analytic functions are smooth in the sense that $g^{(k)} \in L^2$ for $k \in \mathbb{N}$, which leads to $g \in H^k(G)$ for $k \in \mathbb{N}$. This means for analytic functions we expect an arbitrarily high algebraic convergence in p, the uniform polynomial degree on all elements from K_0 to K_N, and this convergence can be understood as exponential.

Theorem 9 *With* u_s *and* g *from (2.4) the p-FEM will have error bounds*

$$\|g - u_s\|_{L^2} \leq C (1 + \ln p)^2 p^2 e^{-\lambda p},$$

for some constants C*,* λ *that only depend on* g*.*

Proof. See [MeSw98, Corrolary 4.9]. ∎

Theorem 10 *With* u_s *and* g *from (2.4) the error bound for the h-FEM will be*

$$\|g - u_s\|_{L^2} \leq C_{u,p} h^{-p-1}, \quad \text{for analytic } g, \tag{2.7}$$

where h *is a uniform mesh width on* G*, and all all rates of convergence will be shown below.*

Proof. See [So99, Theorem 4.]. ∎

Numerical results of rates of convergences shown in *Figure 2.6 (right)* and *Figure 2.7 (right)* suggest that the error bound for the h- and p-FEM solution for L^2 projection should be rephrased.

Proposition 11 *With* u_s *and* g *from (2.4) the error bound for the h- and p-FEM is given as*

$$\|g - u_s\|_{L^2} \leq C_1 \exp\left((\ln h - C_2)(p + 1)\right), \quad \text{for analytic } g,$$

for some constants $C_1, C_2 \geq 0$*.*

This means that we observe algebraic convergence in h refinement with the rate of convergence of $p + 1$, and exponential convergence in the polynomial degree with the rate of convergence of $(-\ln h + C_2)$ as shown on *Figure 2.7 (right)* and *Figure 2.10 (right)*.

The condition number of a matrix measures the sensitivity of the solution of a linear system to changes or errors in the data. It gives an indication of the accuracy of the results from matrix inversion and the linear equation solution.[QSS07, Chapter 3.1.1] It plays a crucial role in the accuracy of iterative solvers like the Generalized Minimal RESidual algorithm [SaSc86].

We can observe that the condition number of the mass matrix in (2.5) increases algebraically with p but stays constant as h decreases (after a certain point), and static condensation reduces it significantly, as displayed in *Figure 2.5*.

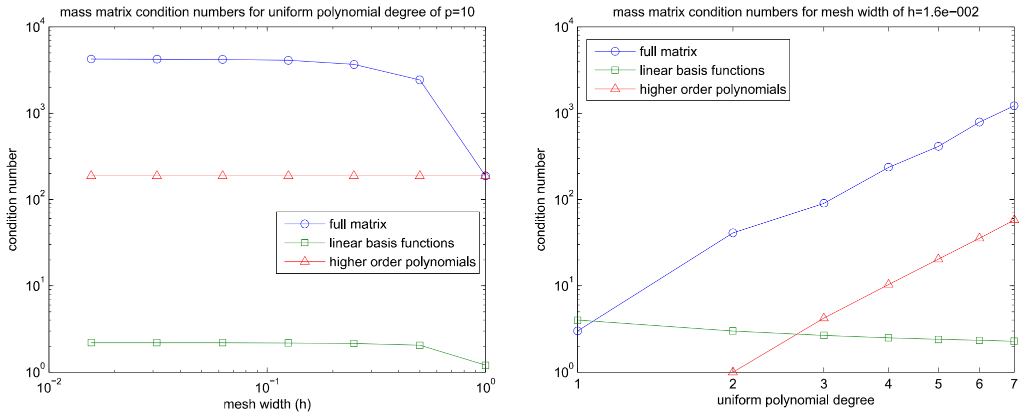


Figure 2.5: Condition number of the mass matrix as mesh size decreases (left) and as the polynomial degree increases (right).

Displaying the results of (2.5) for $u(x) = \sin(\pi x)$ on $G = (0, 1)$ in *Figure 2.6* for various uniform polynomial orders we observe an algebraic convergence in the mesh size displayed as linear lines in a log-log plot, and the rate of convergence being $p + 1$, as described in *Proposition 11*.

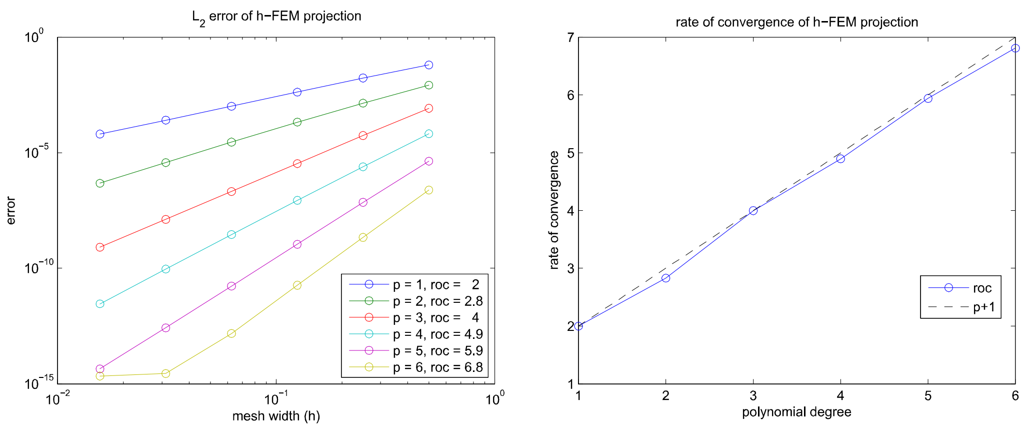


Figure 2.6: Algebraic L^2 converge with mesh refinement for L^2-projection (left) and the rate of convergence with different uniform polynomial degrees (p)

Then using a uniform mesh on $[0, 1]$, we observe exponential convergence by increasing uniform polynomial order on each element, displayed as linear lines in a semi-log plot in *Figure 2.7*. The observed exponential rate of convergence for every mesh size is of $(-\ln h + C_2)$ as suggested in *Proposition 11*.

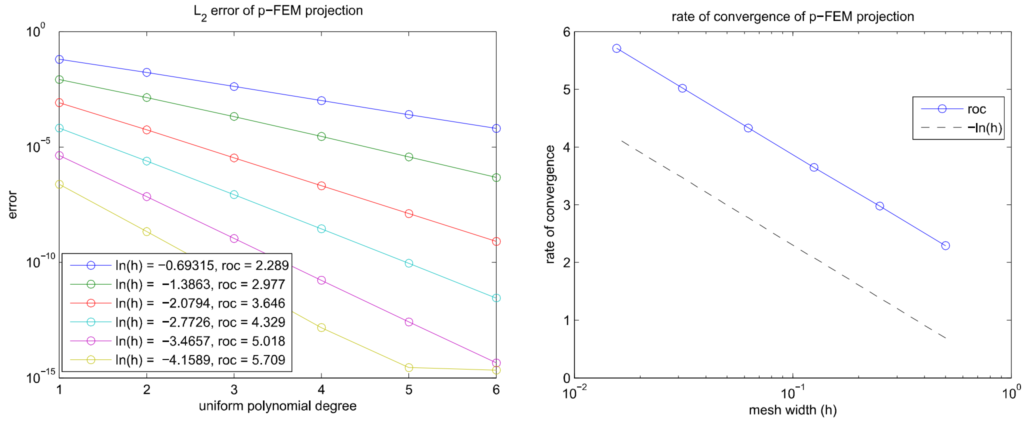


Figure 2.7: Exponential L^2 converge with the increase of the uniform polynomial degrees (p) for L^2-projection (left) and the rate of convergence with different mesh sizes.

2.1.6 Poisson's equation

Solving for u

$$\begin{aligned} -u'' &= f && \text{on } G \\ u(x) &= 0 && \text{on } \partial G, \end{aligned} \tag{2.8}$$

with $f \in L^2(G)$ leads to the variational formulation that reads:

Find $u \in V = \left\{ u \in H^1(G) : u(x) = 0,\ x \in \partial G \right\}$ such that

$$\langle u', v' \rangle_{L^2} = \langle f, v \rangle_{L^2} \quad \forall v \in V,$$

which can be discretized using the FEM space $S^p(G, \tau_h)$ to

$$\mathbf{A}\underline{u} = \underline{f}, \tag{2.9}$$

where $\mathbf{A}_{i,j} = \left\langle b_j', b_i' \right\rangle_{L^2}$, and $\underline{f}_i = \langle f, b_i \rangle_{L^2}$.

Let $\widehat{\mathbf{A}}^{p_i}$ be a local matrix for the Laplace operator on the element K_i with entries

$$\widehat{\mathbf{A}}^{p_i}_{k,l} = \int_{-1}^{1} \nabla \hat{N}_l(\xi) \nabla \hat{N}_k(\xi)\, d\xi \quad k, l \in \{0, ..., p\}.$$

Then the stiffness matrix $\mathbf{A}$ can be assembled from the local matrices defined on the elements by adding up the contributions of the shape functions as

$$\mathbf{A} = \frac{2}{h_0} \begin{pmatrix} \widehat{\mathbf{A}}^{p_0} & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + \frac{2}{h_1} \begin{pmatrix} 0 & & \\ & \widehat{\mathbf{A}}^{p_1} & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + \cdots + \frac{2}{h_{N+1}} \begin{pmatrix} 0 & & \\ & 0 & \\ & & \ddots & \\ & & & \widehat{\mathbf{A}}^{p_{N+1}} \end{pmatrix}.$$

From the orthogonality property of the Legendre polynomials in (2.1) we will have $\left\langle \nabla N_l^j, \nabla N_k^i \right\rangle_{L^2} = \delta_{\{i=j\}} \delta_{\{k=l\}}$ for $i, j \geq 2$. This means that the stiffness matrix for the higher order polynomials,

$\mathbf{A}_{11}$ after static condensation, will be a diagonal matrix with entries of $2/h_i$, which is constant when using uniform meshes, so its condition number will be

$$\mathrm{cond}\left(\mathbf{A}_{11}\right) = 1.$$

Therefore the condition number of the system solving for the linear shape functions equals the condition number of the full system

$$\mathrm{cond}\left(\mathbf{A}_{00} - \mathbf{A}_{10}\left(\mathbf{A}_{11}\right)^{-1}\mathbf{A}_{01}\right) = \mathrm{cond}\left(\mathbf{A}\right).$$

It can be observed that the condition number of (2.9) increases algebraically as h decreases, as displayed in *Figure 2.8 (left)* and it is constant in p, as displayed in *Figure 2.8 (right).*

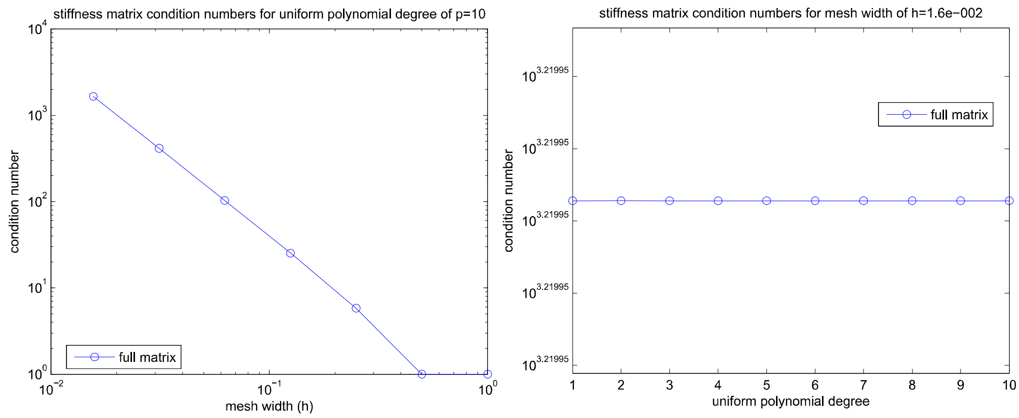


Figure 2.8: Condition number of the stiffness matrix derived from the Lalpace operator increasing in mesh size (h) and staying constant in the uniform polynomial degree (p)

Following *Proposition 11*, results of solving the system with uniform meshes and uniform polynomial orders on every element will lead to the approximation

$$u_s = \underset{u_s \in S^p(G,\tau_h)}{\arg\min} \; \|u - g\|_{L^2}$$

with g being the strong solution of (2.8), to have error bounds

$$\|u_s - g\|_{L^2} \leq C_1 \exp\left(\left(\ln\left(h\right) - C_2\right)\left(p+1\right)\right), \quad \text{for analytic } g.$$

Displaying the results for $g\left(x\right) = \sin\left(\pi x\right)$ on $[0,1]$, and observing an algebraic convergence in the mesh size displayed as linear lines in a log-log plot in *Figure 2.9.*

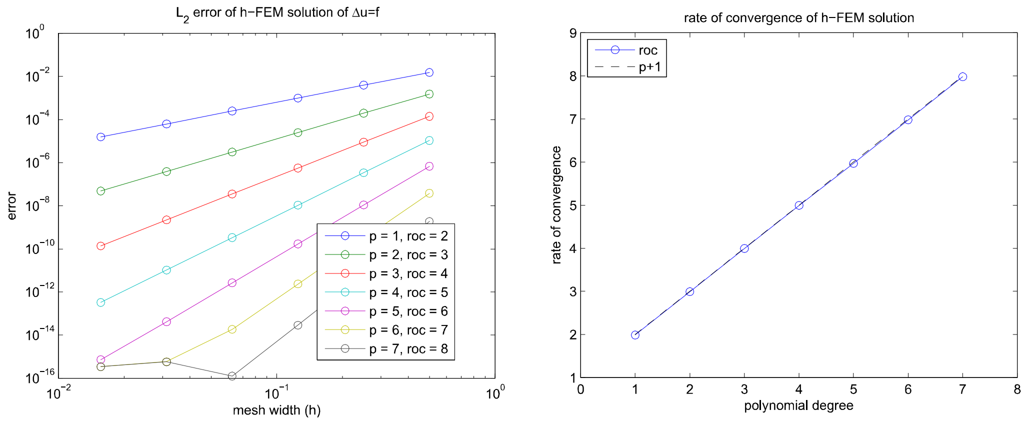


Figure 2.9: Algebraic rate of convergence in L^2-norm with mesh refinement for Laplace equation (left) and the rate of convergence with different uniform polynomial degrees (p).

Then using a uniform mesh on $[0, 1]$, and having exponential convergence by increasing uniformly the polynomial order on the elements, displayed as linear lines in a semi-log plot in *Figure 2.10.*

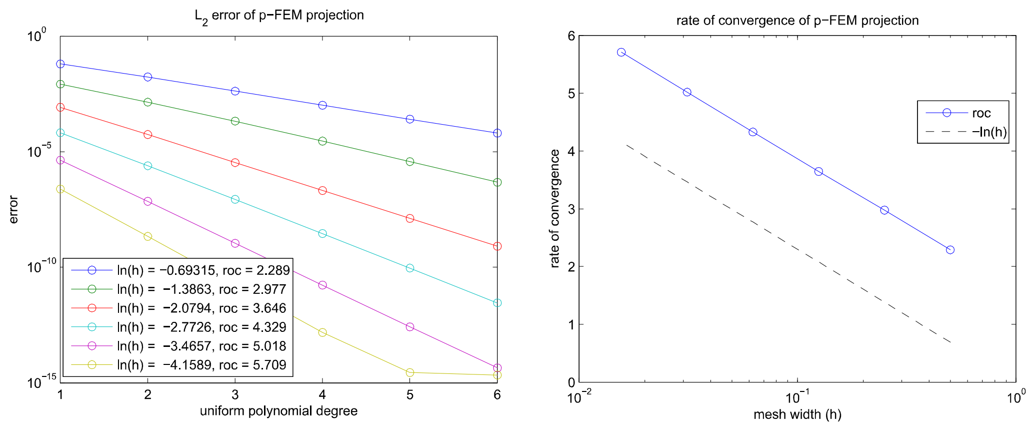


Figure 2.10: Exponential rate of convergence in L^2-norm with the increase of the uniform polynomial degrees (p) for Laplace equation (left) and the rate of convergence with different mesh sizes.

2.1.7 Lévy operator equation

Solving the equation

In order to be able to solve (1.5) we consider

$$\mathcal{A}^L u = f, \tag{2.10}$$

where

$$\mathcal{A}^L u = \int_{\mathbb{R}\setminus\{0\}} \left(u\left(x+y\right) - u\left(x\right) - u'\left(x\right)y\right)\nu\left(dy\right),$$

meaning that $\mathcal{A}^L$ is the generator of a pure jump process characterized by the Lévy triplet $(0, 0, \nu)$.

In a symmetric CGMY setting this equals

$$\mathcal{A}^L u = \int_{\mathbb{R}\setminus\{0\}} \left(u\left(x+y\right) - u\left(x\right) - u'\left(x\right)y\right)\frac{e^{-\beta|y|}}{|y|^{\alpha+1}}dy.$$

We discretize problem in (2.10) using the basis functions from *Section 2.1.2* to

$$\mathbf{S}\underline{u} = \underline{f}, \tag{2.11}$$

where

$$\begin{aligned}\mathbf{S}_{i,j} &= \left\langle \mathcal{A}^L b_j, b_i\right\rangle_{L^2} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}\setminus\{0\}} \left(b_j\left(x+y\right) - b_j\left(x\right) - b_j'\left(x\right)y\right)\nu\left(dy\right)b_i\left(x\right)dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}\setminus\{x\}} b_j'\left(y\right)b_i'\left(x\right)k^{(-2)}\left(x-y\right)dydx,\end{aligned} \tag{2.12}$$

and

$$\underline{f}_i = \left\langle f, b_i\right\rangle_{L^2}.$$

Computing elements of S numerically

Using polynomial basis functions the integrals in (2.12) will not be available in closed form. Therefore we need to compute the elements of $\mathbf{S}$ numerically.

In order to do this we transform each shape function to the reference element, N_k^i being the k-th shape function on the element K_i. We will take care of the task of combining the neighbouring linear shape functions to hat functions in the assembly of the global matrix by adding up their contributions.

We can now define a local stiffness matrix on the element pair $K_i \times K_j$ as $\hat{\mathbf{S}}_{i,j}$, with entries

$$\begin{aligned}\hat{\mathbf{S}}_{(i,k),(j,l)} &= \int_{-1}^{1}\int_{-1}^{1} \nabla\hat{N}_l\left(\xi_y\right)\nabla\hat{N}_k\left(\xi_x\right)\ k^{(-2)}\left(\varphi_i\left(\xi_x\right) - \varphi_j\left(\xi_y\right)\right)\ d\xi_y d\xi_x, \\ k &\in \left\{0,\ldots,p_i\right\}, l \in \left\{0,\ldots,p_j\right\}, \\ \text{for } i,j &\in \left\{1,\ldots,N+1\right\}\end{aligned}$$

where p_i and p_j are the highest order of polynomial shape functions on the elements K_i and K_j respectively.

For numerical integration on the reference element we will distinguish four cases: diagonal blocks $\hat{\mathbf{S}}_{i,i}$ on $K_i \times K_i$; super-diagonal blocks $\hat{\mathbf{S}}_{i,i+1}$ on $K_i \times K_{i+1}$; sub-diagonal blocks $\hat{\mathbf{S}}_{i,i-1}$ on $K_i \times K_{i-1}$ and far from diagonal blocks of $\hat{\mathbf{S}}_{i,j}$ on $K_i \times K_j$ for $|i-j| \geq 2$. We discuss the the computation of the stiffness matrix more in detail in *Appendix B*.

Since the Lévy operator is not a local operator, the assembly of the stiffness matrix is more involved than for local operators, as sown in (2.13)

$$
\begin{aligned}
\mathbf{S} \;=\; & \begin{pmatrix} \hat{\mathbf{S}}_{0,0} & \cdots & 0 \\ \vdots & & \\ 0 & & \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & \hat{\mathbf{S}}_{0,j} & \cdots & 0 \\ & & \vdots & & \\ & & 0 & & \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & \hat{\mathbf{S}}_{0,N+1} \\ & & \vdots \\ & & 0 \end{pmatrix} \\
& + \begin{pmatrix} 0 & & \\ \vdots & & \\ \hat{\mathbf{S}}_{i,0} & \cdots & 0 \\ \vdots & & \\ 0 & & \end{pmatrix} + \cdots + \begin{pmatrix} & & 0 & & \\ & & \vdots & & \\ 0 & \cdots & \hat{\mathbf{S}}_{i,j} & \cdots & 0 \\ & & \vdots & & \\ & & 0 & & \end{pmatrix} + \cdots + \begin{pmatrix} & & 0 \\ & & \vdots \\ 0 & \cdots & \hat{\mathbf{S}}_{i,N+1} \\ & & \vdots \\ & & 0 \end{pmatrix} \\
& + \begin{pmatrix} 0 & & \\ \vdots & & \\ \hat{\mathbf{S}}_{N+1,0} & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} & & 0 & & \\ & & \vdots & & \\ 0 & \cdots & \hat{\mathbf{S}}_{N+1,j} & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} & & 0 \\ & & \vdots \\ 0 & \cdots & \hat{\mathbf{S}}_{N+1,N+1} \end{pmatrix}
\end{aligned}
\tag{2.13}
$$

Numerical integration on the reference element of diagonal blocks

As an example we will now show the calculation of a stiffness matrix for a CGMY process on $G = (-1, 1)$, with a single element $K_0 = G$.

Since the Lévy density k for the CGMY model is strongly singular at 0, i.e. $k(x) \sim x^{-(Y+1)}$ as $x \to 0$ and so $k^{(-2)}(x-y)$ might be discontinuous on the diagonal of $(-1,1)^2$, the second antiderivative has the form as in [Po08, Appendix B], and behaves near 0 as $k^{(-2)}(x) \sim x^{1-Y}$.

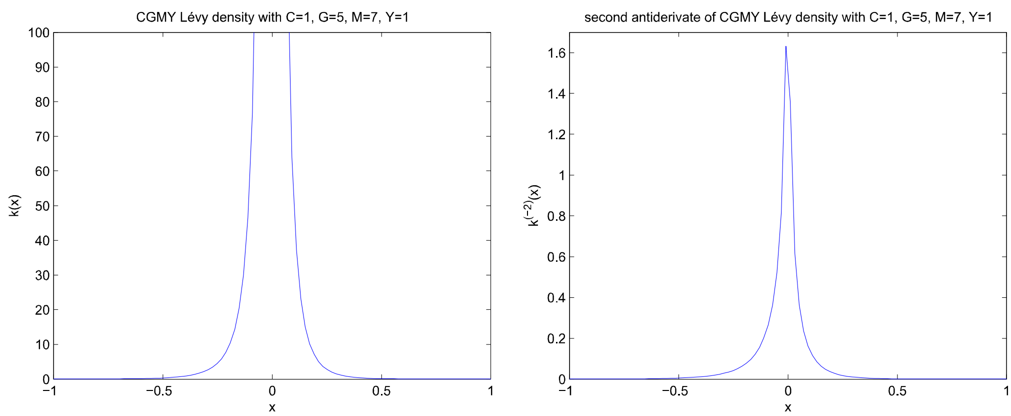


Figure 2.11: CGMY Lévy-density (left) and its second antiderivate (right)

To integrate $k^{(-2)}(x-y)$ on $(-1,1)^2$ we apply Duffy's trick [Du82], and divide $(-1,1)^2$ into two triangles where $k^{(-2)}(x-y)$ has singularity only on one side, so we'll have two integrals as

$$\int_{-1}^{1}\int_{-1}^{1} k^{(-2)}(x-y)\,dydx = \int_{-1}^{1}\int_{-1}^{x} k^{(-2)}(x-y)\,dydx + \int_{-1}^{1}\int_{x}^{1} k^{(-2)}(x-y)\,dydx,$$

and for the ease of numerical integration we transform each triangle to a rectangle, with singularity at one side. For the lower triangle we have with the new variable $z = 1 - \frac{y+1}{x+1}$:

$$\begin{aligned} \int_{-1}^{1}\int_{-1}^{x} k^{(-2)}(x-y)\,dydx &= -\int_{-1}^{1}\int_{1}^{0} k^{(-2)}(x-[x-z(x+1)])(x+1)\,dzdx \\ &= \int_{-1}^{1}\int_{0}^{1} k^{(-2)}(x-[x-z(x+1)])(x+1)\,dzdx. \end{aligned}$$

For the upper triangle we have, with the new variable $z = \frac{y-x}{1-x}$

$$\int_{-1}^{1}\int_{x}^{1} k^{(-2)}(x-y)\,dydx = \int_{-1}^{1}\int_{0}^{1} k^{(-2)}(x-[x+z(1-x)])(1-x)\,dzdx.$$

Now both integrals have singularity at $z=0$, so we need a mesh that is refined towards that singularity.

We discuss the numerical integration more in detail with results of convergence in *Appendix A*.

Complexity of the linear system

The condition numbers of the stiffness matrix increase with both N and p, but the condition number of the linear sub-system $\left(\mathbf{S}_{00} - \mathbf{S}_{10}\left(\mathbf{S}_{11}\right)^{-1}\mathbf{S}_{01}\right)$ decreases with the polynomial order p, as shown in *Figure 2.12-2.14* for the CGMY model with $C=1, G=M=5$ and various tail-exponents.

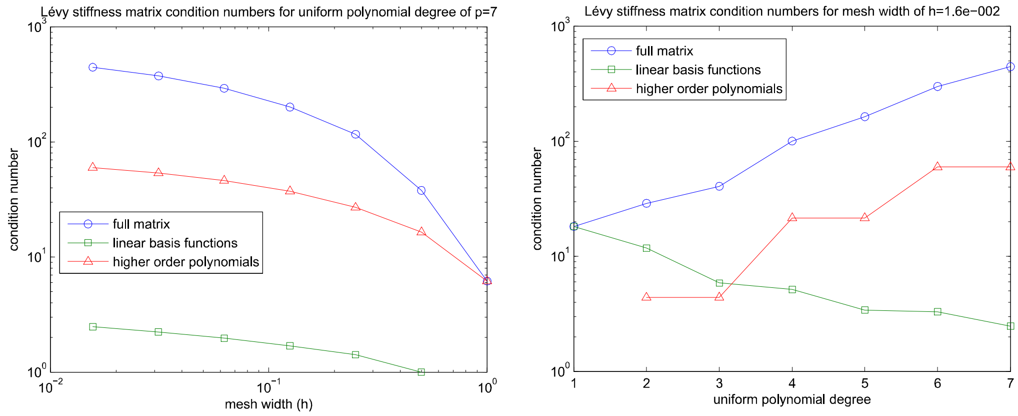


Figure 2.12: Condition numbers for (2.11) with $Y = 0$ for the wole system and the two sub-systems after static condensation as mesh size decreases (left column) and as the uniform polynomial degree increases (right column)

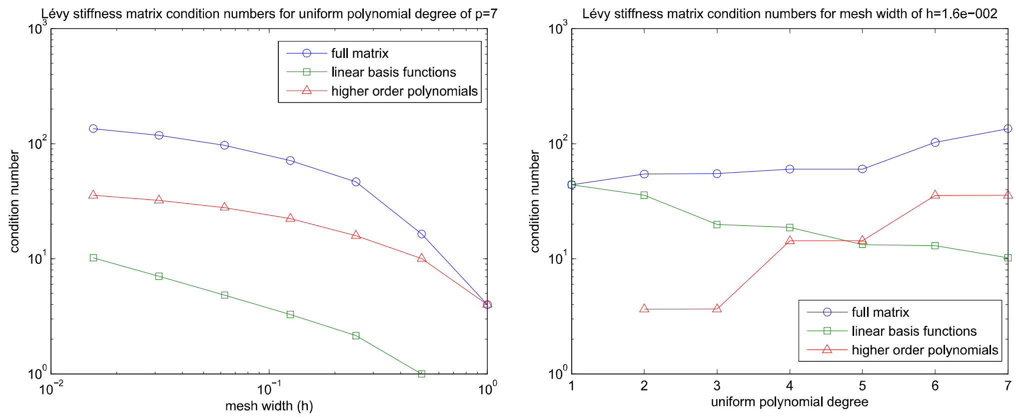


Figure 2.13: Condition numbers for (2.11) with $Y = 0.5$ for the wole system and the two sub-systems after static condensation as mesh size decreases (left) and as the uniform polynomial degree increases (right)

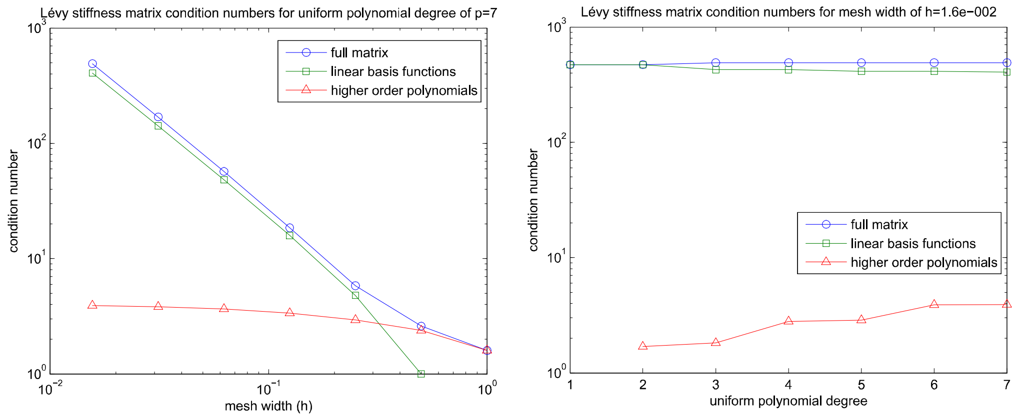


Figure 2.14: Condition numbers for (2.11) with $Y = 1.5$ for the wole system and the two sub-systems after static condensation as mesh size decreases (left) and as the uniform polynomial degree increases (right)

Remark 12 *The CGMY model with parameter $Y > 1$ is of infinite variation, with singularity of the Lévy density of $k(x) \sim x^{Y+1}$ as $x \to 0$.*

Results

Theorem 13 *The solution of the linear system (2.11), $u_{h,p}(x)$, with uniform mesh width and uniform polynomial orders across the elements results in*

$$u_{h,p}(x) = \underset{u \in S^p(G,\tau_h)}{\arg\min} \; \|u - g\|_{L^2} \quad \textit{for analytic } g,$$

where g is the solution of (2.10) the error can be bounded as

$$\|u_{h,p} - g\|_{L^2} \leq C_{u,\nu} h^{p+1}.$$

Proof. See [Sw98, Thm. 3.17]. ■

Displaying the results for the CGMY Lévy operator with parameters $C = 1$, $G = M = 5$, $Y = 0.5$, and for the function $g(x) = x^4 (1-x)^4$, we observe an algebraic convergence in mesh size, shown as a linear line in a log-log plot in *Figure 2.15 (left)*, with rate of convergence $p+1$ as shown in *Figure 2.15 (right)*.

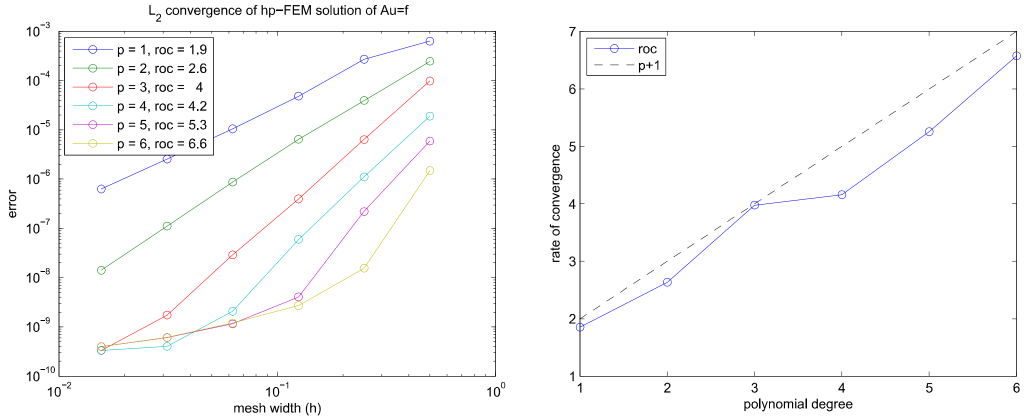


Figure 2.15: L^2 convergence of (2.10) using mesh refinements (left) and the rate of convergence for different uniform polynomial degrees (right).

Remark 14 *The convergence starts to deteriorate around* 10^{-9} *and this is partly due to the fact that the function* $f = \mathcal{A}^L g$*, for a polynomial* g*, has coefficients expressed with confluent hypergeometric functions that are evaluated numerically. The other reason for the deterioration is that the entries of the stiffness matrix are calculated numerically, and for further details on variational crimes turn to [Sw98, Chapter 2.4].*

We observe an exponential convergence in the polynomial degree shown as a linear line in a semi-log plot in *Figure 2.16 (left)* with rate of convergence of $-\ln(h)$ as shown in *Figure 2.16 (right)*.

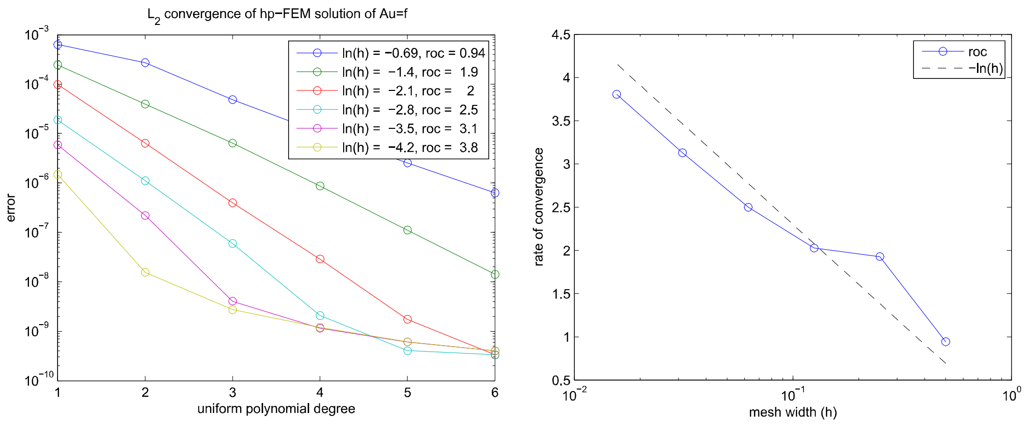


Figure 2.16: L^2 convergence of (2.10) increasing polynomial order (left) and the rate of convergence for different mesh sizes (right).

2.2 *hp* FEM for functions with singularity

Even for the simple plain vanilla options the payoff functions have a singularity at the strike, meaning they are piecewise linear, but not differentiable at the strike. So they are only in $H^{\frac{3}{2}-\varepsilon}(\mathbb{R})$, and not in $H^2(\mathbb{R})$ which would yield optimal, second order convergence for the θ-scheme with $\theta = 0.5$. Our aim is pricing of digital barriers with discontinuity at the barrier, so the payoff is not even in continuous, resulting in the function being only in $H^{\frac{1}{2}-\varepsilon}(\mathbb{R})$.

We have previously shown that using uniform meshes for the p-FEM solver will give us exponential convergence, as described in *Proposition 11*, but only for analytic functions. In order to achieve exponential rate of convergence with functions having singularity we will use *hp* mesh refinement as in [SzBa91, Ch. 4.1.].

Example 15 *A well known example of the superiority of hp FEM is from [Sw98, Chapter 3.4] describing the L^2 projection of a boundary layer function in the form of*

$$u(x) = \exp\left(-\frac{x}{d}\right) \quad \text{on } G = (0,1)$$

where $d \in (0,1]$ is a parameter that can approach 0. The performance of three methods are presented in Figure 2.17*: the h-FEM with hat functions as basis functions (since u is continuous) and a uniform mesh grid with mesh width h; the p-FEM with the single element $K_0 = G$ with increasing uniform polynomial degree and an hp-FEM on two elements $K_0 = (0, dp)$ and $K_1 = (dp, 1)$, where p is the uniform polynomial degree on both elements.*

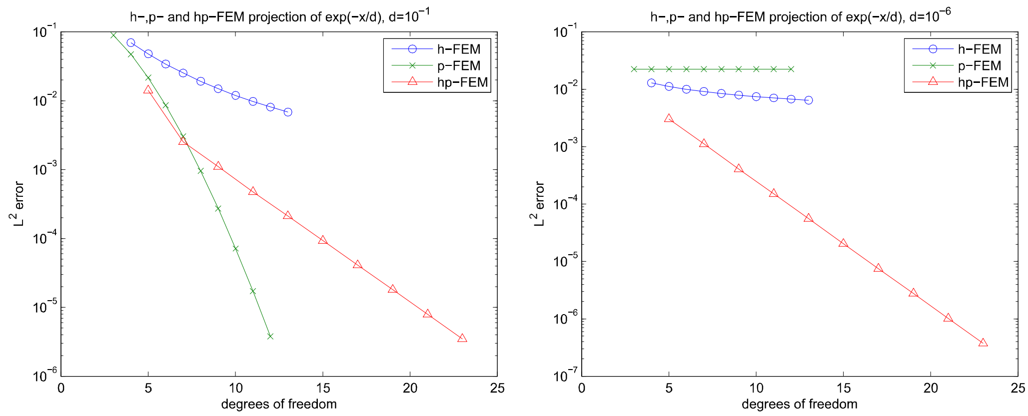


Figure 2.17: Convergence of the L^2 projection of the function $u(x) = \exp(-x/d)$ with $d = 0.09$ (left) and $d = 10^{-6}$ (right).

2.2.1 Meshes

Definition 16 *A geometric graded mesh $\mathcal{G}^N_\sigma(0,1)$ with N refinements and a grading factor $\sigma \in (0,1)$ on $[0,1]$ is, just like a uniform mesh from* Definition 6*, a series of elements $K_k = [x_k, x_{k+1}]$ with the grid points*

$$\begin{aligned} x_0 &= 0, \\ x_k &= \sigma^{N+1-k}, \quad \text{for } k = 1, ..., N+1. \end{aligned}$$

A plot of an example for a geometric graded mesh with $\sigma = 1/2$, $N = 4$ is displayed in *Figure 2.18.*

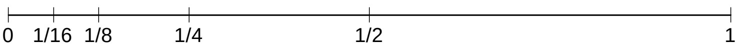


Figure 2.18: A geometric graded mesh with grading factor $\sigma = 0.5$.

From [SzBa91, Ch. 4.1.] we know the grading factor of

$$\sigma^* = \left(\sqrt{2} - 1\right)^2$$

yields an optimal performance for *hp*-FEM when using a geometric graded mesh with grading factor σ^* and polynomial degree on elements

$$p_j = \lceil \mu j \rceil \quad \mu \in \mathbb{R}.$$

Theorem 17 *For the ease of notation suppose that the singularity of the function is at $x_0 = 0$, as in* Example 15*, then we will have exponential convergence in the degrees of freedom as*

$$\inf_{u_s \in S^p(G, \mathcal{G}_\sigma^N)} \|u - u_s\|_{L^2(G)} \leq C_1 \exp\left(-C_2 \sqrt{M}\right),$$

where M is the number basis functions, $C_1, C_2 > 0$.

Proof. See [Sw98, Thm 3.36]. ■

2.2.2 Stiffness matrix for the Lévy operator on a geometric mesh

Discretizing the equation $\mathcal{A}^L u = f$ in the *hp*-FE space leads to a matrix equation

$$\mathbf{S}\underline{u} = \underline{f},$$

where S is the stiffness matrix with elements as

$$\mathbf{S}_{i,j} = \left\langle \mathcal{A}^L b_j, b_i \right\rangle_{L^2},$$

where b_i is the k-th order basis function on the element K_i, as described in *Section 2.1.2*, $\underline{f}$ is a load vector with elements

$$\underline{f}_i = \langle f, b_i \rangle_{L^2}.$$

As shown above, in the case of a Lévy operator the elements of the stiffness matrix can be written as

$$\mathbf{S}_{m,n} = \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} b_n'(y)\, b_m'(x)\, k^{(-2)}(x - y)\, dy dx,$$

where b_n is the k-th order shape function on the element K_i, and b_m is the l-th order shape function on K_j as described in *Section 2.1.2*. By substituting with the diffeomorphism $\varphi_j : \hat{K} \to$

K_j, $\varphi_j(\xi_x) = \frac{x_j+x_{j-1}}{2} + \frac{x_j-x_{j-1}}{2}\xi_x$, and $\varphi_i(\xi_y)$ analogously, we compute the integral on the reference element

$$\mathbf{S}_{m,n} = \int_{-1}^{1}\int_{-1}^{1} b_n'(y)\left(\varphi_i(\xi_y)\right) b_n'\left(\varphi_j(\xi_x)\right)\ k^{(-2)}\left(\varphi_j(\xi_x) - \varphi_i(\xi_y)\right)\ \frac{h_j}{2}d\xi_y \frac{h_i}{2}d\xi_x,$$

and by using that $\nabla \hat{N}_p(x) = \nabla\left(N_p^i(\varphi(x))\right) = \nabla N_p^i(\varphi(x))\,\varphi'(x)\ \forall i$, we have

$$\mathbf{S}_{(i,k)(j,l)} = \int_{-1}^{1}\int_{-1}^{1} \nabla \hat{N}_l(\xi_y)\,\nabla \hat{N}_k(\xi_x)\ k^{(-2)}\left(\varphi_j(\xi_x) - \varphi_i(\xi_y)\right)\ d\xi_y d\xi_x.$$

Proposition 18 *To compute the local blocks $\hat{\mathbf{S}}_{i,j}$ of the stiffness matrix on the reference element $\hat{K}\times\hat{K}$ for $i,j = 1,\ldots,N+1$ define quadratures points and weights as*

$$\begin{aligned}
\underline{\xi}_x &= \underline{g}\otimes\underline{c}^T, \underline{\xi}_y = \underline{g}\otimes\underline{c}^T, \quad \underline{w} = \underline{w_g}\otimes\underline{w_c^T}\cup\underline{w_g}\otimes\underline{w_c^T} \qquad \text{if } i=j, && (2.14)\\
\underline{\xi}_{x,y} &= \underline{c}\otimes\underline{c}^T, \quad \underline{w} = \underline{w_c}\otimes\underline{w_c}^T \qquad \text{if } |i-j| = 1, && (2.15)\\
\underline{\xi}_{x,y} &= \underline{g}\otimes\underline{g}^T, \quad \underline{w} = \underline{w_g}\otimes\underline{w_g}^T \qquad \text{if } |i-j| \geq 2, && (2.16)
\end{aligned}$$

where $\underline{g}$ is a vector of Gauss–Legendre quadrature points of length n with associated weights $\underline{w_g}$ as in [Wi09, Ch. 5.1] and $\underline{c}$ is a vector of composite Gauss quadrature points of length n with associated weights $\underline{w_c}$ as in [Wi09, Ch. 5.2]. Then the error

$$\left|\mathbf{S}_{(i,k)(j,l)} - \left\langle \underline{w}, \nabla\hat{N}_l\left(\underline{\xi}_y\right)\nabla\hat{N}_k\left(\underline{\xi}_x\right)\ k^{(-2)}\left(\varphi_j\left(\underline{\xi}_x\right) - \varphi_i\left(\underline{\xi}_y\right)\right)\right\rangle\right|$$

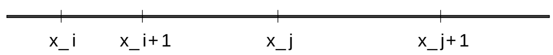
will convergence to 0 *exponentially in n.*

Proof. To compute the integrals derived from the Lévy operator on blocks further from the diagonal, we can use simple Gauss–Legendre quadratures since $k^{(-2)}$ decays exponentially from 0 to make $k^{(-2)}(x-y)$ relatively constant on element pairs that are not neighbours. For this we need the relative distance of the elements to decay fast enough

The length of the element in one dimension is given as

$$|K_i| = |x_i - x_{i+1}| = \sigma^{N+1-i} - \sigma^{N+1-(i+1)} = \sigma^{N+2-i}\left(\frac{1}{\sigma} - 1\right).$$

For numerical integration on $K_i \times K_j$, the two elements K_i and K_j (w.l.o.g. suppose $i < j$) can be illustrated as



Then the relative distance is

$$\begin{aligned}
\delta_{i,j} &= \frac{\operatorname{dist}(K_i, K_j)}{\max\{|K_i|, |K_j|\}} \quad \text{assume } i < j \\
&= \frac{x_j - x_{i+1}}{x_{j+1} - x_j} \quad \text{and since } |i-j| \geq 2 \\
&\geq \frac{\sigma^{N+1-j}(1-\sigma)}{\sigma^{N+1-(j+1)}(1-\sigma)} \\
&= \frac{1}{\sigma}.
\end{aligned}$$

As a result from [Wi09, Theorem 6.3.1.], when the relative distance of elements decays fast enough (by choosing $\sigma < C_f/4$), using simple Gauss–Legendre quadrature rules yields exponential convergence in quadrature points for the integration of the local stiffness matrices.

For the computation of local stiffness matrices next to or on the diagonal of $\mathbf{S}$ we need composite-Gaussian quadrature rule for optimal convergence. ■

The assembly is carried out as in (2.13) in *Section 2.1.7.*

Condition numbers for the full system $\mathbf{S}$, and subsystems after static condensation is depicted in *Figure 2.19* for different degrees of singularity, using geometric meshes with grading factor $\sigma = \sigma^*$ and polynomial increase slope $\mu = 1$.

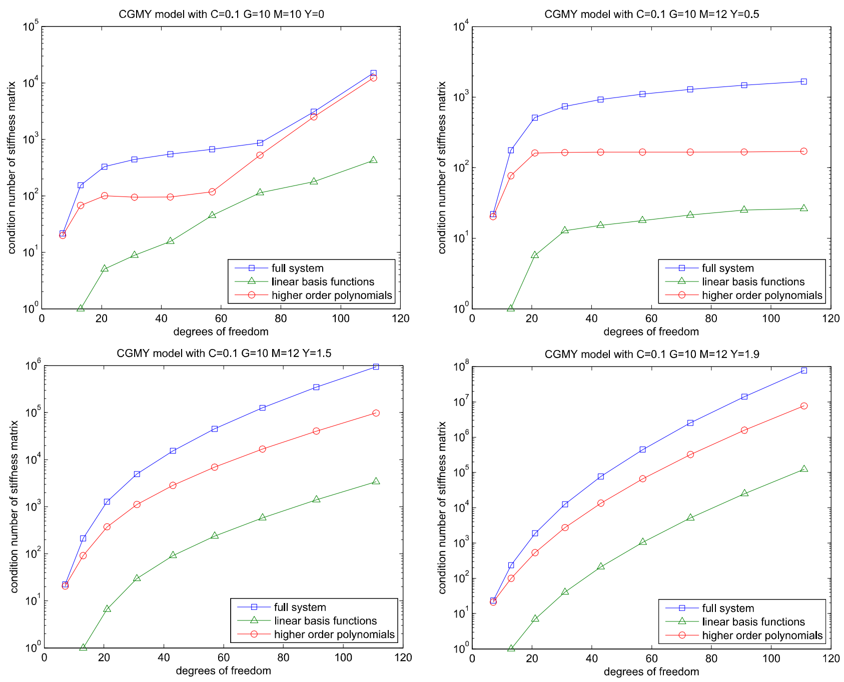
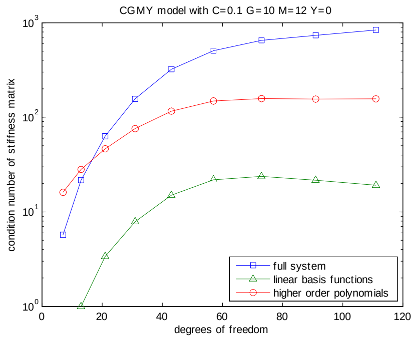


Figure 2.19: Condition numbers of the Lévy stiffness matrix with varios degrees of singularity.

In the VG case with $Y = 0$, using underrefinement, meaning using a larger grading factor, stops the explosion of the condition number, so with $\sigma = 0.5$ we have



Results of using *hp*-FEM with for Lévy operators on functions with singularities is discussed in the next chapter.

Chapter 3

Numerical Examples of hp-FEM from Option Pricing

3.1 Option pricing in Black–Scholes setting

From the pricing equation in (1.1) we will have in log-price $x = \ln S$

$$\begin{aligned} \partial_t u(t,x) + \frac{\sigma^2}{2}\partial_{xx} u(t,x) + \left(\frac{\sigma^2}{2} - r\right)\partial_x u(t,x) + ru(t,x) &= 0, && \text{on } (0,T)\times G \\ u(0,x) &= g(x). && \text{on } G, \end{aligned}$$

with $g(x) = (e^x - K)_+$ for European call contracts.

3.1.1 Variational formulation

The variational formulation reads:
Find $u \in L^2\left((0,T), H^1(\mathbb{R})\right) \cap H^1\left((0,T), L^2(\mathbb{R})\right)$ such that

$$\begin{aligned} \langle \partial_t u, v\rangle_{L^2} + \frac{\sigma^2}{2}\langle u', v'\rangle_{L^2} + \left(\frac{\sigma^2}{2} - r\right)\langle u', v\rangle_{L^2} + r\langle u, v\rangle_{L^2} &= 0, && \text{on } (0,T)\times\mathbb{R}, \\ \langle u, v\rangle_{L^2} &= \langle g, v\rangle_{L^2}, && \text{on } \mathbb{R}, \end{aligned}$$

for all test functions $v \in H^1(\mathbb{R})$.

3.1.2 Localization

We localize the unbounded log-price domain to $G_R = (-R, R)$, and solve for $u \in L^2\left((0,T), H_0^1(G_R)\right) \cap H^1\left((0,T), L^2(G_R)\right)$ such that

$$\begin{aligned} \langle \partial_t u, v\rangle_{L^2} + \frac{\sigma^2}{2}\langle u', v'\rangle_{L^2} + \left(\frac{\sigma^2}{2} - r\right)\langle u', v\rangle_{L^2} + r\langle u, v\rangle_{L^2} &= 0, && \text{on } (0,T)\times G_R, \\ \langle u, v\rangle_{L^2} &= \langle g, v\rangle_{L^2}, && \text{on } G_R, \end{aligned}$$

for all test functions $v \in H_0^1(G_R)$, where $H_0^1(G_R) = \left\{v \in H^1(G_R) : v(x) = 0 \text{ for } x \in \partial G_R\right\}$.

3.1.3 Discretization in space

We use basis functions of $S_0^p(G_R, \tau_h)$ as in *Section 2.1.2*, then following the steps as in *Section 2.1.5* and *Section 2.1.6* we get to the semi-discrete formulation

$$\begin{aligned} \mathbf{M}\underline{\dot{u}} + \mathbf{A}\underline{u} &= 0, \\ \mathbf{M}\underline{u}(0) &= \underline{g}. \end{aligned} \tag{3.1}$$

3.1.4 θ-scheme

We can discretize the system of ODEs in **(3.1)** in time using the θ-scheme. In the time domain we take a sequence of timesteps $\{k_m\}_{m=1}^M$, with $\sum k_i = T$, and define a time-grid as $t_0 = 0$, $t_m = \sum_{i=1}^m k_i$, so that $t_M = T$. By setting $\underline{u}^m = \underline{u}(t_m)$ we get to a fully discrete scheme, which in matrix form reads

$$\begin{aligned} \frac{1}{k_{m+1}}\mathbf{M}\left(\underline{u}^{m+1} - \underline{u}^m\right) + \mathbf{A}\left(\theta\underline{u}^{m+1} + (1-\theta)\underline{u}^m\right) &= 0, \\ \mathbf{M}\underline{u}^0 &= \underline{g}, \end{aligned}$$

which can be rephrased to a recursion as

$$\begin{aligned} (\mathbf{M} + k_{m+1}\theta\mathbf{A})\underline{u}^{m+1} &= (\mathbf{M} - k_{m+1}\theta\mathbf{A})\underline{u}^m, \\ \mathbf{M}\underline{u}^0 &= \underline{g}, \end{aligned}$$

where $\underline{g}_i = \langle g, b_i\rangle_{L^2}$, with $g : G \to \mathbb{R}$ being the initial condition of the PDE, and $\mathbf{M}\underline{u}^0 = \underline{g}$ stands for the projection of the initial condition to $S_0^p(G_R, \tau_h)$.

Using a uniform mesh for both space and time discretization is far from optimal, we only get second order convergence as shown in *Figure 3.1*.

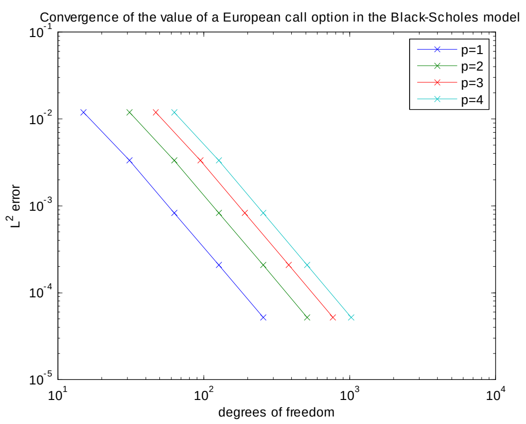


Figure 3.1: Second order convergece in degrees of of freedom using uniform meshes and different uniform polynomial orders.

3.1.5 Refined method

As shown in *Figure 3.1* using higher order polynomial basis functions did not increase the rate of convergence due to the singularity of the payoff function at values x which are near at-the-money. To achieve exponential convergence we move to a different problem defined with the variational formulation

$$\begin{aligned} \langle \partial_t \widetilde{u}, v \rangle_{L^2} + \frac{\sigma^2}{2} \langle \widetilde{u}', v' \rangle_{L^2} &= 0, \qquad \text{on } (0,T) \times \mathbb{R}, \\ \langle \widetilde{u}, v \rangle_{L^2} &= \langle g, v \rangle_{L^2}, \ \text{on } \mathbb{R}, \end{aligned} \tag{3.2}$$

for which we have

$$e^{-rt} \widetilde{u} \left(t, x + \left(\frac{\sigma^2}{2} - r \right) (T-t) \right) = u(t,x).$$

The formulation in (3.2) has no drift and no penalty terms and so the singularity of $\widetilde{u}(t,z)$ will stay at $z=0$ for $t \geq 0$.

Therefore we used two geometric graded meshes in log-price, both refining towards 0, and used a *hp*-version of the discontinuous Galerkin (*hp*-dG) time discretization scheme described in [SoSw99] with a geometric graded time-mesh refining towards maturity, i.e. $t=0$. As *Figure 3.2* shows we have exponential convergence with respect to the degrees of freedom in space. We used the parameters for the model $\sigma_{BS} = 0.3$ and $r = 0.05$; and for the factor for grading the meshes in space $\sigma = 0.5$ and in time $\beta = 0.3$, with polynomial increase slope of $\mu = 0.5$.

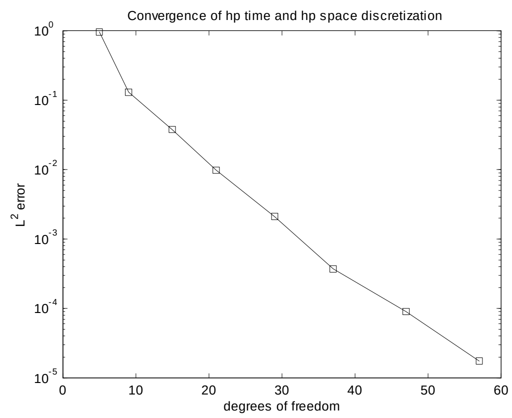


Figure 3.2: Exponential convergence using *hp*-FEM in both space and time for a European call option in the Black–Scholes model

3.2 Option pricing in Lévy setting

From u that solves the pricing equation in (1.5) for the CGMY model, we remove the drift and the discounting to get

$$e^{-rt} \widetilde{u} \left(t, x + \left(\frac{\sigma^2}{2} - r \right) (T-t) \right) = u(t,x)$$

which satisfies

$$
\begin{aligned}
\partial_t \widetilde{u}(t,x) + \int_{\mathbf{R}\setminus\{0\}} \left(\widetilde{u}(t,x+y) - \widetilde{u}(t,x) - \partial_x \widetilde{u}(x)\, y\right) k(y)\, dy &= 0, \\
\widetilde{u}(0,x) &= g(x), \; x \in \mathbb{R}.
\end{aligned}
\tag{3.3}
$$

3.2.1 Variational Formulation

The variational formulation of (3.3) reads:
Find $u \in L^2((0,T), H^\rho(\mathbb{R})) \cap H^1\left((0,T), L^2(\mathbb{R})\right)$ such that

$$
\begin{aligned}
\langle \partial_t \widetilde{u}, v\rangle_{L^2} + a^L(\widetilde{u}, v) &= 0, \qquad \text{on } (0,T)\times\mathbb{R} \\
\langle \widetilde{u}, v\rangle_{L^2} &= \langle g, v\rangle_{L^2}, \quad \text{on } \mathbb{R}, \\
\rho &= \begin{cases} 1 & \text{if } \sigma > 0 \\ \alpha/2 & \text{if } \sigma = 0 \end{cases},
\end{aligned}
\tag{3.4}
$$

for all $v \in H^\rho(\mathbb{R})$, where $a^L(u,v) = \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{x\}} u'(t,x)\, v'(y)\, k^{(-2)}(x-y)\, dydx$ and $H^s(\mathbb{R})$ is a Sobolev space of fractional order s defined through the Fourier space with the norm for $0 \le s \le 1$

$$
\|u\|_{H^s(\mathbb{R})}^2 = \int_{\mathbb{R}} (1+|\xi|)^{2s} |\widehat{u}(\xi)|^2\, d\xi,
$$

where $\widehat{u}$ is the Fourier transform of u.

In Levy setting α governs the singularity of the Levy density as $z \to 0$ in the sense that

$$
|k(z)| \le C \frac{1}{|z|^{\alpha+1}} \quad \text{for } |z| \le 1. \tag{3.5}
$$

3.2.2 Localization

As in the Black–Scholes case we localize in the unbounded log-price domain to $G_R = (-R, R)$ with the space $H_0^\rho(G_R) = \left\{ v \in H^\rho(\mathbb{R}) : u|_{\mathbb{R}\setminus\overline{G}} = 0 \right\}$. Then the localized problem reads:

Find $u \in L^2((0,T), H_0^\rho(G_R)) \cap H^1\left((0,T), L^2(G_R)\right)$ such that

$$
\begin{aligned}
\langle \partial_t \widetilde{u}, v\rangle_{L^2(G_R)} + a_R^L(\widetilde{u}, v) &= 0, \quad \forall v \in H^\rho(G_R), \quad \text{a.e. in } (0,T) \\
\langle \widetilde{u}(0), v\rangle_{L^2(G_R)} &= \langle g, v\rangle_{L^2(G_R)}, \quad \forall v \in H^\rho(G_R), \\
\widetilde{u}(t,x) &= 0 \quad \text{on } (0,T)\times G_R^c,
\end{aligned}
\tag{3.6}
$$

where $a_R^L(u,v) = \int_{G_R} \int_{G_R\setminus\{x\}} u'(t,x)\, v'(y)\, k^{(-2)}(x-y)\, dydx$.

3.2.3 Discretization in space and in time using the θ-scheme

Discretization of (3.6) in space is analogous to *Section 3.1.3* and in time using the θ-scheme is analogous to *Section 3.1.4*.

The results in *Figure 3.3* show that the convergence of the θ-scheme is of second order, regardless of the uniform polynomial degree on the elements.

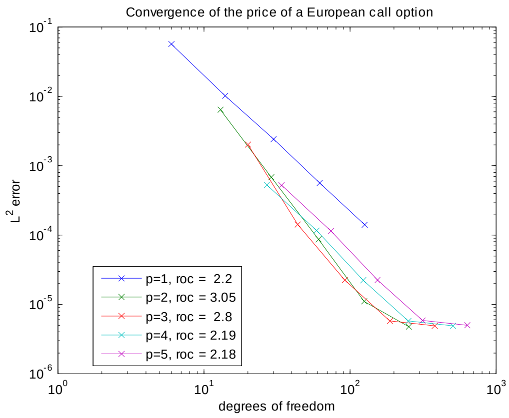


Figure 3.3: L^2 convergence of a European call price using uniform mesh grid and uniform polynomial degrees on the elements

3.2.4 Refined method

Just as in *Section 3.1.5* to achieve exponential convergence we discretized the log-price space with two geometrically graded meshes refining towards 0 with grading factor $\sigma = 0.5$ and increasing polynomial degree on elements away from 0 with linear slope μ. Time domain was discretized by the $hp - dG$ method on a graded time mesh with grading factor $\beta = 0.5$.

The results for Variance Gamma with characteristic parameters $\sigma = 0.3$, $\nu = 0.01$ and $\theta = 0.1$ is shown in *Figure 3.4*.

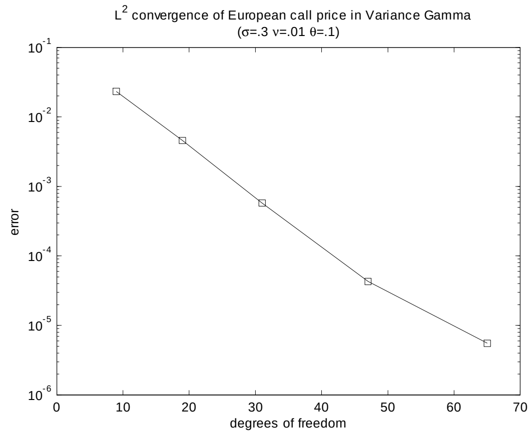


Figure 3.4: Exponential convergence in L^2 for European call price in Variance Gamma model.

Also in the CGMY model we observe exponential convergence in *Figure 3.5* with $Y = 0.5$ and in *Figure 3.6* with $Y = 1.5$.

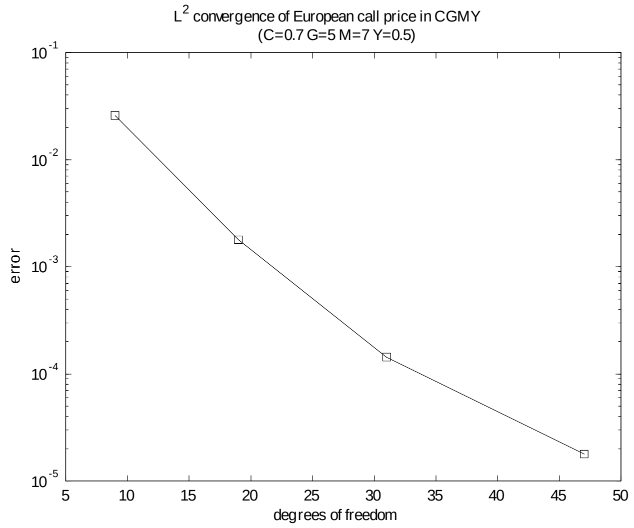


Figure 3.5: Exponential convergence in L^2 for European call price in CGMY model with $Y = 0.5$.

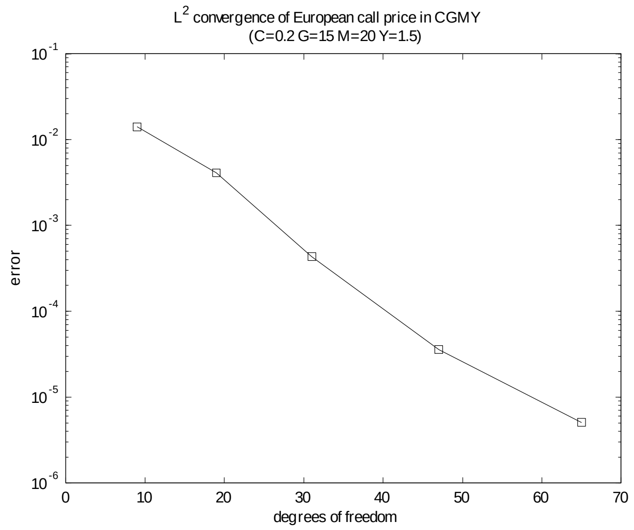


Figure 3.6: Exponential convergence in L^2 for European call price in CGMY model with $Y = 1.5$

Chapter 4

Lookback Options

4.1 Distribution of the supremum of a Lévy-process

The distribution of the supremum of a Lévy process $\hat{L}_t = \sup_{s\in[0,t]} L_s$ is defined as

$$F_{\hat{L}_T}(x) := \mathbf{P}\left[\sup_{t\in[0,T]} L_t \leq x\right].$$

The payoff from a digital barrier option with barrier B at maturity is

$$\Pi_T = \begin{cases} 1 & \text{if } \sup_{t\in[0,T]} S_t < B \\ 0 & \text{if } \sup_{t\in[0,T]} S_t \geq B. \end{cases}$$

Let us denote the price of the digital barrier option by $u^B(t,x) = e^{-r(T-t)}\mathbb{E}\left[\mathbb{1}_{\left\{\sup_{s\in[t,T]} L_t \leq B\right\}}\middle| L_t = x\right]$.

Proposition 19 *The cumulative distribution function of the supremum of a Lévy process satisfies the pricing PIDE*

$$\begin{aligned} \partial_t u - A^L u - ru &= 0 \qquad \textit{on } (0,T)\times\mathbb{R}, \\ u &= 1 \qquad \textit{on } \mathbb{R} \end{aligned}$$

solved on the log-price domain $\mathbb{R}$ *and time domain* $(0,T)$*. With* $u^0(T,x) = e^{-rT}F_{\hat{L}_T}(-x)$*.*

Proof. By using the property of independent increments and the Markov property of the-Lévy process one has

$$\begin{aligned} u^H(t,x) &= \mathbf{P}\left[\hat{L}_T < H\middle| L_t = x\right] \\ &= \mathbf{P}\left[\hat{L}_T < 0\middle| L_t = x - H\right] = u^0(t, x-H), \end{aligned}$$

and then the following holds

$$F_{\hat{L}}(x) = \mathbf{P}\left[\hat{L}_T < x\right] = u^x(T,0) = u^0(T,-x).$$

From that the claimed relation between F and u follows. ■

Therefore the price of a digital barrier option is the discounted value of the cumulative distribution function of the supremum $\sup_{t\in[0,T]} L_t$ of the Lévy process $(L_t)_{t\geq 0}$. This way the distribution function of the supremum can be interpreted as a solution of a modified partial integro-differential equation.

So $F_{\hat{L}}(x)$ can be computed as a solution of

$$\tilde{u}(T-t,x) = e^{r(T-t)}u^0(T-t,x) = \mathbb{E}\left[\mathbb{1}_{\left\{\sup_{s\in[t,T]} L_t\leq 0\right\}}\middle| L_t = x\right] = \mathbf{P}\left[\hat{L}_T < 0\middle| L_t = x\right],$$

where u solves a parabolic equation with initial condition given as

$$\begin{aligned} \partial_t \tilde{u}(t,x) + \mathcal{A}\tilde{u}(t,x) &= 0 \quad \text{on } (0,T)\times\mathbb{R}, \\ \tilde{u}(0,x) &= \mathbb{1}_{x<0}. \end{aligned} \tag{4.1}$$

4.1.1 Variational Formulation

The variational formulation of (4.1), where $\mathcal{A}$ is from (1.6), is analogous to *Section 3.4* and reads:

Find $u \in L^2((0,T), H^\rho(\mathbb{R})) \cap H^1\left((0,T), L^2(\mathbb{R})\right)$, such that

$$\begin{aligned} \langle \partial_t u, v\rangle_{L^2} + a_R^L(u,v) &= 0 \quad \forall v \in H^\rho(\mathbb{R}), \quad \text{a.e. in } (0,T), \\ \langle u(0), v\rangle_{L^2} &= \langle \mathbb{1}_{x<0}, v\rangle_{L^2}, \\ \rho &= \begin{cases} 1 & \text{if } \sigma > 0 \\ \alpha/2 & \text{if } \sigma = 0 \end{cases}, \end{aligned}$$

where α is from (3.5).

4.1.2 Localization

We can localize the problem to a bounded domain of $G_R = (-R, 0)$, and thus impose a zero Dirichlet boundary condition on G_R^c. Then the localized problem is as follows:

Find $u \in L^2(J, H_0^\rho(G_R)) \cap H^1\left(J, L^2(\mathbb{R})\right)$, such that

$$\begin{aligned} \langle \partial_t u, v\rangle_{L^2} + a^L(u,v) &= 0 \quad \text{in } (0,T)\times G_R \\ u(0) &= 1 \quad \text{in } G_R \\ u &= 0 \quad \text{in } (0,T)\times G_R^c \end{aligned} \tag{4.2}$$

4.1.3 Discretization

We discretize (4.2) using the Galerkin method on the space $S^p(G_R, \tau_h)$ with basis functions as in *Section 2.1.2*.

This way the weak formulation in (4.2) can be represented by a system of ODEs in a semi-discrete form using matrix operators.

Find $u \in L^2\left(\bar{J}, S^p(G_R, \tau_h)\right)$, which can be characterized by $\underline{u} \in \mathbb{R}^M$ as $u(x) = \sum_{i=1}^M \underline{u}_i b_i(x)$, and $\underline{u}$ satisfies

$$\mathbf{M}\dot{\underline{u}}(t) + \mathbf{A}\underline{u}(t) = 0, \tag{4.3}$$

where $\mathbf{M}$ is the mass-matrix defined in *Section 2.1.5*, and $\mathbf{A}$ is the stiffness matrix, defined for the Black–Scholes setting in *Section 2.1.6* or for the CGMY setting defined in *Section 2.1.7*.

We can discretize the system of ODEs from (4.3) in time using the θ-scheme as in *Section 3.1.4*, which can be rephrased to a recursion

$$\begin{aligned} (\mathbf{M} + k_{m+1}\theta\mathbf{A})\,\underline{u}_N^{m+1} &= (\mathbf{M} - k_{m+1}\theta\mathbf{A})\,\underline{u}_N^{m}, \\ \underline{u}_N^0 &= \underline{1}. \end{aligned}$$

Black–Scholes setting

The analytical solution of (4.1) in the Black–Scholes model follows from [Sh04], for a an arithmetic Brownian motion $X_t = \alpha t + \sigma W_t$. We define the supremum process as $\hat{X}_t = \sup_{s\in[0,t]} X_t$, then we shall have

$$\mathbf{P}\left[X_T \le m\right] = \mathbf{P}\left[\frac{\alpha}{\sigma}T + W_T \le \frac{m}{\sigma}\right],$$

so by substituting $\alpha = \left(r - \sigma^2/2\right)/\sigma$ and $m = x/\sigma$ into the formula of [Sh04, Corollary 7.2.2.] we will get

$$\mathbf{P}\left[\hat{X}_T \le x\right] = -\exp\left(2\frac{r-\frac{\sigma^2}{2}}{\sigma^2}x\right)\Phi\left(\frac{-x-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{x-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

As show in *Figure 4.1*, without graded meshes the convergence in the $L^2\left(L^2\right)$ norm is far from optimal, and in L^2 the convergence is just of second order.

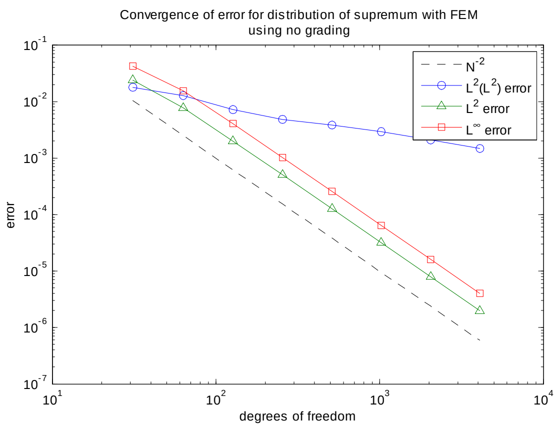


Figure 4.1: Convergence for the supremum of the Black–Scholes model using linear-FEM and θ-scheme. L^2 and L^∞ errors measure the difference of $u\,(T,x)$ and $\mathbf{P}\left[\hat{X}_T \le x\right]$.

Lévy setting

The solution of (4.1) in a CGMY model with $Y = 1.3$ converges also only with second order as shown in *Figure 4.2*.

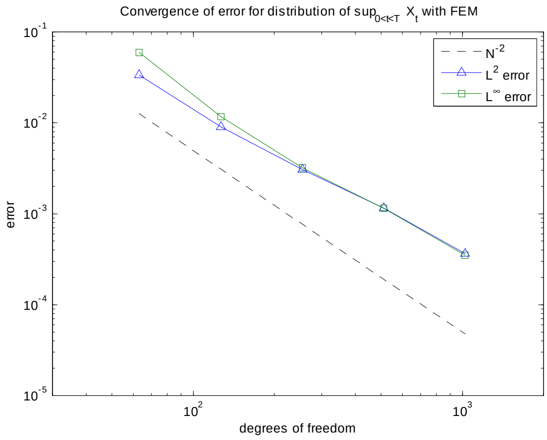


Figure 4.2: Convergence for the supremum of the CGMY model using linear-FEM and θ-scheme.

4.1.4 Refined method

As shown previously in *Section 3.1.5* and *Section 3.2.4*, using geometric graded meshes and higher order polynomials yields exponential convergence even if the initial condition has singularity. To compute the distribution of the supremum we applied a geometric graded mesh for $G_R = (-R, 0)$ with a grading factor of $\sigma = 0.5$, and a geometric mesh in time with grading factor $\beta = \left(\sqrt{2} - 1\right)^2$. The results for the Black–Scholes model are in *Figure 4.3*, and for the CGMY model in *Figure 4.4*.

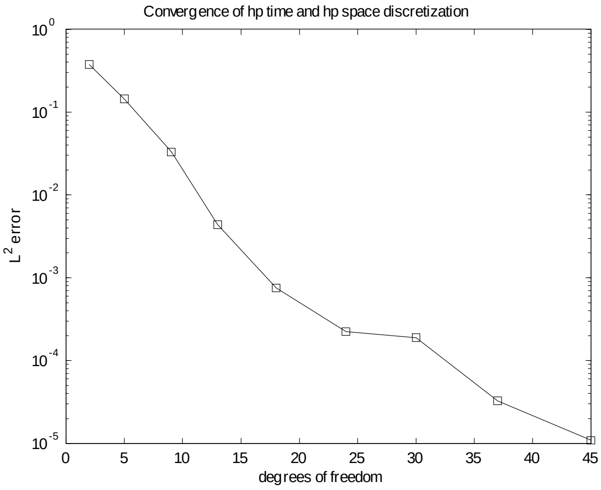


Figure 4.3: Exponential convergence in L^2 for the solution of the supremum of the geometric Brownian motion with hp-FEM in space and time.

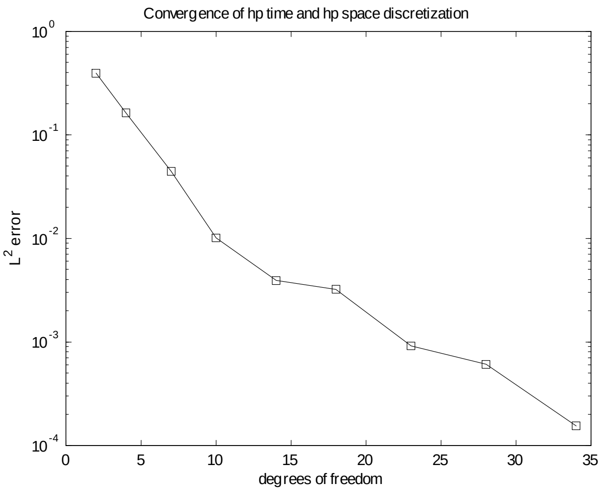


Figure 4.4: Exponential convergence in L^2 for the solution of the supremum of the CGMY process with hp-FEM in space and time.

4.2 Price of a Lookback Option

Theorem 20 *The payoff of a fixed strike lookback call option is a function g of the supremum at maturity of the underlying price process, so the fair price of the option is*

$$v(S_0) = \int_0^\infty \left(1 - F_{\hat{L}_T}(x)\right) d\widetilde{g}(x) + \widetilde{g}(0),$$

where $\widetilde{g}(x) = g(S_0 e^x)$.

Proof. For the price of the lookback-option, v, we have

$$\begin{aligned} v(S_0) &= \mathbb{E}\left[g\left(\hat{S}_T\right)\right] \\ &= \mathbb{E}\left[g\left(S_0 e^{\hat{L}_T}\right)\right] \\ &= \int_0^\infty g(S_0 e^x) F_{\hat{L}_T}(dx). \end{aligned}$$

From [Sa99, Lemma 17.6] we have that for an arbitrary function ξ of bounded variation, and a non-negative random variable X with cumulative distribution function F_X

$$\mathbb{E}[\xi(X)] = \int_c^\infty \xi(x)\, dF_X(x) = \int_c^\infty (1 - F_X(x))\, d\xi(x) + \xi(c),$$

and the relation follows. ■

Theorem 21 *In the CGMY process $\left(1 - F_{\hat{L}}(x)\right)\exp(x)$ is integrable if $G, M \geq 1$ holds.*

Proof. The expression $\int_{\mathbb{R}} \left(1 - F_{\hat{L}_T}(x)\right)\exp(x)\, dx$ is the expectation $\mathbb{E}\left[\exp\left(\hat{L}_T\right)\right]$, for which we can use [Sa99, Theorem 25.18] stating if

$$\mathbb{E}[\xi(|L_t|)] < \infty \qquad \text{for some } t > 0,$$

then

$$\mathbb{E}\left[\xi\left(\hat{L}_t\right)\right] < \infty \qquad \text{for every } t > 0.$$

In the CGMY model, where $\nu(dx) = k(x)\, dx$, and the Lévy density has the form of

$$k(x) = \frac{\exp(-G|x|)}{|x|^{1+Y}} \mathbb{1}_{x<0} + \frac{\exp(-M|x|)}{|x|^{1+Y}} \mathbb{1}_{x>0},$$

we can write

$$\begin{aligned} \mathbb{E}[\xi(|L_t|)] &= \int_{\mathbb{R}} e^{|x|} k(x)\, dx \\ &= \int_{-\infty}^0 e^{|x|} \frac{\exp(-G|x|)}{|x|^{1+Y}} dx + \int_0^\infty e^{|x|} \frac{\exp(-M|x|)}{|x|^{1+Y}} dx \\ &= \int_{-\infty}^0 \frac{\exp(-(G-1)|x|)}{|x|^{1+Y}} dx + \int_0^\infty \frac{\exp(-(M-1)|x|)}{|x|^{1+Y}} dx, \end{aligned}$$

which will be finite if $G - 1 \geq 0$ and $M - 1 \geq 0$, meaning $G, M \geq 1$. ■

4.2.1 Numerical Results

Numerical results in the Black–Scholes case with $\sigma_{BS} = 0.3$ are compared with the closed form solution from [CoVi91], the results are seen in *Figure 4.5* with *hp*-FEM in both time and space. Also the results for the CGMY model with $Y = 1.7$ shown in *Figure 4.6* show exponential convergence in the degrees of freedom.

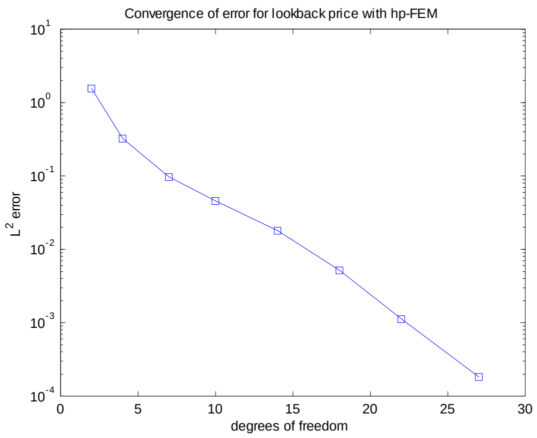


Figure 4.5: Exponential convergence in lookback-option price for the Black–Scholes model using *hp*-FEM in both space and time discretization.

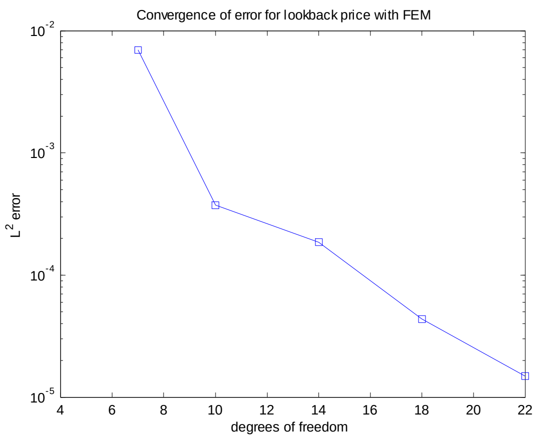


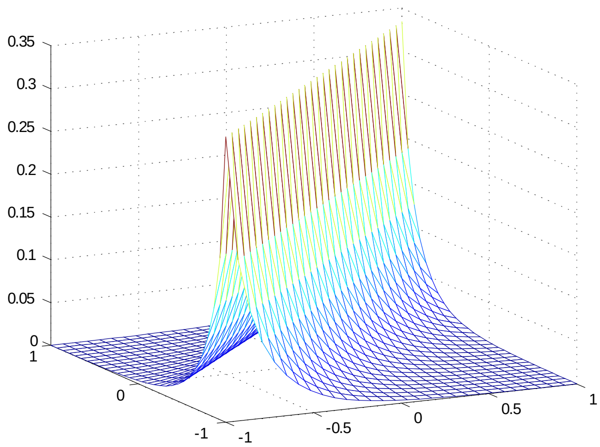
Figure 4.6: Exponential convergence in lookback-option price for the CGMY model using *hp*-FEM in both space and time discretization.

Appendix A

Numerical integration on the reference element

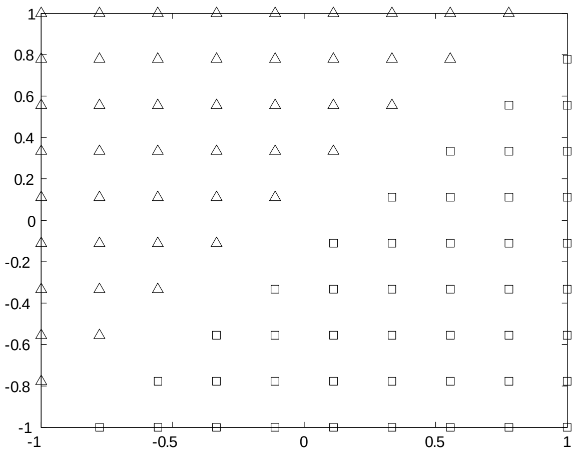
We want to integrate numerically $\int_{-1}^{1}\int_{-1}^{1} k^{(-2)}(x-y)\,dydx$ in order to compute the diagonal elements in the stiffness matrix.

Since k, and $k^{(-i)}$ have singularity at 0, $k^{(-2)}(x-y)$ will have a singularity on the diagonal of $[-1,1]^2$.



So we apply Duffy's trick [Du82], and divide $[-1,1]^2$ into two triangles where $k^{(-2)}(x-y)$

has singularity only on one side, as



So we'll have two integrals as

$$\int_{-1}^{1}\int_{-1}^{1} k^{(-2)}(x-y)\,dydx = \int_{-1}^{1}\int_{-1}^{x} k^{(-2)}(x-y)\,dydx + \int_{-1}^{1}\int_{x}^{1} k^{(-2)}(x-y)\,dydx,$$

and for the ease of numerical integration we transform each triangle to a rectangle, with singularity at one side. For the lower triangle we have with the new variable $z = 1 - \frac{y+1}{x+1}$:

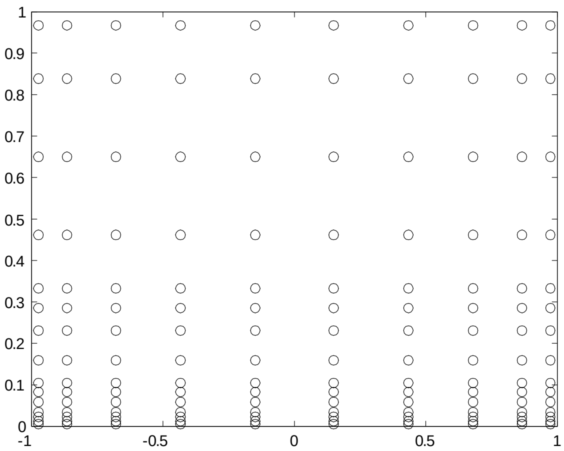
$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{x} k^{(-2)}(x-y)\,dydx &= -\int_{-1}^{1}\int_{1}^{0} k^{(-2)}(x-[x-z(x+1)])(x+1)\,dzdx \\ &= \int_{-1}^{1}\int_{0}^{1} k^{(-2)}(x-[x-z(x+1)])(x+1)\,dzdx.\end{aligned}$$

For the upper triangle we have, with the new variable $z = \frac{y-x}{1-x}$

$$\int_{-1}^{1}\int_{x}^{1} k^{(-2)}(x-y)\,dydx = \int_{-1}^{1}\int_{0}^{1} k^{(-2)}(x-[x+z(1-x)])(1-x)\,dzdx.$$

Now both integrals have singularity at $z = 0$, so we want to have a mesh that is refined towards that singularity.

We chose the composite Gaussian quadrature rule for the z coordinate, as in [Wi09, Ch. 5], and a Gauss–Legendre quadrature rule for x, then for $[-1,1] \times [0,1]$ we'll have a mesh as



Numerically we create the mesh with two matrices defined as

$$X = g \otimes \mathbf{1}_{1\times n_{cg}},$$
$$Z = \mathbf{1}_{n_g \times 1} \otimes cg^T,$$

where g is a vector of 1-d Gauss–Legendre points on $[-1,1]$, cg is a vector of composite Gauss points on $[0,1]$, n_z is the length of cg and n_x is the length of g. The weights of these coordinates are also stored as well, and the algorithm is as follows:

Algorithm 22

```
[x_cg, _w_cg] = compgauss(0, 1, n, sigma, mu, 1);
[x_g, _w_g] = gauleg(m);
X = kron(x_g, ones(1, length(x_cg)));
Z = kron(x_cg, ones(length(x_g), 1));
KL = k2(X - (X - Z. * (X + 1))). * (X + 1);
intL = w_g * KL * w_cg;
KU = k2(X - (Z. * (1 - X) + X)). * (1 - X);
intU = w_g * KU * w_cg;
int = intL + intU;
```

To test the convergence of the code we chose `n=(3:10)` and `mu=2`. The results of the code can be compared with a simple Gauss–Legendre algorithm where cg is replaced with Gauss–Legendre quadrature points of the same length.

To compute the theoretical value to which the code should converge we do the following for the lower triangle with a change of variable $z = x - y$

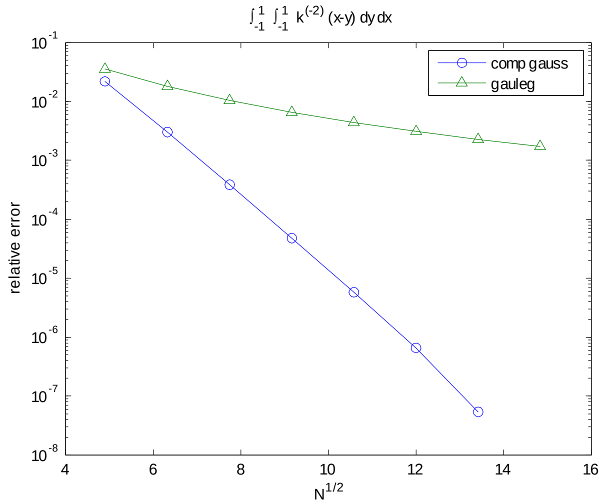
$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{x} k^{(-2)}(x-y)\,dydx &= -\int_{-1}^{1}\int_{x+1}^{0} k^{(-2)}(z)\,dzdx \\
&= \int_{-1}^{1}\int_{0}^{x+1} k^{(-2)}(z)\,dzdx \\
&= \int_{-1}^{1}\left(k^{(-3)}(0+) - k^{(-3)}(x+1)\right)dx \\
&= \int_{-1}^{1} k^{(-3)}(0+)\,dx - \int_{-1}^{1} k^{(-3)}(x+1)\,dx \\
&= 2k^{(-3)}(0+) - \int_{0}^{2} k^{(-3)}(v)\,dv \\
&= 2k^{(-3)}(0+) - \left(k^{(-4)}(0+) - k^{(-4)}(2)\right) \\
&= 2k^{(-3)}(0+) - k^{(-4)}(0+) + k^{(-4)}(2)\,,
\end{aligned}$$

and for the upper triangle it's analogously $2k^{(-3)}(0-) - k^{(-4)}(0-) + k^{(-4)}(-2)$.

Where, in the CGMY model

$$\begin{aligned}
k^{(-3)}(0\pm) &= C\frac{\Gamma(2-\alpha)}{2}\beta_{\pm}^{Y-2} \\
k^{(-4)}(0\pm) &= C\frac{\Gamma(3-\alpha)}{6}\beta_{\pm}^{Y-3},
\end{aligned}$$

with $\beta_+ = M$ and $\beta_- = G$, and the convergence, compared with a simple Gauss-Legendre rule on both axis with the same number of points looks like



Appendix B

Stiffness matrix with the Lévy kernel

Discretizing the equation $\mathcal{A}u = f$ in the hp-FE space leads to a matrix equation

$$\mathbf{S}\mathbf{u} = \mathbf{f},$$

where S is the stiffness matrix with elements as

$$\mathbf{S}_{m,n} = \left\langle \mathcal{A}^L b_n, b_m \right\rangle_{L^2},$$

where b_m is the k-th order basis function on the element K_i. $\mathbf{f}$ is the load vector with elements

$$\mathbf{f}_m = \left\langle f, b_m \right\rangle_{L^2}.$$

As shown above, in the case of a Lévy operator the elements of the stiffness matrix can be written as

$$\mathbf{S}_{(i,k)(j,l)} = \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \nabla N_l^i(y) \nabla N_k^j(x) k^{(-2)}(x-y) \, dy dx,$$

where N_k^i is the p-th order shape function on the element K_i, and $k^{(-2)}$ is the second antiderivate of the Lévy measure $\nu(dx) = k(x)\,dx$. By substituting with the diffeomorphism $\varphi_i(\xi_x) = \frac{x_i + x_{i-1}}{2} + \frac{x_i - x_{i-1}}{2}\xi_x$, and ξ_y analogously, we move to the reference element

$$\mathbf{S}_{(i,k)(j,l)} = \int_{-1}^{1}\int_{-1}^{1} \nabla N_l^i\left(\varphi_i\left(\xi_y\right)\right) \nabla N_k^j\left(\varphi_j\left(\xi_x\right)\right) \ \kappa\left(\varphi_j\left(\xi_x\right) - \varphi_i\left(\xi_y\right)\right) \ \frac{h_j}{2} d\xi_y \frac{h_i}{2} d\xi_x,$$

and by using that $\nabla \hat{N}(x) = \nabla\left(N\left(f(x)\right)\right) = \nabla N\left(f(x)\right) f'(x)$, we'll have

$$\mathbf{S}_{(i,k)(j,l)} = \int_{-1}^{1}\int_{-1}^{1} \nabla \hat{N}_l\left(\xi_y\right) \nabla \hat{N}_k\left(\xi_x\right) \ \kappa\left(\varphi_j\left(\xi_x\right) - \varphi_i\left(\xi_y\right)\right) \ d\xi_y d\xi_x.$$

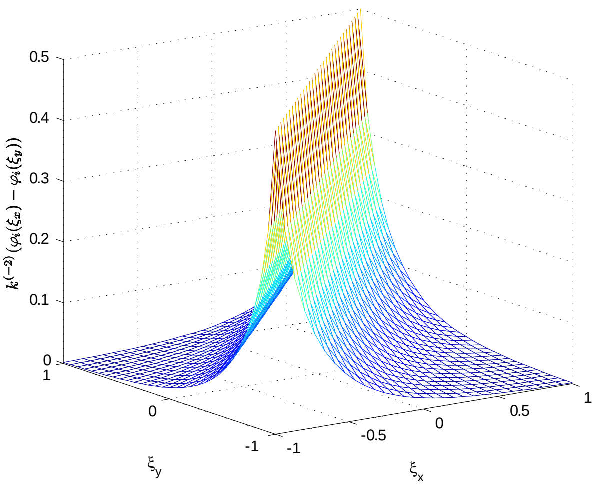
From here we will distinguish four cases for the local stiffness matrices defined element-vise

1. diagonal blocks of $\mathbf{S}_{i,i}$

2. sup-diagonal blocks of $\mathbf{S}_{i,i+1}$
3. sub-diagonal blocks of $\mathbf{S}_{i,i-1}$
4. off diagonal blocks of $\mathbf{S}_{i,j}$, where $|i-j| \geq 2$.

B.1 Diagonal blocks

In this case we have integrals of the type $\int\int \nabla\hat{N}_l\left(\xi_y\right)\nabla\hat{N}_k\left(\xi_x\right)\ \kappa\left(\varphi_i\left(\xi_x\right)-\varphi_i\left(\xi_y\right)\right)\ d\xi_y d\xi_x$, and since the kernel κ has singularity at 0, the integral will have singularity at the vertex,



so we apply Duffy's trick [Du82], and have

$$\begin{aligned}\mathbf{S}_{(i,k)(i,l)} &= \int_{-1}^{1}\int_{-1}^{\xi_x}\nabla\hat{N}_l\left(\xi_y\right)\nabla\hat{N}_k\left(\xi_x\right)\ \kappa\left(\varphi_j\left(\xi_x\right)-\varphi_i\left(\xi_y\right)\right)\ d\xi_y d\xi_x \\ &\quad+\int_{-1}^{1}\int_{\xi_x}^{1}\nabla\hat{N}_l\left(\xi_y\right)\nabla\hat{N}_k\left(\xi_x\right)\ \kappa\left(\varphi_j\left(\xi_x\right)-\varphi_i\left(\xi_y\right)\right)\ d\xi_y d\xi_x,\end{aligned}$$

then do a change of variables for the lover triangle as $\xi_y = -u\left(1+\xi_x\right)+\xi_x$ and for the upper triangle as $\xi_y = v\left(1-\xi_x\right)+\xi_x$, to have

$$\begin{aligned}\mathbf{S}_{(i,k)(i,l)} &= \int_{-1}^{1}\int_{0}^{1}\nabla\hat{N}_l\left(-u\left(1+\xi_x\right)+\xi_x\right)\nabla\hat{N}_k\left(\xi_x\right)\ \kappa\left(\varphi_j\left(\xi_x\right)-\varphi_i\left(-u\left(1+\xi_x\right)+\xi_x\right)\right)\ \left(1+\xi_x\right)d\xi_y d\xi_x \\ &\quad+\int_{-1}^{1}\int_{0}^{1}\nabla\hat{N}_l\left(v\left(1-\xi_x\right)+\xi_x\right)\nabla\hat{N}_k\left(\xi_x\right)\ \kappa\left(\varphi_j\left(\xi_x\right)-\varphi_i\left(v\left(1-\xi_x\right)+\xi_x\right)\right)\ \left(1-\xi_x\right)d\xi_y d\xi_x,\end{aligned}$$

where now both integrals are on a rectangle with singularity on the lower edge, so a composite Gaussian quadrature rule is used to discretize both u and v, and a simple Gauss–Legendre is used to discretize along ξ_x.

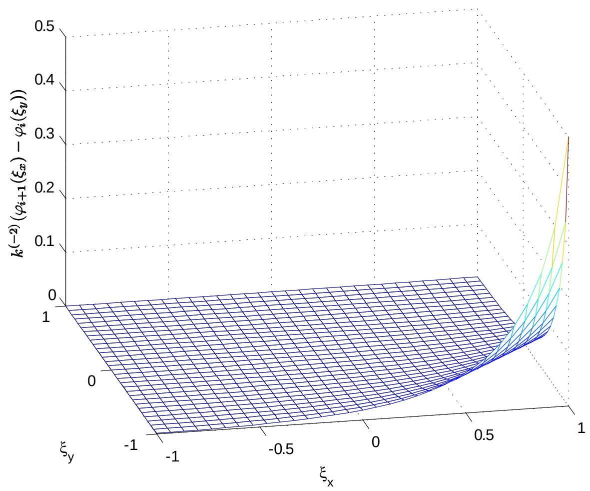
The algorithm looks like

Algorithm 23

```
[x_cg, w_cg] = comp_gauss(n);
[x_g, w_g] = gau_leg(m);
X = kron(x_g, ones(1, n));
Z = kron(ones(m, 1), x_cg);
Y_u = X + Z. * (1 - X);
Y_l = X - Z. * (1 + X);
grad N_x = legendre_grad(X);
grad N_yu = legendre_grad(Y_u);
grad N_yl = legendre_grad(Y_l);
K_u = k2([m_x + h_x/2 * X] - [m_y + h_y/2 * Y_u]). * (1 - X);
K_u = k2([m_x + h_x/2 * X] - [m_y + h_y/2 * Y_l]). * (1 + X);
% elements
for i = 1 : p_y + 1
 for i = 1 : p_x + 1
  S(i, j) = w_g'*(grad N_yu(i). * grad N_x(j). * K_u + grad N_yl(i). * grad N_x(j). * K_l) * w_cg;
 end
end
```

B.2 Sup-diagonal blocks

For the blocks of $\mathbf{S}_{i,i+1}$, the kernel $k^{(-2)}$ in the integral $\int\int \nabla\hat{N}_l\left(\xi_y\right)\nabla\hat{N}_k\left(\xi_x\right)\,\kappa\left(\varphi_{i+1}\left(\xi_x\right)-\varphi_i\left(\xi_y\right)\right)\,d\xi_y d\xi_x$ will have a singularity in one corner of $\hat{K}\times\hat{K}$



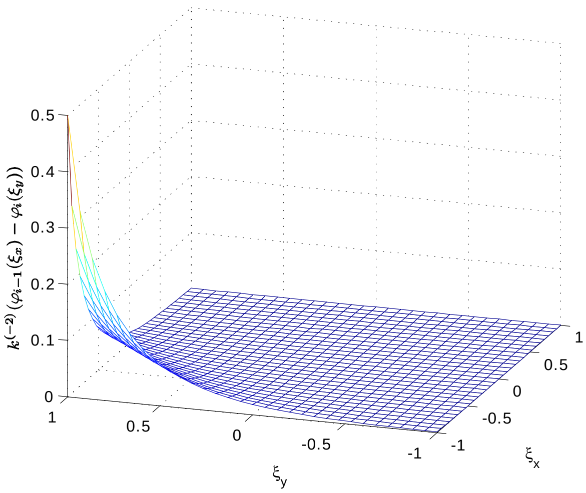
and so we discretize both ξ_x and ξ_y with a composite Gauss quadrature, refining towards $\xi_x \nearrow 1$ and $\xi_y \searrow -1$.

Algorithm 24

```
[x_cg, w_cg] = comp_gauss(n);
X = kron(1 - 2 * x_cg, ones(1, n));
Y = kron(ones(n, 1), 2 * x_cg - 1);
grad N_x = legendre_grad(X);
grad N_y = legendre_grad(Y);
K = k2((m_x + h_x/2 * X) - (m_y + h_y/2 * Y));
% elements
for i = 1 : p_y + 1
 for i = 1 : p_x + 1
  S(i, j) = -4 * w_cg'*(grad N_y(i). * grad N_x(j). * K) * w_cg;
 end
end
```

B.3 Sub-diagonal blocks

For the blocks of $\mathbf{S}_{i,i-1}$, the kernel $k^{(-2)}$ in the integral $\int\int \nabla \hat{N}_l\left(\xi_y\right) \nabla \hat{N}_k\left(\xi_x\right)\ \kappa\left(\varphi_{i-1}\left(\xi_x\right) - \varphi_i\left(\xi_y\right)\right)\ d\xi_y d\xi_x$ will have a singularity in one corner of $\hat{K} \times \hat{K}$



and so we discretize both ξ_x and ξ_y with a composite Gauss quadrature, refining towards $\xi_y \nearrow 1$ and $\xi_x \searrow -1$.

Algorithm 25

```
[x_cg, w_cg] = comp_gauss(n);
X = kron(2 * x_cg - 1, ones(1, n));
Y = kron(ones(n, 1), 1 - 2 * x_cg);
grad N_x = legendre_grad(X);
grad N_y = legendre_grad(Y);
K = k2((m_x + h_x/2 * X) - (m_y + h_y/2 * Y));
% elements
for i = 1 : p_y + 1
```

```
  for i = 1 : p_x + 1
   S(i, j) = −4 ∗ w_cg′∗(grad N_y(i). ∗ grad N_x(j). ∗ K) ∗ w_cg;
  end
end
```

B.4 Off diagonal blocks

To compute the integrals derived from the Lévy operator on blocks further from the diagonal, we can use simple quadratures since $k^{(-2)}$ decays fast enough away from 0 to make $k^{(-2)}(x-y)$ relatively constant on element pairs that are not neighbours. For this property we need the relative distance of the elements to decay fast enough, and it is shown in *Proposition 18*, but with

$$\begin{aligned} \delta_{i,j}^{rel} &= \frac{\operatorname{dist}(K_i, K_j)}{\max\{|K_i|, |K_j|\}} \quad \text{w.l.o.g. suppose } i < j \\ &= \frac{x_j - x_i}{h} \quad \text{since } |i-j| \geq 2 \text{ for off-diagonal blocks} \\ &\geq \frac{h}{h} = 1 \end{aligned}$$

B.5 Assembly of the global stiffness matrix

We index the basis functions in a way that we first number the external shape functions from 1 to $N+1$, so the neighbouring N_0 and N_1 linear basis functions form hat functions, and so their contributions will add up during assembly. Then we number the internal shape functions from $N+2$ to M, where M is the number of all shape functions in all elements, and

$$M = 1 + \sum_{i=1}^{N} p_i,$$

where p_i is the polynomial degree on element K_i, and we have altogether $N+1$ linear hat functions (two of which are "half-hats").

Then the assembly algorithm looks like

Algorithm 26 (Assembly of integral-operator stiffness matrix)

```
for i = 1 : nElements
 for j = 1 : nElements
  idx_i = mesh.basisnumbering(i, :);
  idx_j = mesh.basisnumbering(j, :);
  Aloc = local_stiffness(mesh.coord(i), mesh.coord(j), mesh.poly(i), mesh.poly(j));
  A(idx_i, idx_j) = A(idx_i, idx_j) + Aloc;
 end
end
```

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